The Hirsch Conjecture and its relatives (part I of III)

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OR

Hirsch Wars Trilogy

Slides (Seville version, March 2012):

`http://personales.unican.es/santosf/Hirsch/Wars`

1. Episode I: The Phantom Conjecture. (Today)
2. Episode II: Attack of the Prismatoids + Episode III: Revenge of the Linear Bound. (Tomorrow)
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Polyhedra and polytopes

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

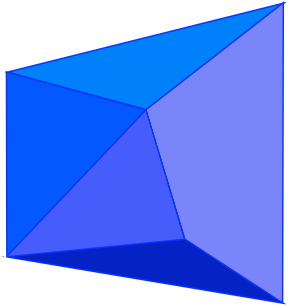
A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.



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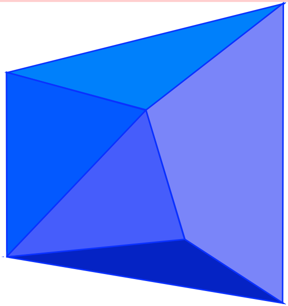
Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, every bounded polyhedron is a polytope.



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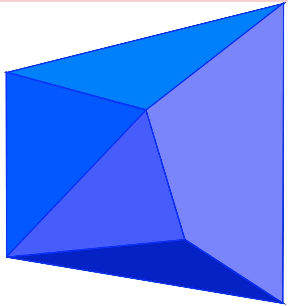

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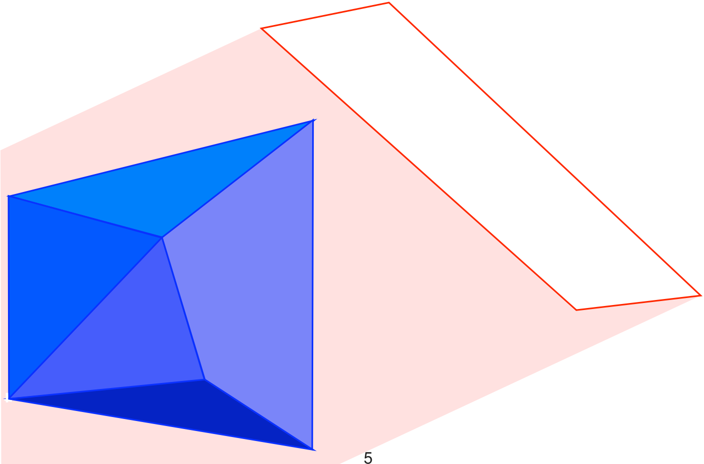
Faces of *P*

Let *P* be a polytope (or polyhedron) and let *H* be a hyperplane not cutting, but touching *P*.



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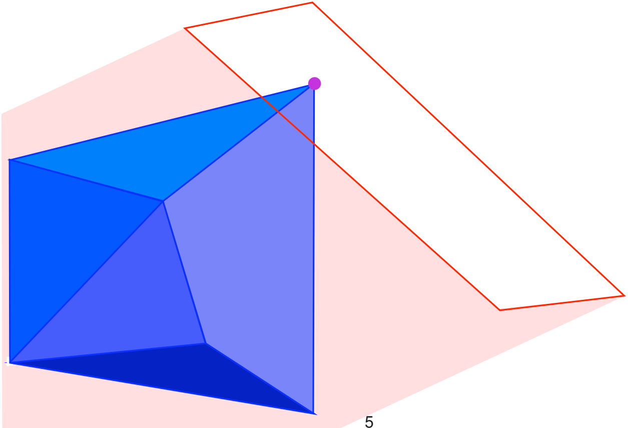
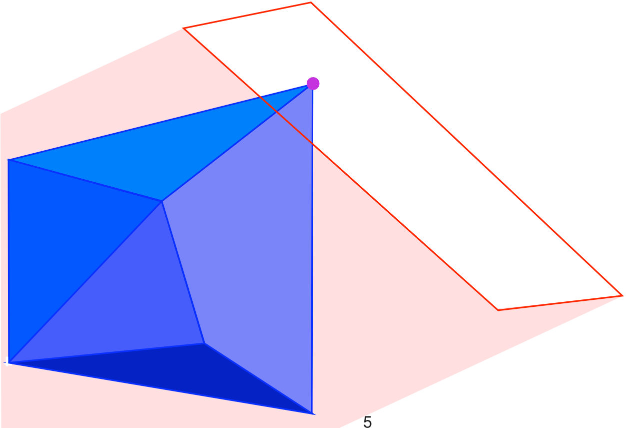


