Plactic-like monoids and Hopf algebras

The 70$^{\text{th}}$ Séminaire Lotharingien de Combinatoire
Ellwangen

Jean-Baptiste Priez

Laboratoire de Recherche en Informatique
Faculté des sciences d'Orsay
Université Paris-Sud

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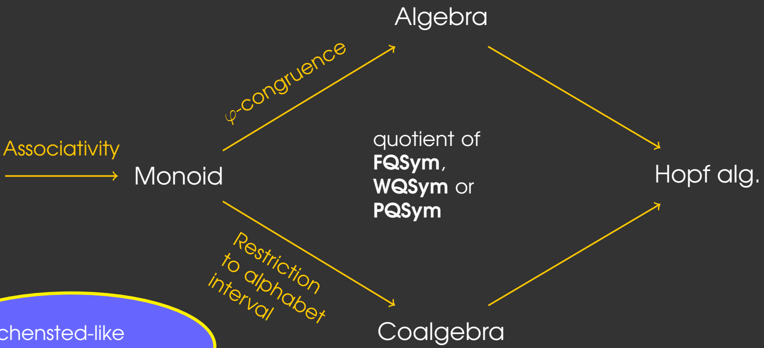
Outline of the talk

Build Hopf algebra from plactic-like monoids: [P., 13]

Relations :

- Plactic
- Sylvester
- Stalactic
- Baxter
- . . .

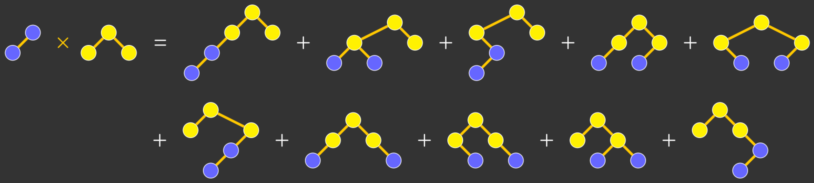


Schensted-like correspondance

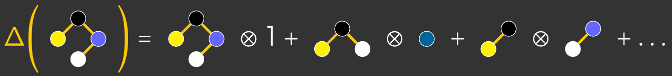
Hopf algebra

Combinatorial Hopf algebra (CHA):

- vector space $\mathcal{H}$
- with *basis* indexed by a *combinatorial class*
- $\times$: the product *assembles* elements

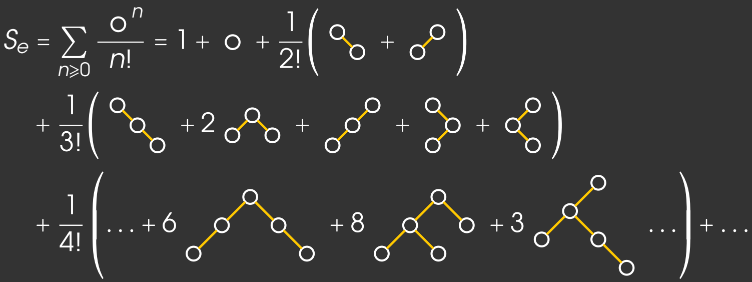


- Δ : the coproduct *disassembles* elements



Improve the formal series expressivity

Replace the notion of size by the objects :



More information : Tree T coefficients ?

→ *The number of permutations giving T by binary search tree insertion*

Polynomial realization (1)

Alphabet :

Let $\mathfrak{A}$ be a *totally ordered* and *infinite alphabet*.

φ-maps :

Let $\mathcal{C}$ be a *combinatorial class* :

- permutations $\mathfrak{S}$
- packed words (or ordered set partitions) **PW**
- parking functions **PF**

The map $\varphi : \mathfrak{A}^* \to \mathcal{C}$ is a *φ-map* if it is :

- *std* : $\mathfrak{A}^* \to \mathfrak{S}$: the *standardization* map (***FQSym***)
- *pack* : $\mathfrak{A}^* \to$ **PW** : the *packing* map (***WQSym***)
- *park* : $\mathfrak{A}^* \to$ **PF** : the *parking function* map (***PQSym***)

φ-maps

Let $w := 5753388783888$ be a word on $\mathbb{N}_{>0}$.

