The Hirsch Conjecture and its relatives (part II of III)

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Why is $n - d$ a "reasonable" bound?

- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

> For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).
> We call these "Hirsch-sharp" polytopes.

- For every $n > d$, it is easy to construct unbounded polyhedra where the bound is tight.
- $H(n, d)$ is weakly monotone w.r.t. $(n - d, d)$, not to (n, d).

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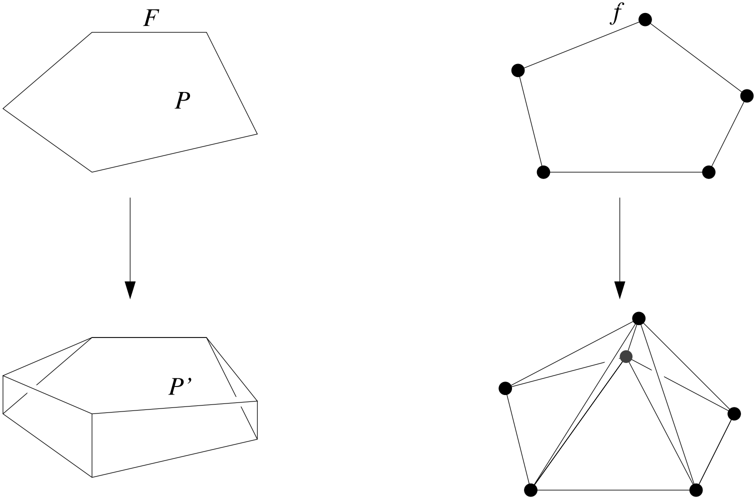

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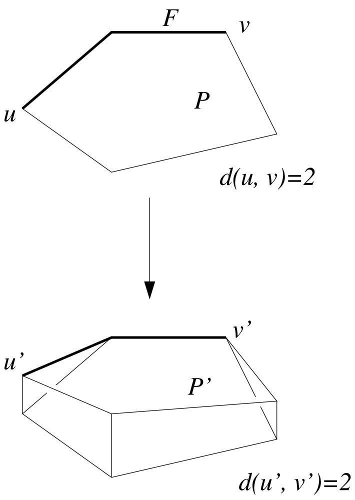

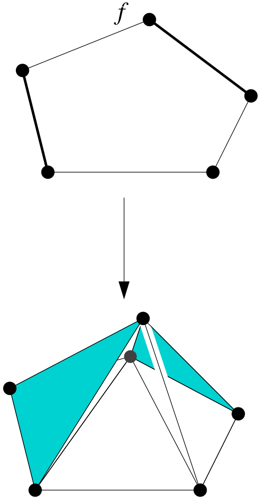

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Proof: Let $H(n, d) = \max\{\delta(P) : P \text{ is a } d\text{-polytope with } n \text{ facets}\}$. The basic idea is:

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- If $n < 2d$, then $H(n, d) \leq H(n-1, d-1)$ because every pair of vertices u and v lie in a common facet F, which is a polytope with one less dimension and (at least) one less facet (induction on n and $n - d$).

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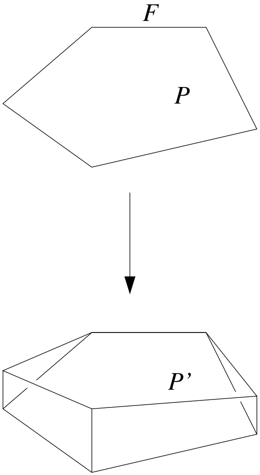

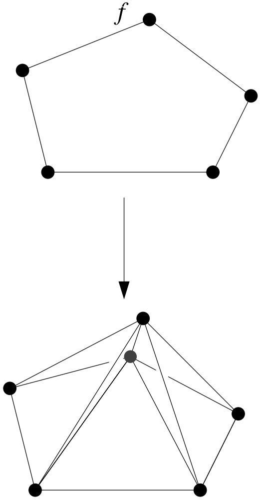

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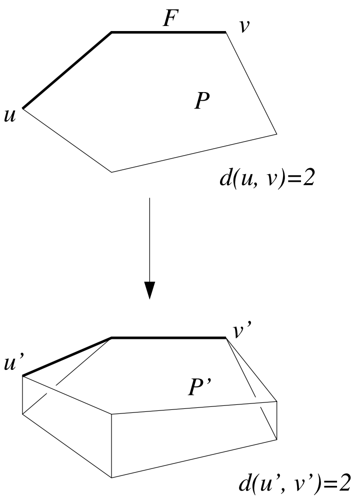

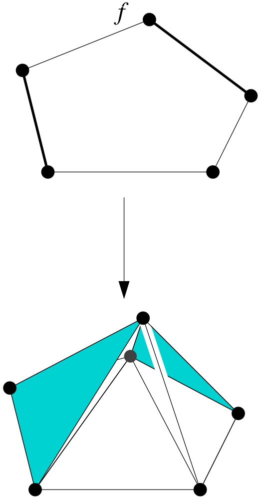

Two important remarks

The d-step Theorem follows from and implies (respectively) the following:

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For every d-polytope P with n facets and diameter δ there is a $d+1$-polytope with one more facet and the same diameter δ.

