The Hirsch Conjecture and its relatives (part III of III)

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If you cannot beat'em, generalize'em

Instead of looking at (simplicial) polytopes, why not look at the maximum diameter of more general complexes? We can, for example, consider:

- Pure simplicial complexes. $H_C(n, d)$
- Pseudo-manifolds (w. or wo. bdry). $H_{\overline{pm}}(n, d)$, $H_{pm}(n, d)$
- Simplicial manifolds (w. or wo. bdry). $H_{\overline{M}}(n, d)$, $H_M(n, d)$
- Simplicial spheres (or balls). $H_S(n, d)$, $H_B(n, d)$,
- ...

Remark, in all definitions, n is the number of vertices and $d-1$ is the dimension.

$H_\bullet(n, d)$ is the (dual) diameter; two simplices are considered adjacent if they differ by a single vertex.

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Some easy remarks and a toy example

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In dimension one (graphs):

$$H_C(n,2) = H_{\overline{pm}}(n,2) = H_{\overline{M}}(n,2) = H_B(n,2) = n-1,$$

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Corollary

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In fact: $H_C(n, d) =$ length of the maximum induced path in the *Johnson graph $J(n, d)$*.

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In dimension two:

Theorem (S. 2013+)

$$\frac{2}{9}(n-1)^2 < H_C(n,3) = H_{\overline{pm}}(n,3) < \frac{1}{4}n^2.$$

In higher dimension:

Theorem (S. 2013+)

$$H_C(kn,kd) > \frac{1}{2^{k-1}} H_C(n,d)^k.$$

Corollary (S. 2013+)

$$\Omega\left(\left(\frac{n}{d}-1\right)^{\frac{2d}{3}}\right) < H_C(n,d) = H_{\overline{pm}}(n,d) < \binom{n}{d-1}.$$

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Theorem: $H_{\overline{pm}}(n, 3) > \frac{2}{9}(n-1)^2$

Proof.

1. Without loss of generality assume $n = 3k + 1$.
2. With the first $2k + 1$ vertices, construct k disjoint cycles of length $2k + 1$ (That is, decompose K_{2k+1} into k disjoint Hamiltonian cycles).
3. Remove an edge from each cycle to make it a chain, and join each chain to each of the remaining k vertices.
4. Glue together the k chains using $k - 1$ triangles.

In this way we get a chain of triangles of length

$$(2k + 1)k - 2 > \frac{2}{9}(3k)^2.$$

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Theorem: $H_C(kn, kd) > \frac{1}{2^{k-1}} H_C(n, d)^k$

Proof.

1. Let Δ be a complex achieving $H_C(n, d)$. W.l.o.g. assume its dual graph is a path.
2. Take the join Δ^{*k} of k copies of Δ. Δ^{*k} is a complex of dimension $kd - 1$, with kn vertices and whose dual graph is a k-dimensional grid of size $H_C(n, d)$. (It has $(H_C(n, d) + 1)^k$ maximal simplices).
3. In this grid consider a maximal induced path. This can be done using more than $\frac{1}{2^{k-1}}$ of the vertices.

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Normal simplicial complexes

Definition

A pure simplicial complex is called normal if the dual graph of every link is connected. (That is, if every link is strongly connected)

Theorem

Let K be a pure, normal simplicial complex of dimension $d-1$ with n vertices. Then:

1. $\operatorname{diam}(K) \leq n^{\log d+2}$ *[Kalai-Kleitman 1992, Eisenbrand et al. 2010]*
2. $\operatorname{diam}(K) \leq 2^{d-1} n$ *[Larman 1970 , Eisenbrand et al. 2010]*
3. *If K is, moreover, flag then* $\operatorname{diam}(K) \leq n-d$ *(Hirsch bound!) [Adiprasito-Benedetti 2013+]*

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Flag normal simplicial complexes

Definition

A simplicial complex is flag if every "minimal non-simplex" has two elements. That is, if $\partial u \subset K$ for some $u \subset [n]$ with $|u| \geq 3$ then $u \in K$. Equivalently, the Stanley-Reisner ring of K is generated in degree two.

The Adiprasito-Benedetti result follows from:

- If K is flag then, with the "spherical right-angled metric" for every simplex, every star in K is *geodesically convex* [Gromov'87]
- Hence, every geodesic path γ between the interior of two simplices u and v of K is non-revisiting (it never abandons a star and then enter it again).
- The fact that K is normal (and flag) guarantees that such paths can be perturbed to not cross simplices of codimension two or higher, hence they induce non-revisiting paths in the dual graph.

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A simplicial complex is flag if every "minimal non-simplex" has two elements. That is, if $\partial u \subset K$ for some $u \subset [n]$ with $|u| \geq 3$ then $u \in K$. Equivalently, the Stanley-Reisner ring of K is generated in degree two.