Standardization :

Build the *unique* $\sigma \in \mathfrak{S}$ with same *length* and same *inversion set*.

$$std(w) = 46512897\mathtt{A3BCD}$$

Packing : [Duchamp-Hivert-Thibon, 02]

Build the *unique* $u \in$ **PW** with same *length* k and with same *ordered set partitions of* $[k]$.

$$pack(w) = 2321144341444$$

Polynomial realization (2)

[Hivert-Novelli-Thibon (05), Novelli-Thibon (06)]

Let $\mathcal{H}$ be a Hopf algebra with basis indexed by a *combinatorial class* $\mathcal{C}$.

A *polynomial realization* $r_{\mathfrak{A}} : \mathcal{H} \to \mathbb{K}\langle\mathfrak{A}\rangle$ is an injective morphism : for any element basis m_σ

$$r_{\mathfrak{A}}(m_\sigma) := \sum_{\substack{w \in \mathfrak{A}^* \\ \varphi(w)=\sigma}} w\,.$$

Why use polynomial realization :

- use *plactic-like congruences* on $\mathfrak{A}^*$ to
- define a *quotient Hopf algebra*

Polynomial realization of **FQSym**

Let $\sigma := 1423$ be a permutation.
Let $\mathbb{N}_{>0}$ be the alphabet of non-negative integers.

$$r_{\mathbb{N}_{>0}}(\mathbb{G}_\sigma) = \sum_{\substack{w \in \mathbb{N}^*_{>0} \\ std(w)=\sigma}} w$$

$$= 1211 + 1312 + 1322 + 1411 + \cdots + 2322 + \cdots + 4967 + \cdots$$

- If $\sigma := 123\ldots = Id_n$ the identity of size n,

$$r_{\mathbb{N}_{>0}}(\mathbb{G}_\sigma) = \sum_{u_1 \leqslant \ldots \leqslant u_n} u_1 \ldots u_n$$

- If $\sigma := n(n-1)\ldots 1$ the $\max(\mathfrak{S}_n)$,

$$r_{\mathbb{N}_{>0}}(\mathbb{G}_\sigma) = \sum_{u_1 > \ldots > u_n} u_1 \ldots u_n$$

Why plactic-like ?

[Poirier-Reutenauer (95), Duchamp-Hivert-Thibon (02)]

Plactic relations :

$$u \cdot acb \cdot v \equiv_p u \cdot cab \cdot v \qquad \text{for } a \leqslant b < c$$

$$u \cdot bac \cdot v \equiv_p u \cdot bca \cdot v \qquad \text{for } a < b \leqslant c$$

The *Schensted algorithm* computes classes under the *plactic relations*.

$$(11,8,10,12,2,5,9,1,3,4,6,7) \rightarrow$$

11				
8	10	12		
2	5	9		
1	3	4	6	7

Realize ***FSym*** : the *Hopf algebra of tableaux* as the ***FQSym*** quotient by the *plactic relations*

Congruence and φ-congruence

Let u,v,w and x be words.

Congruence :

An *equivalence relation* $\equiv$ is a *congruence* if

$$u \equiv v \text{ and } w \equiv x \text{ then } u \cdot w \equiv v \cdot w.$$

φ-congruence :

A *congruence* is a *φ-congruence* if

$$u \equiv v \text{ if and only if } \begin{cases} \varphi(u) \equiv \varphi(v) \\ ev(u) = ev(v) \end{cases}$$

where $ev(w)$ counts the number of occurrence of each letter.

Remark : $\mathfrak{S}$, **PW** and **PF** could be identify as set of word-like
Recall : φ is *std*, *pack* or *park*

Congruence and φ-congruence

Le

C

A

φ

A

w

R

R

Example : φ-congruence

$$2725 \equiv_p 2275$$

$$328 \equiv_p 382$$

$$\Downarrow \cdot$$

$$2725 \cdot 328 \equiv_p 2275 \cdot 382$$

$$\Downarrow \mathit{std}$$

$$1423 \equiv_p 1243,\ 213 \equiv_p 231,$$

$$1625437 \equiv_p 1265473$$

Compatibility with restriction to alphabet intervals

[Lascoux-Schützenberger, 81]

A *congruence* is compatible with *restriction to alphabet intervals* if for any interval $I \subset \mathfrak{A}$,

$$\text{if } u \equiv v \text{ then } u_I \equiv v_I$$

where w_I is word restricted to the alphabet I.