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Attack of the prismatoids

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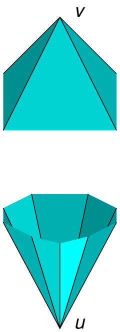
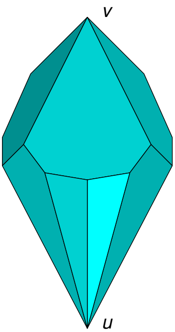
A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).





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3-spindles have length ≤ 3.

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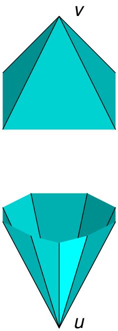

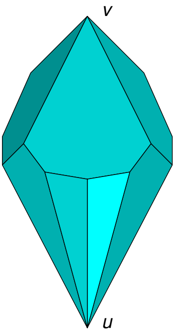

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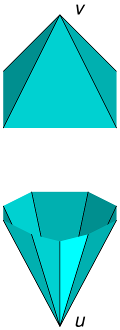

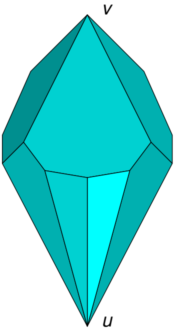

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That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

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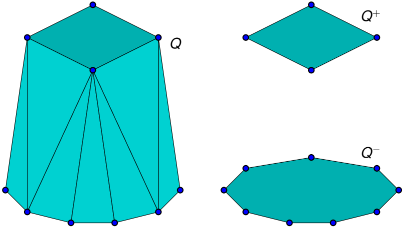

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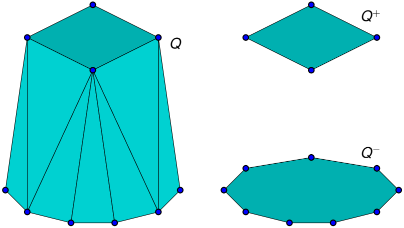

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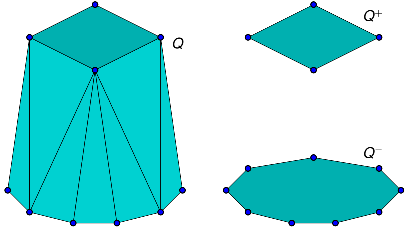

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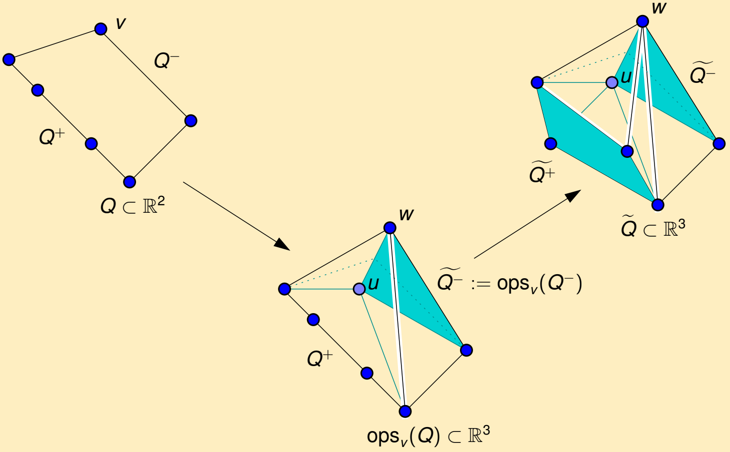
That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$.

Corollary

In particular, if a prismatoid Q has width $> d$ then there is another prismatoid Q' (of dimension $n-d$, with $2n-2d$ vertices, and width $\geq \delta + n - 2d > n - d$) that violates (the dual of) the Hirsch conjecture.

d-step theorem for prismatoids

Proof.



□

Width of prismatoids

So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
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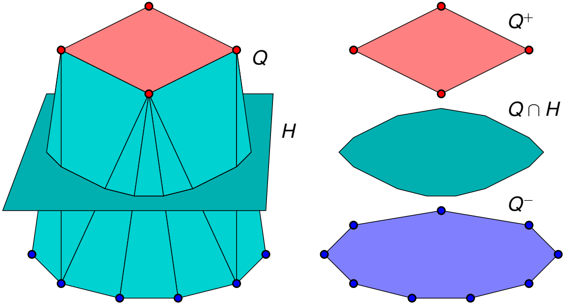
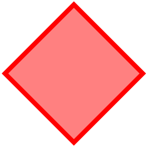
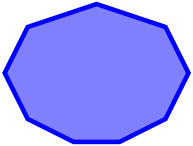
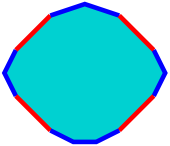
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice ...