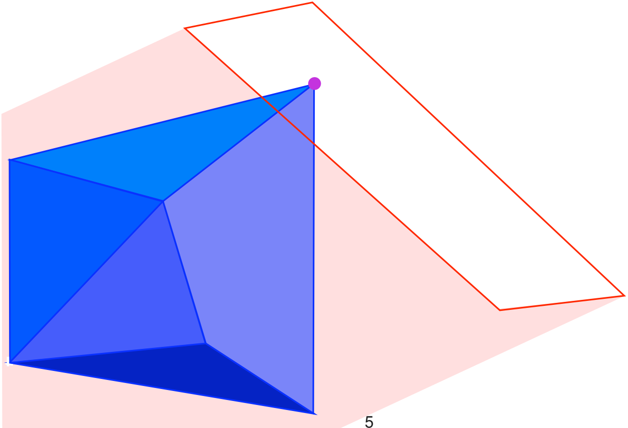
Faces of P

We say that $H \cap P$ is a face of P.



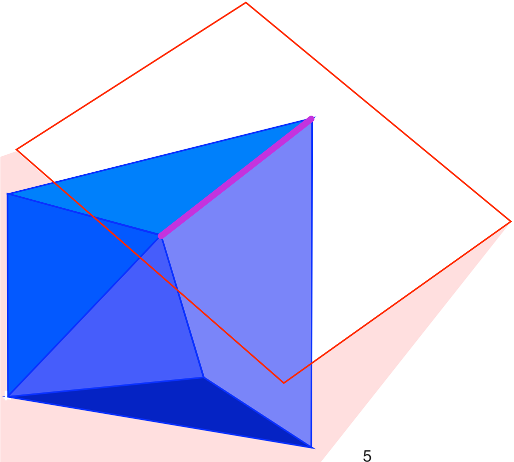
Faces of P

Faces of dimension 0 are called vertices.



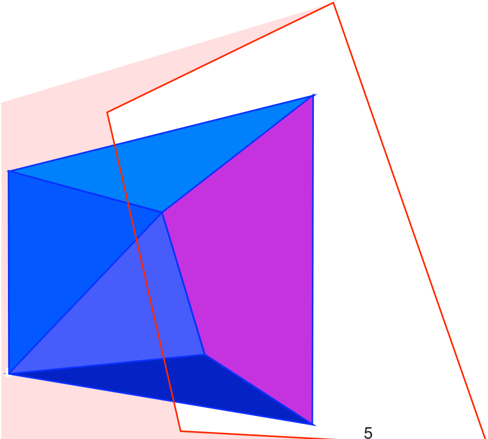
Faces of P

Faces of dimension 1 are called edges.



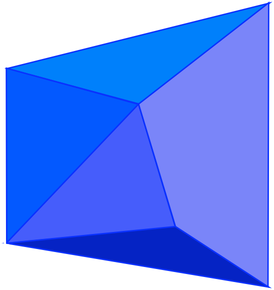
Faces of P

Faces of dimension $d-1$ are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a graph (finite, undirected)



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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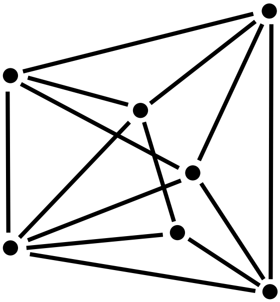

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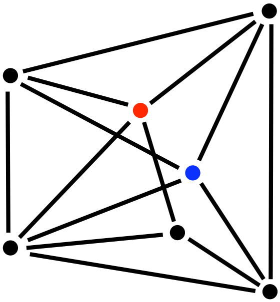

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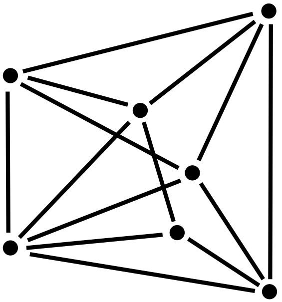

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\mathrm{diam}(P) = \max\{d(u, v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\mathrm{diam}(P) \leq n - d.$$

polytope	facets	dimension	$n-d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k+2$	3	$k-1$	$\lfloor k/2 \rfloor + 1$
n-cube	$2n$	n	n	n

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Brief history of the conjecture