Compatibility with restriction to alphabet intervals

Example : Restriction to alphabet intervals

$$2725328 \equiv_p 7522382$$

restrict to $[3,7]$: $753 \equiv_p 753$

restrict to $[2,5]$: $22532 \equiv_p 52232$

restrict to $[14,\infty]$: $\varepsilon \equiv_p \varepsilon$

$\vdots$

Plactic-like congruence

A *congruence* $\equiv$ is a *φ-plactic-like congruence* if $\equiv$

- is *φ-congruence* and
- is compatible with *restriction to alphabet intervals*

Theorem ([P., 13])

Let $\mathcal{H}$ be a *Hopf algebra* with basis indexed by a *combinatorial class $\mathcal{C}$*.
Let $\varphi : \mathfrak{A}^* \to \mathcal{C}$ be its *φ-maps* associated.

For any *φ-plactic-like congruence* $\equiv$, the quotient $\mathcal{H}/\equiv$ is a *Hopf algebra quotient* with basis indexed by $\mathcal{C}/\equiv$.

Operations on plactic-like congruences

Let $\sim$ and $\approx$ be both *plactic-like congruences*.

The *union* $\sim \vee \approx$: the *smallest congruence* containing both.

The *intersection* $\sim \wedge \approx$: the congruence $\equiv$ such that $u \equiv v$ if $u \sim v$ *and* $u \approx v$.

Theorem ([P., 13])

The *union* and the *intersection* of two *plactic-like congruences* are *plactic-like congruences*.

Corollary

That implies a *lattice structure* on *Hopf algebras*.

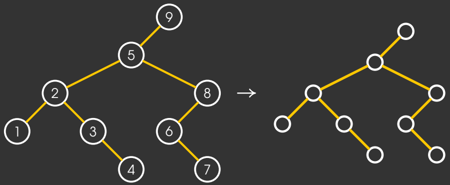
Sylvester congruence [Hivert-Novelli-Thibon, 05]

Relation :

$$u \cdot ac \cdot w \cdot b \cdot v \equiv_{sylv} u \cdot ca \cdot w \cdot b \cdot v \qquad \text{for } a \leqslant b < c$$

Schensted-like algorithm : Binary Search Tree (BST) insertion

(476312859) →



Serie : Catalan

Realize the Loday-Ronco *Hopf algebra* (***PBT***)

Stalactic congruence [Hivert-Novelli-Thibon, 06]

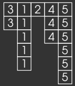
Relation :

$$u \cdot ba \cdot w \cdot b \cdot v \equiv_{stal} u \cdot ab \cdot w \cdot b \cdot v$$

Schensted-like algorithm : Stalactic insertion

Sketch : last letter occurrence + multiplicity

$(5154315114531245 5) \rightarrow$



$D_n := \sum_{k=1}^{n} k! \binom{n-1}{k-1}$ ($k = \#$ different letters).

Serie : $St(z) = \sum_{n \geqslant 0} D_n z^n = 1 + z + 3z^2 + 11z^3 + 49z^4 + 261z^5 + 1631z^6 + \cdots$

Realize the *Stalactic Hopf algebra* (***WQSym***$/\equiv_{stal}$)

Sylvester union Stalactic [P., 13]

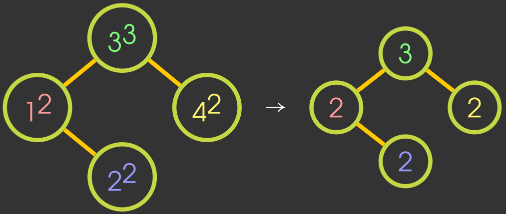
Relations :

$$u \cdot ac \cdot w \cdot b \cdot v \equiv_{sylv} u \cdot ca \cdot w \cdot b \cdot v \qquad \text{for } a \leqslant b < c$$

$$u \cdot ba \cdot w \cdot b \cdot v \equiv_{stal} u \cdot ab \cdot w \cdot b \cdot v$$