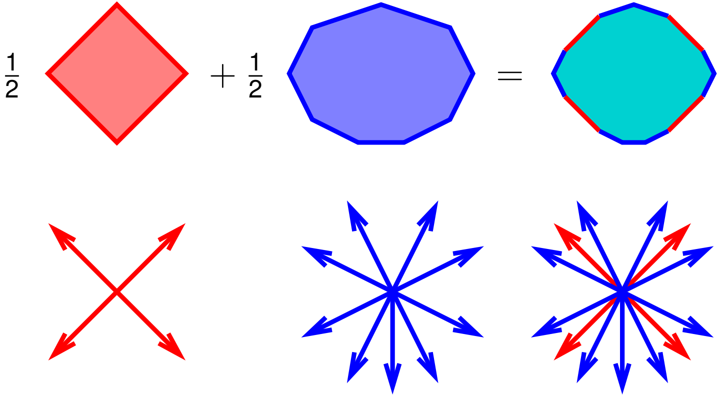
Combinatorics of prismatoids

…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.

$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

Combinatorics of prismatoids

...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the “superposition” of those of Q^+ and Q^-.



Combinatorics of prismatoids

So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Theorem

Let Q be a d-prismatoid with bases Q^+ and Q^- and let G^+ and G^- be the corresponding maps in the $(d-2)$-sphere (central projection of the normal fans of Q^+ and Q^-). Then, the width of Q equals 2 plus the minimum number of steps needed to go from a vertex of G^+ to a vertex of G^- in (the graph of) the superposition of the two maps.

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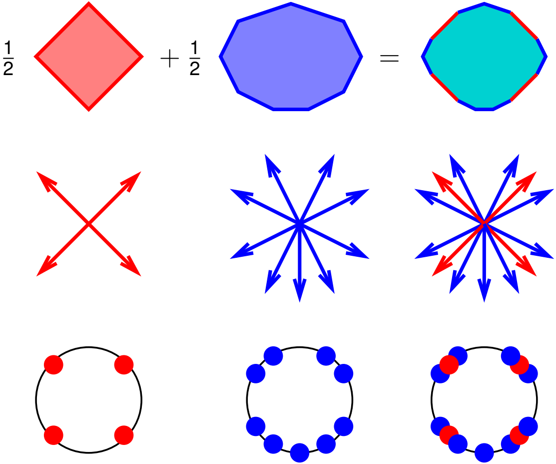

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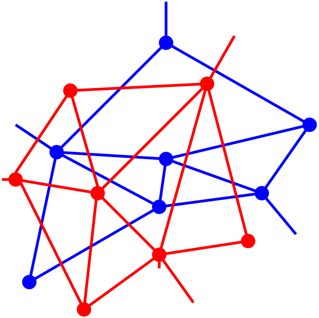

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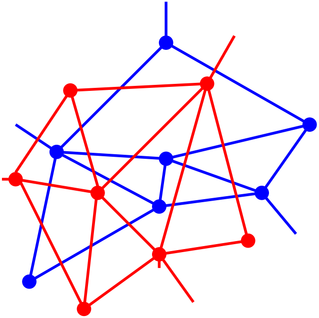

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Remember that Klee and Walkup, in 1967, disproved the Hirsch conjecture:

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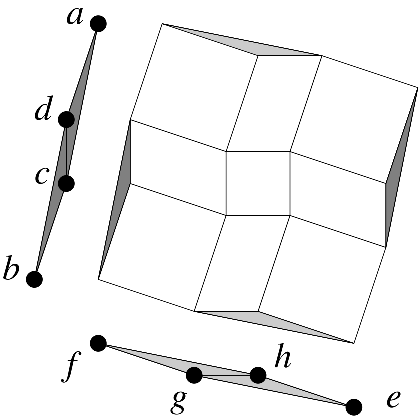

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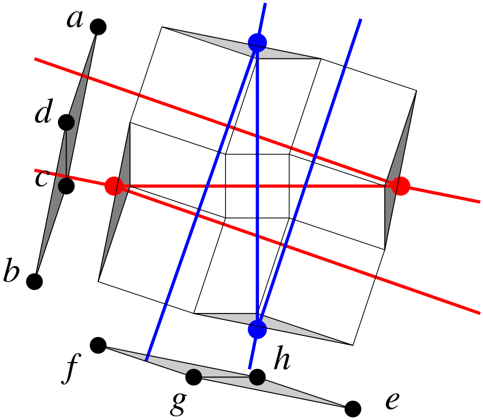

