Combinatorics of asymptotic representation theory

Part 1

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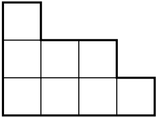
plan: asymptotic representation theory 1

representations of the symmetric groups S_n for $n \to \infty$

$\lambda =$  $\longrightarrow \infty$

$$\rho^\lambda =?$$

$$\frac{\operatorname{Tr} \rho^\lambda(\pi)}{\dim \rho^\lambda} =? \qquad \text{relative characters}$$

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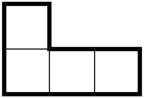
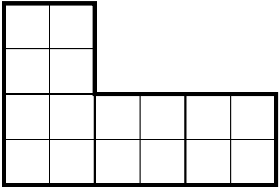
characters $\longleftrightarrow$ shape of the Young diagram

plan: asymptotic representation theory 2



Young diagram λ



dilated diagram 2λ

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study $r\lambda$ for $r \to \infty$

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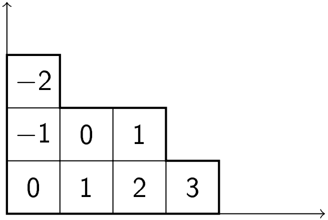
characters $\longleftrightarrow$ shape of the Young diagram

content of a box

$$\text{content}(\square) = (x\text{-coordinate}) - (y\text{-coordinate})$$

Example:

$\lambda =$



$$\text{content}(\lambda) = \underbrace{(-2, -1, 0, 0, 1, 1, 2, 3)}_{\text{multiset}}$$

Jucys-Murphy elements and contents

for $1 \le j \le n$ we define

$$X_j := \underbrace{(1,j) + (2,j) + \cdots + (j-1,j)}_{\text{sum of transpositions}} = \sum_{i<j} (i,j) \in \mathbb{C}(S_n);$$

elements $X_1, \dots, X_n \in \mathbb{C}(S_n)$ commute.

Theorem

If $P(x_1, \dots, x_n)$ is a symmetric polynomial, $\lambda \vdash n$, content$(\lambda) = (c_1, \dots, c_n)$ *then*

$$\frac{\operatorname{Tr} \rho^\lambda \big(P(X_1, \dots, X_n)\big)}{\dim \rho^\lambda} = P(c_1, \dots, c_n)$$

important example, part 1

for a clever choice...

$$P(x_1,\dots,x_n) = \underbrace{\left(\sum_{1\le j\le n} x_j^2\right)}_{\text{main term}} - \underbrace{\binom{n}{2}}_{\text{correction term}}$$

...we obtain...

$$P(X_1,\dots,X_n) = \left(\sum_{1\le j\le n} \underbrace{\sum_{i_1,i_2<j} (i_1 j)(i_2 j)}_{X_j^2}\right) - \binom{n}{2} =$$

$$\underbrace{\left(\sum_{1\le j\le n} \sum_{\substack{i_1,i_2<j\\ i_1\neq i_2}} \underbrace{(i_1 j)(i_2 j)}_{(j i_2 i_1)}\right)}_{\text{sum of all cycles of length 3}} + \underbrace{\left(\sum_{1\le j\le n} \sum_{i<j} \underbrace{(ij)(ij)}_{\text{id}}\right) - \binom{n}{2}}_{0}$$

important example, part 2

$$\frac{\operatorname{Tr} \rho^\lambda(\text{sum of all cycles of length 3})}{\dim \rho^\lambda} = c_1^2 + \cdots + c_n^2 - \binom{n}{2},$$

$$\frac{\operatorname{Tr} \rho^\lambda\big((1,2,3)\big)}{\dim \rho^\lambda} = \frac{c_1^2 + \cdots + c_n^2 - \binom{n}{2}}{\frac{1}{3} n(n-1)(n-2)}$$

Morals:

- if we choose P in a clever way, we get something useful,
- $\frac{\operatorname{Tr} \rho^\lambda\big((1,2,3)\big)}{\dim \rho^\lambda}$ is an ugly quantity,
- use $n(n-1)(n-2)\frac{\operatorname{Tr} \rho^\lambda\big((1,2,3)\big)}{\dim \rho^\lambda}$ instead,
- character $\longleftrightarrow$ shape,
- fixed conjugacy class, arbitrary λ $\to$dual combinatorics

normalized characters

for $\pi \in S_k$ and $\lambda \vdash n$ we define normalized character

$$\mathrm{Ch}_\pi(\lambda) := \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\dim \rho^\lambda}$$

→ Kerov & Olshanski

Example

$$\underbrace{\mathrm{Ch}_3}_{\text{"character on cycle of length 3"}} := \mathrm{Ch}_{\underbrace{(1,2,3)}_{\in S_3}} = 3\left(c_1^2 + \cdots + c_n^2 - \binom{n}{2}\right)$$

---

algebra of polynomial functions is defined as the linear span of (Ch_π) over all choices of π

Exercise:

$$\mathrm{Ch}_2 \cdot \mathrm{Ch}_2 = \mathrm{Ch}_{2,2} + 3\,\mathrm{Ch}_3 + 2\,\mathrm{Ch}_{1,1}$$

discrete functionals of shape

for $k \geq 0$

$$p_k(\lambda) := \sum_{\square \in \lambda} (\text{content}\,\square)^k$$

$p_0, p_1, \dots$ form an algebraic basis of the algebra of polynomial functions

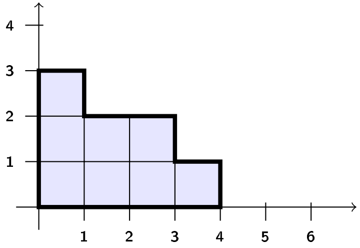
Example:

$$\mathrm{Ch}_3 = 3\left[\underbrace{p_2}_{c_1^2+\cdots+c_n^2} - \left(\underbrace{\frac{1}{2}p_0^2 - \frac{1}{2}p_0}_{\binom{n}{2}} \right) \right]$$

continuous functionals of shape 1

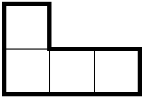
for $k \geq 2$

$$S_k(\lambda) := (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2}\, dx\, dy$$

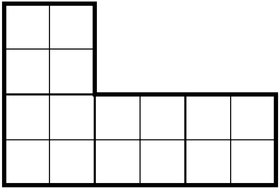


$S_2, S_3, \ldots$ form an algebraic basis of the algebra of polynomial functions

continuous functionals of shape 2



Young diagram λ



dilated diagram 2λ

S_k is homogeneous of degree k:

$$S_k(r\lambda) = r^k S_k(\lambda)$$

outlook

characters $\longleftrightarrow$ shape of the Young diagram

$$\mathrm{Ch}_1 = \underbrace{S_2}_{\text{degree } 2},$$

$$\mathrm{Ch}_2 = \underbrace{S_3}_{\text{degree } 3},$$

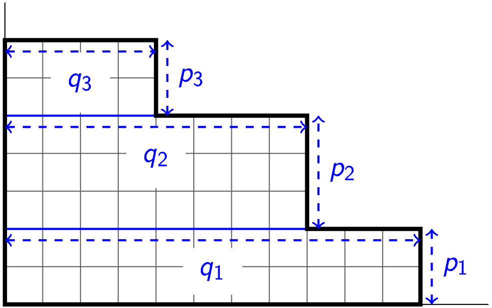
$$\mathrm{Ch}_3 = \underbrace{S_4 - \frac{3}{2} S_2^2}_{\text{degree } 4} + \underbrace{S_2}_{\text{degree } 2},$$

$$\mathrm{Ch}_4 = \underbrace{S_5 - 4 S_2 S_3}_{\text{degree } 5} + \underbrace{5 S_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_5 = \underbrace{S_6 - 5 S_2 S_4 - \frac{5}{2} S_3^2 + \frac{25}{6} S_2^3}_{\text{degree } 6} + \underbrace{15 S_4 - \frac{35}{2} S_2^2}_{\text{degree } 4} + \underbrace{8 S_2}_{\text{degree } 2}.$$