1. It was communicated by W. M. Hirsch to G. Dantzig in 1957 (Dantzig had recently invented the simplex method for linear programming).
2. Several special cases have been proved: $d \leq 3$, $n - d \leq 6$, $0/1$-polytopes, ...
3. But in the general case **we do not even know of a polynomial bound** for diam(P) in terms of n and d.
4. In 1967, Klee and Walkup disproved the unbounded case.
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Linear programming

A linear program is the problem of maximization (or minimization) of a linear functional subject to linear inequality constraints. That is:

Given

- a system $Mx \leq b$ of linear inequalities ($b \in \mathbb{R}^n$, $M \in \mathbb{R}^{d \times n}$), and
- an objective function $c^t \in \mathbb{R}^d$

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- $\max\{c^t \cdot x : x \in \mathbb{R}^d, Mx \leq b\}$ (and a point x where the maximum is attained).

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- Around 1980 two polynomial time algorithms for linear programming were proposed by Khachiyan and Karmakar (*ellipsoid* and *interior point* method).
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Connection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets and d dimensions.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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In this sense, more important than the standard Hirsch conjecture (which is false) is the following "polynomial version" of it:

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Let $H(n,d)$ denote the maximum diameter of d-polyhedra with n facets. There is a constant k such that:

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"As simple as possible"

Definition

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A d-polytope is simplicial if every facet has exactly d vertices. That is, if every proper face is a simplex. ($\simeq$ vertices are "in general position").

Of course, the (polar) dual of a simple polytope is simplicial, and vice-versa.

Lemma (Klee 1964)

For every n and d the maximum diameter of d-polytopes / d-polyhedra with n facets is achieved at a simple one.

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A d-polytope is simplicial if every facet has exactly d vertices. That is, if every proper face is a simplex. ($\simeq$ vertices are "in general position").

Of course, the (polar) dual of a simple polytope is simplicial, and vice-versa.

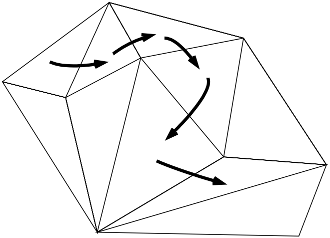
Lemma (Klee 1964)

For every n and d the maximum diameter of d-polytopes / d-polyhedra with n facets is achieved at a simple one.

"As simple as possible"

Remark

We will often dualize the problem. We want to travel from one facet to another of a polytope Q (the polar of P) along the "dual graph", whose edges correspond to *ridges* of Q.



By the Klee lemma we can restrict our attention to simplicial polytopes; their face lattices are *simplicial complexes* with the topology of a $(d-1)$-sphere. (*Simplicial* $(d-1)$*-spheres*).

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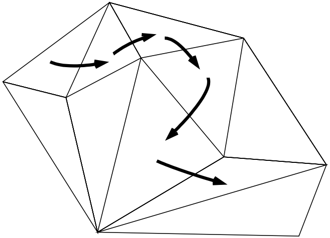

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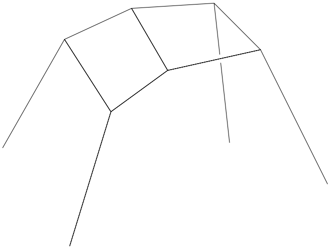

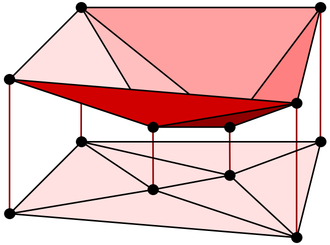

The polar of an unbounded d-polyhedron with n facets "is" a regular triangulation of n points in $\mathbb{R}^{d-1}$.

The Klee-Walkup non-Hirsch (8,4)-polyhedron

Klee and Walkup proved:

Theorem (Klee-Walkup 1967)

There is a 4-dimensional unbounded polyhedron with 8 facets and diameter 5.

Let us prove the following equivalent version:

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There is a regular triangulation of 8 points in $\mathbb{R}^3$ that has two tetrahedra at distance five from one another.

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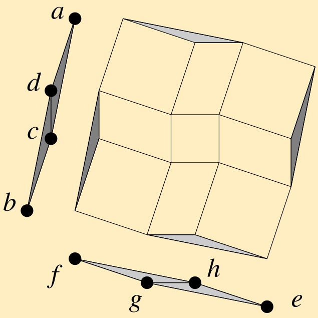

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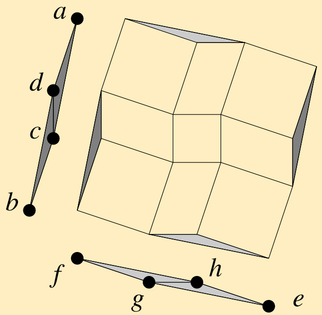

Three steps are needed to go from any light triangle to any dark triangle.