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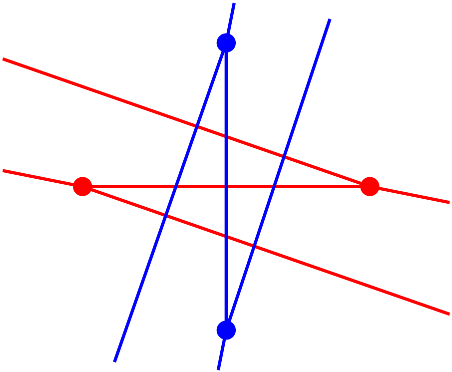

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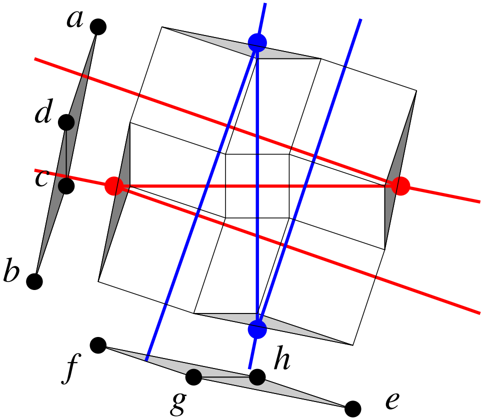

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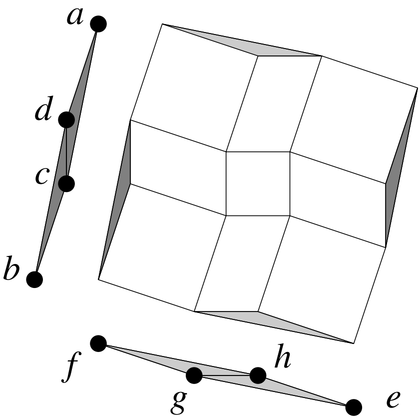

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In every transversal pair of maps in the sphere there is a path of length two from some blue vertex to some red vertex.

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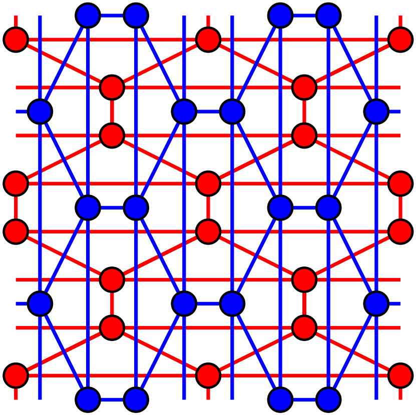

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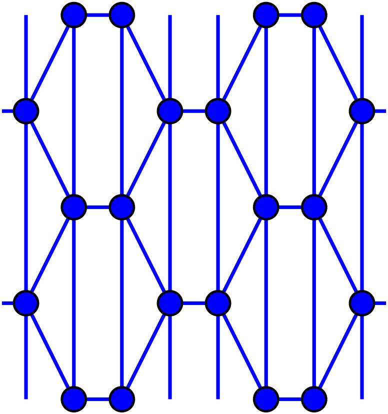

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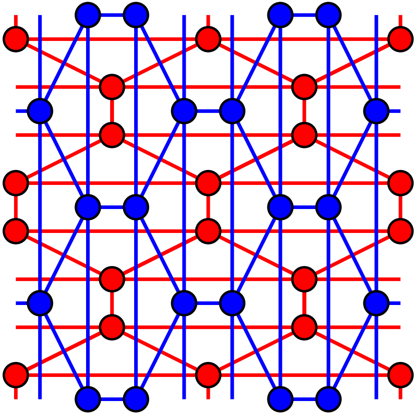

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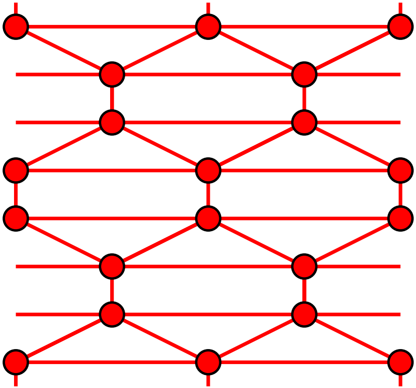