Stanley coordinates

$\mathbf{p} \times \mathbf{q} =$



if $\lambda \mapsto F(\lambda)$ is an nice on Young diagrams, it is a good idea to study the polynomial

$$F(\mathbf{p} \times \mathbf{q})$$

and the **p**-square free coefficients

$$[p_1 q_1^{k_1} \cdots p_r q_r^{k_r}] F(\mathbf{p} \times \mathbf{q})$$

$\mathbf{p}$-square-free terms 1

Theorem

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k_1, \dots, k_r \geq 2$

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_r}} F \Big|_{S_2=S_3=\cdots=0} = [p_1 q_1^{k_1-1} \cdots p_r q^{k_r-1}] F(\mathbf{p} \times \mathbf{q})$$

Example: for any $k, k_1, k_2 \geq 2$:

$$\begin{aligned} [S_k]F =& [p_1 q_1^{k-1}] F(\mathbf{p} \times \mathbf{q}), \\ [S_{k_1} S_{k_2}]F =& [p_1 q_1^{k_1-1} p_2 q_2^{k_2-1}] F(\mathbf{p} \times \mathbf{q}) \qquad \text{if } k_1 \neq k_2, \\ 2 \cdot [S_k^2]F =& [p_1 q_1^k p_2 q_2^k] F(\mathbf{p} \times \mathbf{q}), \end{aligned}$$

p-square-free terms 1

Theorem

if $F = F(\lambda)$ *is a polynomial in* $S_2, S_3, \ldots$ *then for any* $k_1, \ldots, k_r \geq 2$

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_r}} F \bigg|_{S_2 = S_3 = \cdots = 0} = [p_1 q_1^{k_1 - 1} \cdots p_r q^{k_r - 1}] F(\mathbf{p} \times \mathbf{q})$$

p-square-free terms 1

Theorem

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Hint: for $i_1 < \cdots < i_r$, $r \geq 0$

$$[p_{i_1} \cdots p_{i_r}] \underbrace{S_k(\mathbf{p} \times \mathbf{q})}_{\text{as polynomial in } \mathbf{p}} = \begin{cases} (-1)^{r-1} \, (k-1)_{r-1} \, \overbrace{q_{i_r}^{k-r}}^{\text{exponent at least 1}} & \text{if } 1 \leq r \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

p-square-free terms 2

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \ldots$ then for any $k_1, k_2 \geq 2$

$$[p_1 q_1^{k_1 - 1} p_2 q_2^{k_2 - 1}] F(\mathbf{p} \times \mathbf{q}) = \frac{\partial}{\partial S_{k_1}} \frac{\partial}{\partial S_{k_2}} F \bigg|_{S_2 = S_3 = \cdots = 0} = \frac{\partial}{\partial S_{k_2}} \frac{\partial}{\partial S_{k_1}} F \bigg|_{S_2 = S_3 = \cdots = 0} = [p_1 q_1^{k_2 - 1} p_2 q_2^{k_1 - 1}] F(\mathbf{p} \times \mathbf{q})$$

$\mathbf{p}$-square-free terms 3

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \ldots$ then for any $k \geq 3$

$$[p_1 p_2 q_2^{k-2}]F = -(k-1)\ [S_k]F = -(k-1)[p_1 q_1^{k-1}]F$$

Hint: for $i_1 < \cdots < i_r,\ r \geq 0$

$$[p_{i_1} \cdots p_{i_r}] \underbrace{S_k(\mathbf{p} \times \mathbf{q})}_{\text{as polynomial in } \mathbf{p}} = \begin{cases} (-1)^{r-1}\ (k-1)_{r-1}\ \overbrace{q_{i_r}^{k-r}}^{\text{exponent at least 1}} & \text{if } 1 \leq r \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