Schensted-like algorithm : $\mathcal{BST}_m$:= analogous BST insertion

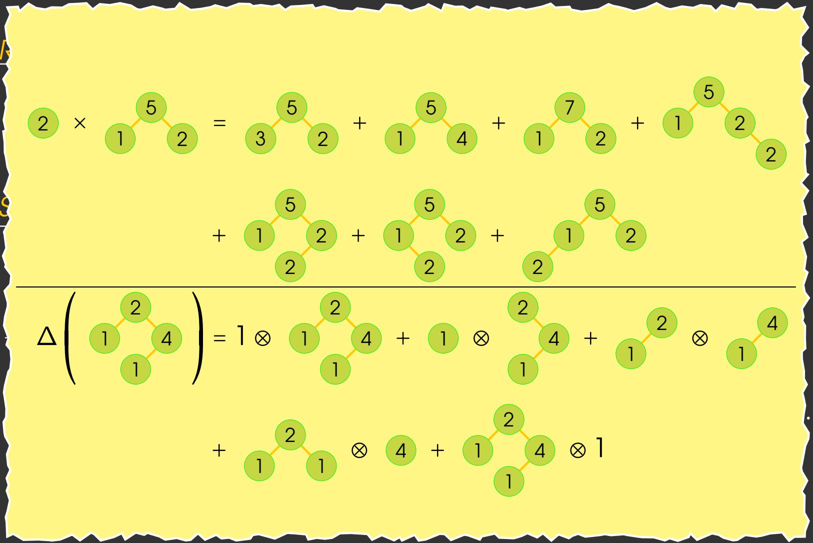
$\mathcal{BST}_m(122344133) =$



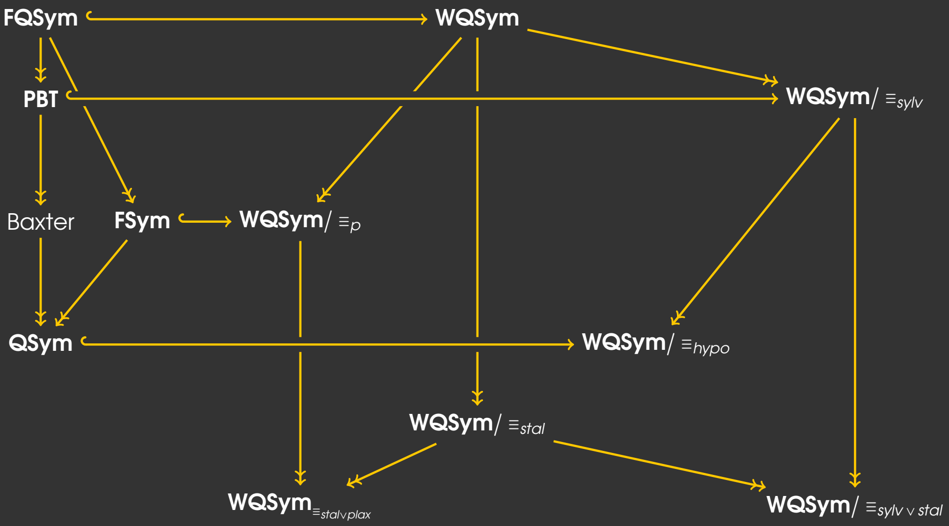
Other results :

- *Hook length formula* (generating series proof)
- *Serie* : $S(t) = \frac{1-t-\sqrt{5t^2-6t+1}}{2t} = 1 + t + 3t^2 + 10t^3 + 36t^4 + 137t^5 + \ldots$

Sylvester union Stalactic [P., 13]



Lattice structure on Hopf algebras



Hopf algebra

Polynomial realization

Plactic-like congruences

Examples of plactic-like congruences and operations

Lattice

Perspectives

Perspectives

- grow the lattice structure
  - *Stalactic* $\vee$ *Plactic* ?

$$SG(t) = 1 + t + 3t^2 + 8t^4 + 23t^5 + 67t^6 + \dots$$

- links with N. READING works
- generalize pattern to CONNES-KREIMER Hopf algebra
- Dendriform links

Hook length formula : BTm

Formula :

$$h_l(T) = |T| \left(\prod_{t \in T} |T| \times (e(t) - 1)! \right)^{-1}$$

Generating series proof :

$$x = a + \sum_{k \geqslant 1} B_k(x, x)$$

with $B_k(x, y)$ a bilinear map and its valuation is strictly greater than $ev(x) + ev(y)$.

Dendriform results

Lemma

The $\equiv$ carries the *dendriform structure* of the *convolution product*.

Let $\equiv$ a *plactic-like congruence*.

Lemma

If for any u, v such that $u \equiv v$ we have $last(u) = last(v)$ then the congruence carries the *codendriform structure* of Δ.

Work in progress : "Theorem"

If $\equiv$ respects the previous Lemma then $\mathcal{H}/\equiv$ must be *autodual*...