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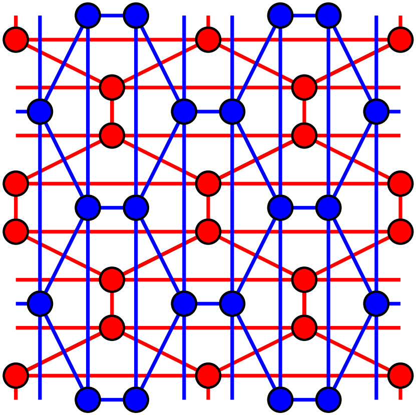

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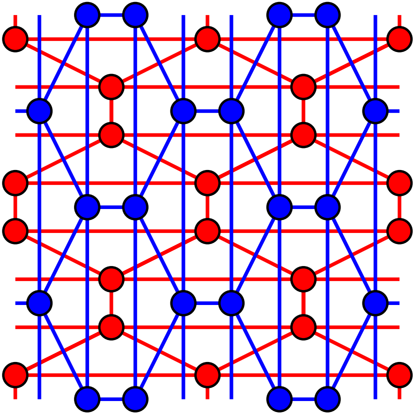

. . . or with finite ones in the torus!

5-prismatoids of width > 5

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That is, we want a pair of (geodesic, polytopal) cell decompositions of the 3-sphere such that if we draw them one on top of the other (common refinement) there is no path of length ≤ 3 from a blue vertex to a red vertex.

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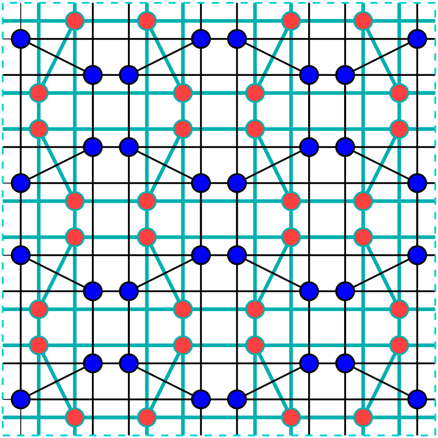

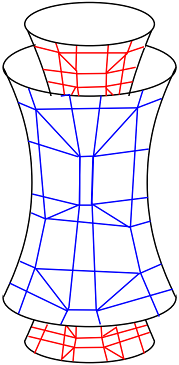

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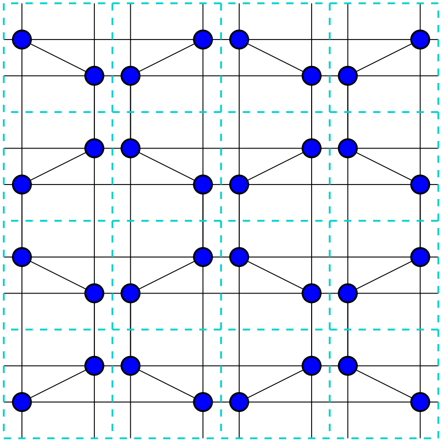

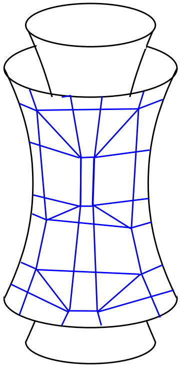

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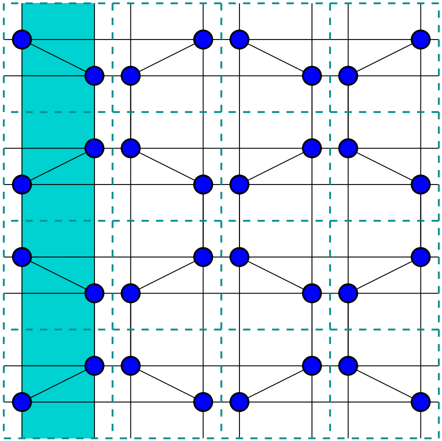

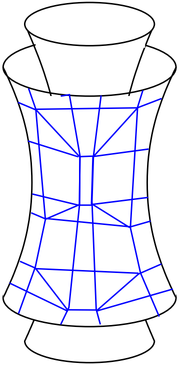

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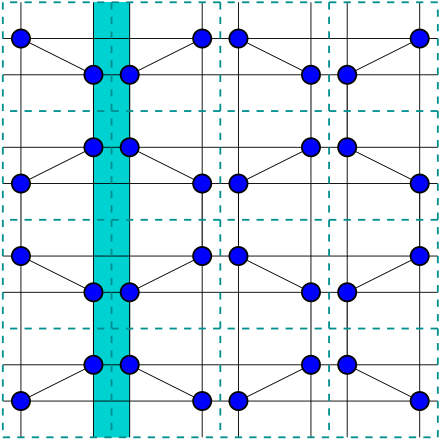