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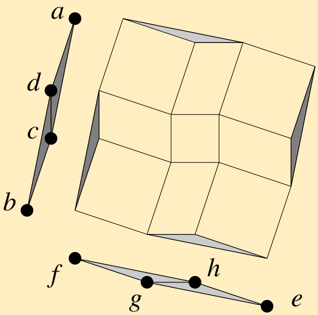

Gluing two more tetrahedra (one on top, one on bottom), we get diameter 5.

□

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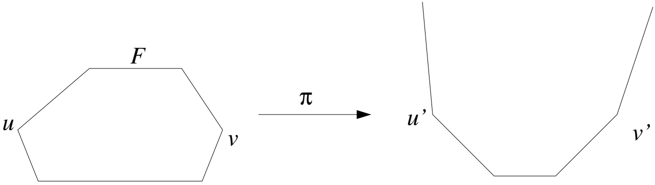
bounded Hirsch-sharp $\Rightarrow$ unbounded non-Hirsch

From a bounded (9,4)-polytope we get an unbounded (8,4)-polyhedron with (at least) the same diameter by projectively sending the "9th facet" to infinity. ($9 = n > 2d = 8$ is needed)



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bounded Hirsch-sharp $\Leftarrow$ unbounded non-Hirsch

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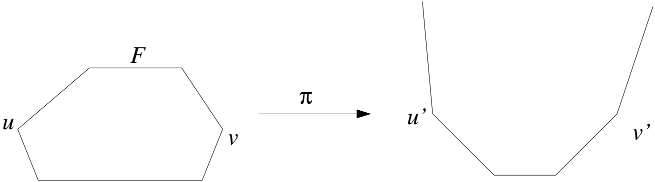

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Theorem [Kalai-Kleitman 1992], "quasi-polynomial"

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A pure simplicial complex is called normal if the dual graph of every link is connected. (That is, if every link is strongly connected)

The Kalai-Kleitman bound follows from the following recursion (where, now, $H(n, d)$ denotes the max. diameter among normal and pure simplicial $(d-1)$-complexes with n vertices):

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Proof.

Let u, v be two simplices in a normal, pure simplic. complex K.

- For each $i \in \mathbb{N}$, let U_i be the i-neighborhood of u (the subcomplex consisting of all simplices at distance at most i from u). Let V_j the j-neighborhood of v.
- Let i_0 (resp. j_0) be the smallest value such that U_{i_0} (resp. V_{j_0}) contains more than half of the vertices. This implies $i_0 - 1$ and $j_0 - 1$ are at most $H(\lfloor n/2 \rfloor, d)$.
- Let $u' \in U_{i_0}$ and $v' \in V_{j_0}$ having a common vertex. Then:

$$\mathrm{dist}(u', v') \leq H(n-1, d-1).$$

- So: $d(u,v) \leq \mathrm{dist}(u,u') + \mathrm{dist}(u',v') + \mathrm{dist}(v',v) \leq$

$$\leq 2H(\lfloor n/2 \rfloor, d) + H(n-1, d-1) + 2.$$

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Why is $n - d$ a "reasonable" bound?

- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

  For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).
  We call these "Hirsch-sharp" polytopes.

- For every $n > d$, it is easy to construct unbounded polyhedra where the bound is tight.
- $H(n, d)$ is weakly monotone w.r.t. $(n - d, d)$, not to (n, d).

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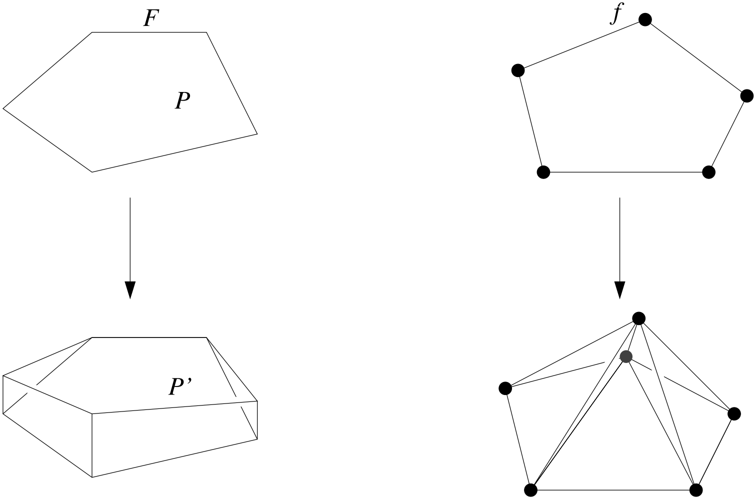
Wedging, a.k.a. one-point-suspension