outlook

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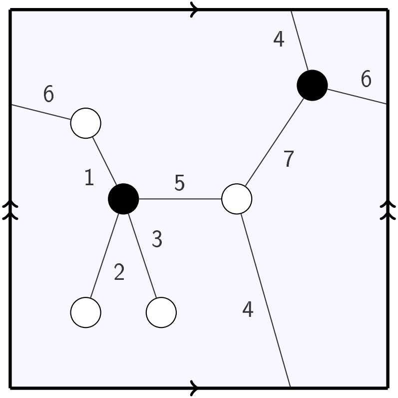
maps

map

- is a graph drawn on an oriented surface,
- bipartite,
- with one face,
- labeled,



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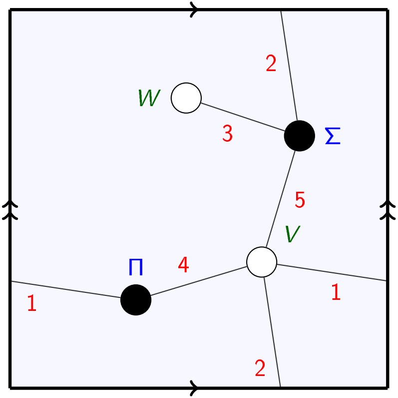
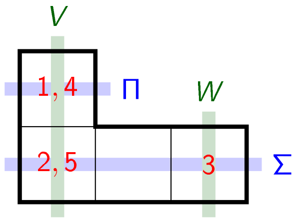
map with k edges $\longleftrightarrow$ $\sigma_1, \sigma_2 \in S_k$ such that $\sigma_1\sigma_2 = (1, 2, \ldots, k)$

Stanley's character formula



→Stanley, Féray, Śniady

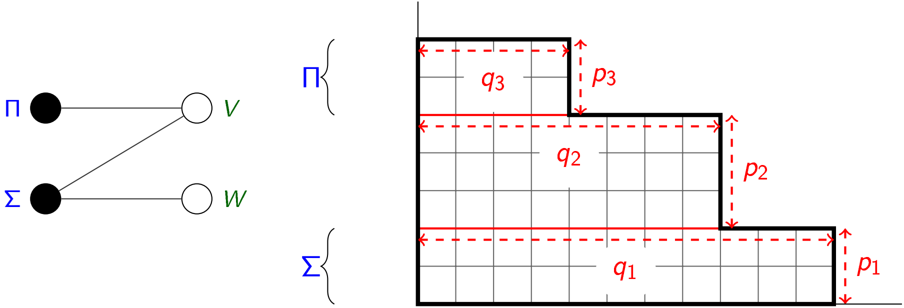


$N_M(\lambda) = \#$ embeddings of M to λ

$$\mathrm{Ch}_k(\lambda) = \sum_M (-1)^{k-\#\text{white vertices}} N_M(\lambda),$$

where the sum runs over maps M with k edges

N_M in Stanley coordinates



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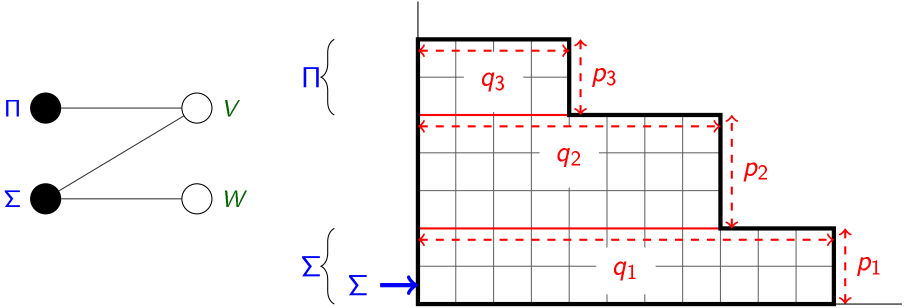