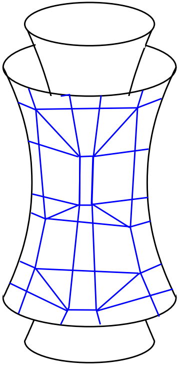

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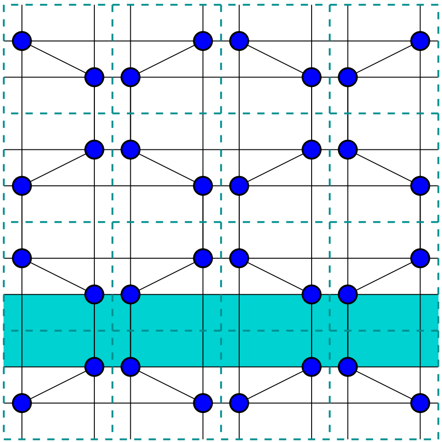

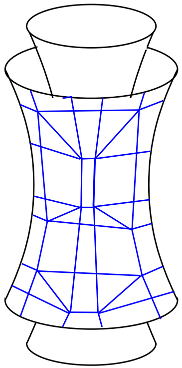

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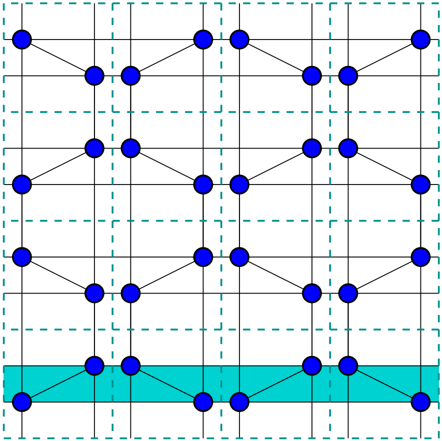

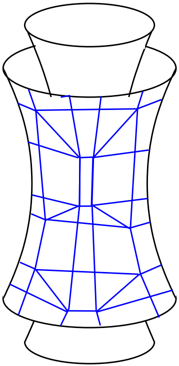

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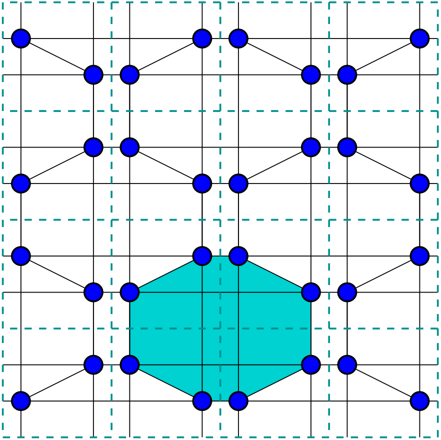

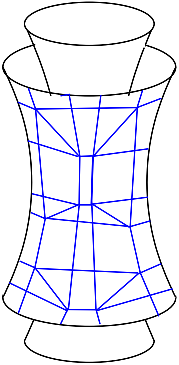

A 5-prismatoid of width > 5

Theorem (S. 2012)

The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

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A 5-prismatoid of width > 5

Theorem (S. 2012)

The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

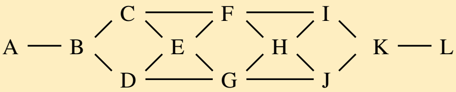
Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

A 5-prismatoid of width > 5

Proof 1.

It has been verified computationally that the dual graph of Q (modulo symmetry) has the following structure:

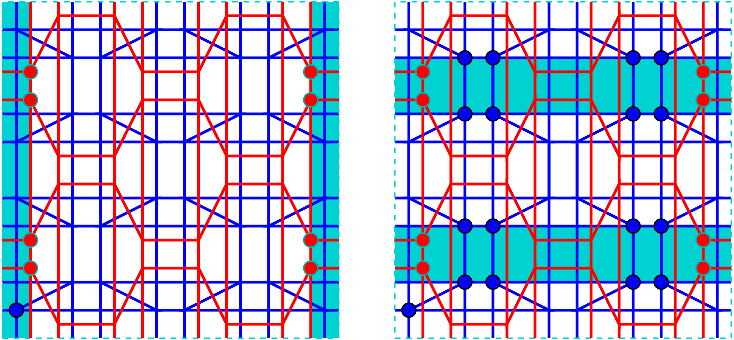


□

A 5-prismatoid of width > 5

Proof 2.