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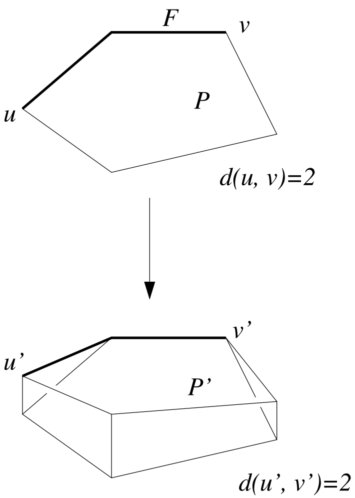

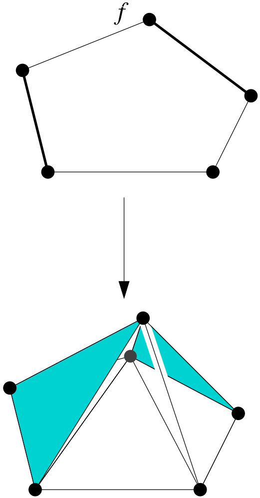

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It is possible to go from u to v so that at each step we abandon a facet containing u and we enter a facet containing v.

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Proof: Let $H(n, d) = \max\{\delta(P) : P \text{ is a } d\text{-polytope with } n \text{ facets}\}$. The basic idea is:

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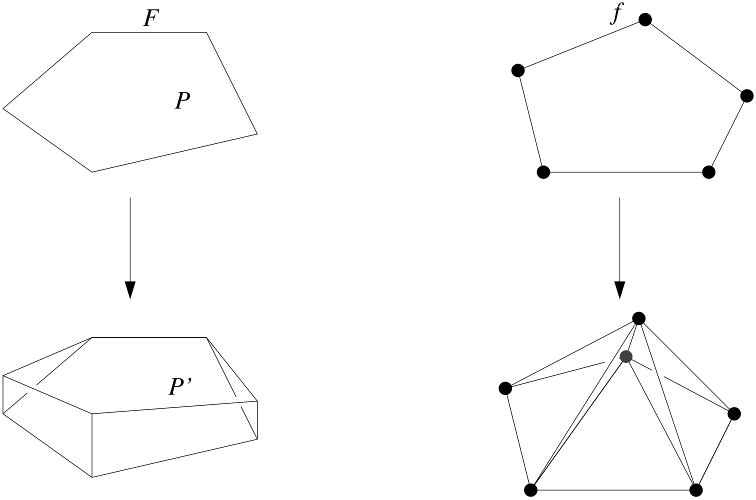

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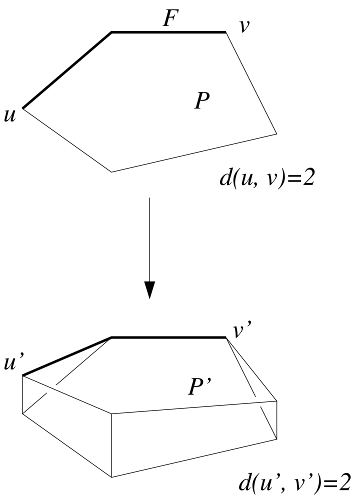

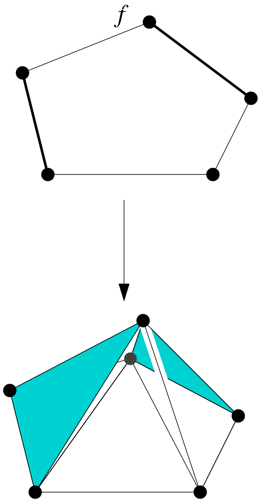

Two important remarks

The *d*-step Theorem follows from and implies (respectively) the following:

Lemma

For every d-polytope P with n facets and diameter δ there is a $d+1$-polytope with one more facet and the same diameter δ.

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There is a function $f(k) := H(2k, k)$ such that

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Thank you

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