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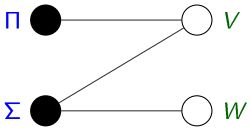
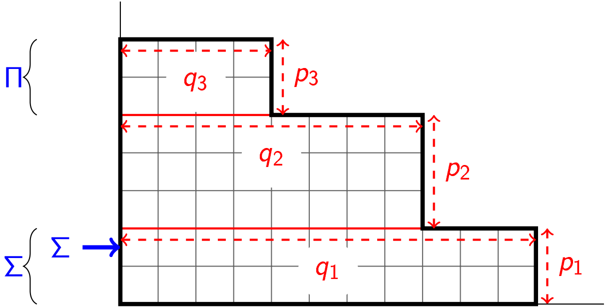
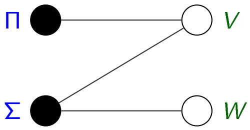
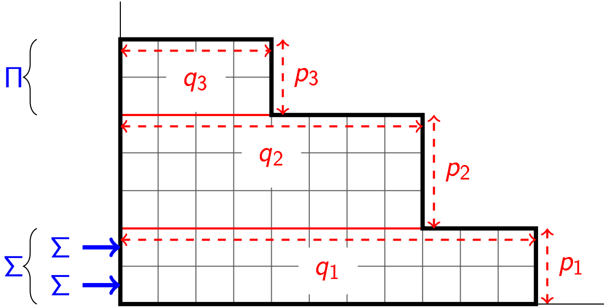




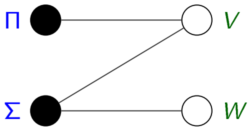
N_M in Stanley coordinates

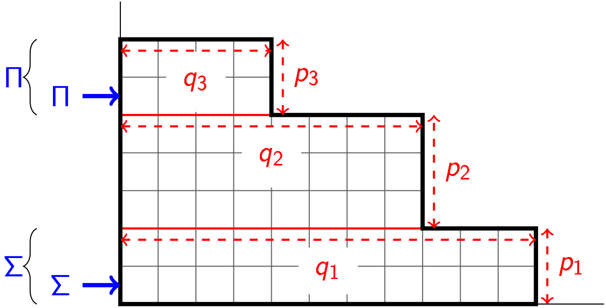




$p_1 \times$

N_M in Stanley coordinates





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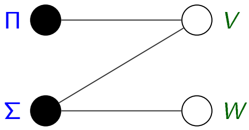
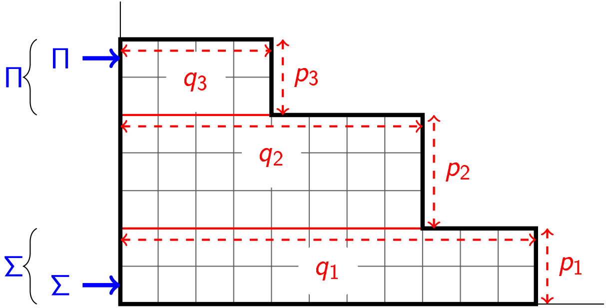
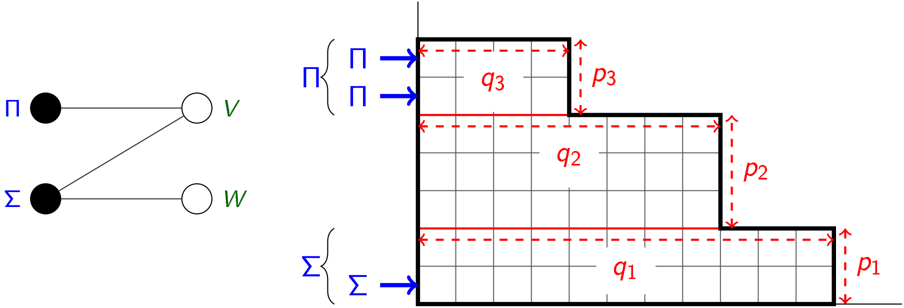
N_M in Stanley coordinates