Check that there are no blue vertex a and red vertex b such that a is a vertex of the blue cell containing b and b is a vertex of the red cell containing a. □



Smaller 5-prismatoids of width > 5

With the same ideas

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The following 5*-prismatoid with* 28 *vertices (and 274 facets) has width* 6*.*

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There is a non-Hirsch polytope of dimension 23 *with* 46 *facets.*

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And with some more work:

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There is a 5*-prismatoid with* 25 *vertices and of width* 6*.*

Corollary

There is a non-Hirsch polytope of dimension 20 *with* 40 *facets.*

This one has been explicitly computed. It has $36,442$ vertices, and diameter 21.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1 1 0 0 21 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0
1 1 0 0 16 -8 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0
1 1 0 0 0 -25 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0
1 1 0 0 -16 -10 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
1 1 0 0 -21 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
1 1 0 0 -20 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 39 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 19 7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 3/100 -1/50 0 -23 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/100 -1/50 0 37 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/2000 7/2000 0 388/10 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 100000000000 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 -10000000 0 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 -10000000 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 -10000000 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 -10000000000 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 -100000000000 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 -100000000000 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 -100000000000 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 -100000000000 1 0 0 0 0 0 0
1 -1 30 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0
1 -1 4 -15 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
1 -1 0 -33/2 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0
1 -1 -1 -16 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0
1 -1 -55/2 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0
1 -1 -17 18 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
1 -1 0 38 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 22 17 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -10 0 1/5 -1/5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 2999/100 0 -3/25 -1/5 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 299999/10000 0 0 1/100 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -2745/100 0 1/5000 1/800 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 100000000 1000000000
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 -100000 0 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 -10000000 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 -10000000 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 -10000000 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 -100000000 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 -100000000 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 100000000 -1000000000
end
allbases
printcobasis
```

Asymptotic width in dimension five

Theorem (Matschke-Santos-Weibel, 2013+)

There are 5-dimensional prismatoids with n vertices and width $\Omega(\sqrt{n})$.

Sketch of proof

Start with the following "simple, yet more drastic" pair of maps in the torus.

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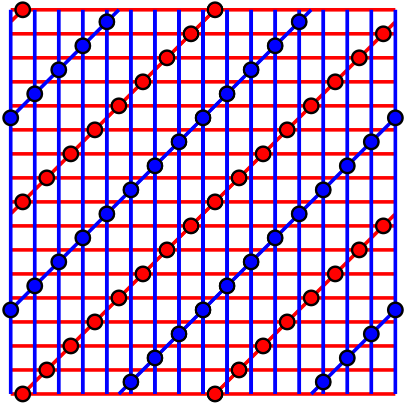

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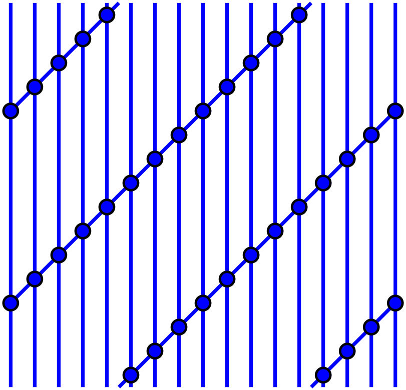

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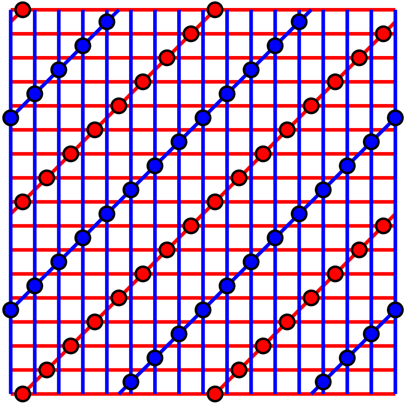