The Adiprasito-Benedetti result follows from:

- If K is flag then, with the "spherical right-angled metric" for every simplex, every star in K is *geodesically convex* [Gromov'87]
- Hence, every geodesic path γ between the interior of two simplices u and v of K is non-revisiting (it never abandons a star and then enter it again).
- The fact that K is normal (and flag) guarantees that such paths can be perturbed to not cross simplices of codimension two or higher, hence they induce non-revisiting paths in the dual graph.

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Connected layer families

The Kalai-Kleitman and Larman bounds follow from more general arguments. They are actually valid for connected layer families.

Definition (Eisenbrand et al. 2010)

A connected layer family (CLF) of rank d on n symbols is a pure simplicial complex Δ of dimension $d-1$ with n vertices, together with a map

$$\lambda : \text{facets}(\Delta) \to \mathbb{Z}$$

with the following property: for every simplex (of whatever dimension) $\tau \in \Delta$ the values taken by λ in the star of τ form an interval.

The length of a CLF is the difference between the maximum and the minimum values taken by λ.

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The length of a CLF is the difference between the maximum and the minimum values taken by λ.

Example: A CLF of rank 2 and length $\sim 3n/2$

λ	0	1	2	3	4	5	6	7	8	9
		13	14		35	36		57	58	
Δ	12			34			56			78
		24	23		46	45		68	67	

Let $H_{clf}(n, d) :=$ max length of a CLF of rank d on n symbols.
The example shows that:

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Two properties of c.l.f.'s

- The clf property is hereditary via links: If Δ is a clf, every link in it (together with the same map λ) is a clf.

- "Conversely", if a pure simplicial complex Δ is normal (every link has a connected dual graph), then Δ is a clf with respect to the map

$$\lambda(v) = d(u, v),$$

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Let $H_{clf}(n, d)$ be the maximum length of clf's of rank d on n elements.

$H_{clf}(n,d) \le n^{\log_2 d+2}$ (Kalai-Kleitman bound)

The Kalai-Kleitman bound follows from the following recursion:

$$H_{clf}(n,d) \le 2H_{clf}(\lfloor n/2 \rfloor, d) + H_{clf}(n-1, d-1) + 2.$$

To prove the recursion:

- Let u and v be simplices in the first and last layer, respectively. For each $i \in \mathbb{N}$, let U_i be the i-neighborhood of u (the union of the first $i+1$ layers, that is, those at distance at most i from u). Call V_j the j-neighborhood of v.
- Let i_0 and j_0 be the smallest values such that U_{i_0} and V_{j_0} contain more than half of the vertices. This implies $i_0 - 1$ and $j_0 - 1$ are at most $H_{clf}(\lfloor n/2 \rfloor, d)$.
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$H_{clf}(n,d) \le 2^{d-1}n$ (Larman bound)

By induction on d. The case $d = 1$ is trivial. For higher d:
Let U_1 be the maximum interval of layers starting with the first one and such that all layers in U_1 use some common element.
Let U_2 be the maximum interval of layers starting with the first one after U_1 and such that all layers in U_2 use some common element. Etc.
Let k be the number of pieces U_i that we get. Let n_i be the number of elements used in the i-th piece U_i. Then:

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- Each element is used in at most two of the $U_i\Delta$'s $\Rightarrow \sum n_i \le 2n$.

Hence:

$$\begin{aligned}\text{length}(\Delta) &\le \sum \text{length}(U_i) + (k-1)\\ &\le 2^{d-2}(2n-k) + (k-1) \le 2^{d-1}n\end{aligned}$$

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Let U_2 be the maximum interval of layers starting with the first one after U_1 and such that all layers in U_2 use some common element. Etc.
Let k be the number of pieces U_i that we get. Let n_i be the number of elements used in the i-th piece U_i. Then:

- $\text{length}(U_i) \leq H_{clf}(n_i - 1, d-1) \leq 2^{d-2}(n_i - 1)$
- Each element is used in at most two of the $U_i\Delta$'s $\Rightarrow \sum n_i \leq 2n$.

Hence:

$$\begin{aligned} \text{length}(\Delta) &\leq \sum \text{length}(U_i) + (k-1) \\ &\leq 2^{d-2}(2n-k) + (k-1) \leq 2^{d-1} n \end{aligned}$$

$H_{clf}(n, d) \leq 2^{d-1} n$ (Larman bound)

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A further generalization:

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A connected layer multifamily (CLMF) of rank d on n symbols is the same as a CLF, except we allow a pure simplicial multicomplex Δ (simplices are multisets of vertices, with repetitions allowed)

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Complete and injective clmf's

"Complete" and "injective" clmf's are (the) two extremal cases.

It turns out that in these two cases:

Theorem (Hähnle et al@polymath3, 2010)

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Hähnle's Conjecture has been checked for all the values of n and d satisfying $n \leq 3$, $d \leq 2$, $n + d \leq 11$, or $6n + d \leq 37$.

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