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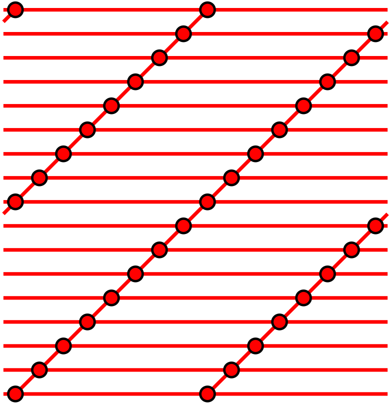

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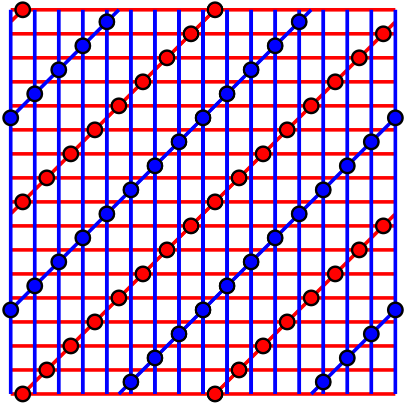
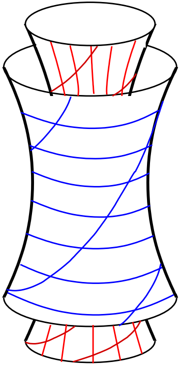


Asymptotic width in dimension five

Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps. □

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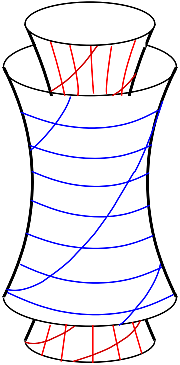

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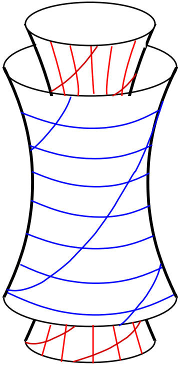

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Many non-Hirsch polytopes

Once we have a non-Hirsch polytope we can derive more via:

1. Products of several copies of it (dimension increases).
2. Gluing several copies of it (dimension is fixed).

To analyze the asymptotics of these operations, we call excess of a d-polytope P with n facets and diameter δ the number

$$\epsilon(P) := \frac{\delta}{n-d} - 1 = \frac{\delta - (n-d)}{n-d}.$$

E. g.: The excess of our non-Hirsch polytope with $n - d = 20$ and with diameter 21 is

$$\frac{21-20}{20} = 5\%.$$

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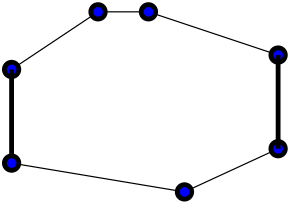

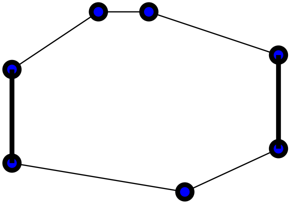

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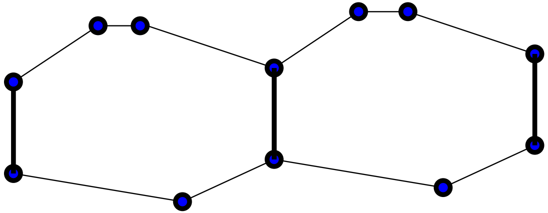

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$$\frac{\delta_1}{n_1 - d} - 1 = \frac{\delta_2}{n_2 - d} - 1 = \epsilon \quad \Rightarrow \quad \frac{\delta}{n - d} - 1 = \epsilon - \frac{1}{(n_1 - d) + (n_2 - d)}.$$

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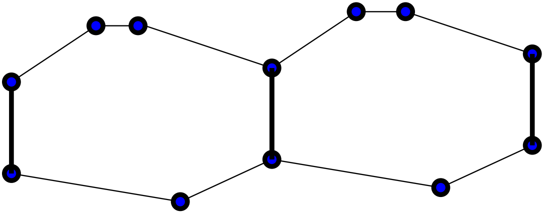

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Corollary

For each $k \in \mathbb{N}$ there is an infinite family of non-Hirsch polytopes of fixed dimension $20k$ and with excess (tending to)

$$0.05\left(1 - \frac{1}{k}\right).$$

The excess of a prismatoid

But we know there are “worst” prismatoids: 5-prismatoids of arbitrarily large width. Will those produce non-Hirsch polytopes with worst excess?

To analyze the asymptotics of this, let us call *excess* of a prismatoid of width δ with n vertices and dimension d the quantity

$$\frac{\delta - d}{n - d}$$

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Lemma

Via the strong d-step Theorem, a prismatoid of a certain excess produces non-Hirsch polytopes of that same excess.

Proof.

The dimension, number of facets and diameter of the non-Hirsch polytope produced by the strong d-step Theorem are

$$n-d, \qquad 2(n-d), \qquad \delta+(n-2d).$$

So, its excess is

$$\frac{\delta+(n-2d)-(n-d)}{n-d}=\frac{\delta-d}{n-d}.$$

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Prismatoids of large width won't help (much)

In dimension 5, we know how to construct polytopes of arbitrarily large width $\delta \sim \sqrt{n}$. . . but their excess tends to zero:

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Let us be optimistic and suppose that we could construct 5-prismatoids with n vertices and linear width $\simeq \alpha n$.

Their excess will now tend to α. So, we still get only polytopes that violate Hirsch by a constant ("linear" Hirsch bound).

OK, let us try to be more optimistic.

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$$\lim \frac{\delta - 5}{n - 5} = \lim \frac{\sqrt{n} - 5}{n - 5} = 0.$$

Let us be optimistic and suppose that we could construct 5-prismatoids with n vertices and linear width $\simeq \alpha n$.

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In fixed dimension, certainly not:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

Proof.

This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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Corollary

Using the Strong d-step Theorem for 5-prismatoids it is impossible to violate the Hirsch conjecture by more than 33%.

Thank you

T H E   E N D

O F   T H E   G E O M E T R I C   T R I L O G Y

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Thank you

THE END

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