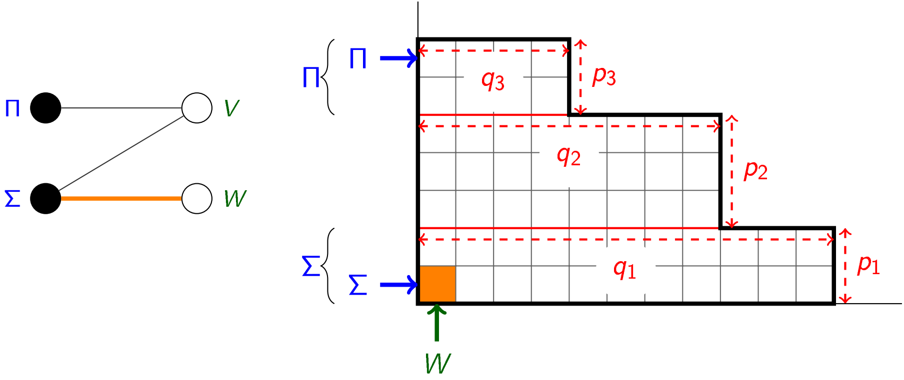
$p_1 \times$

N_M in Stanley coordinates



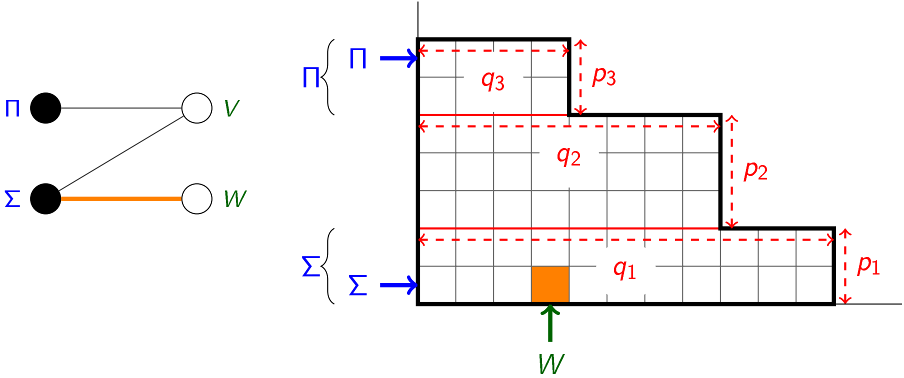
$p_1 \times p_3 \times$

N_M in Stanley coordinates



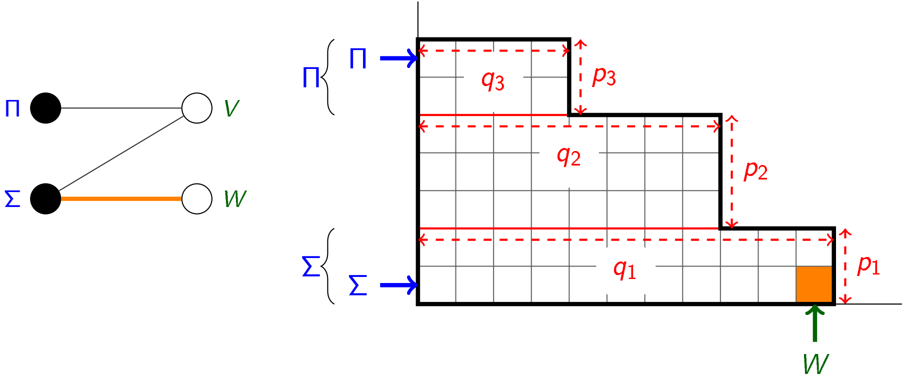
$p_1 \times p_3 \times$

N_M in Stanley coordinates



$p_1 \times p_3 \times$

N_M in Stanley coordinates



$p_1 \times p_3 \times$

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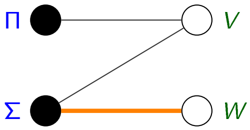
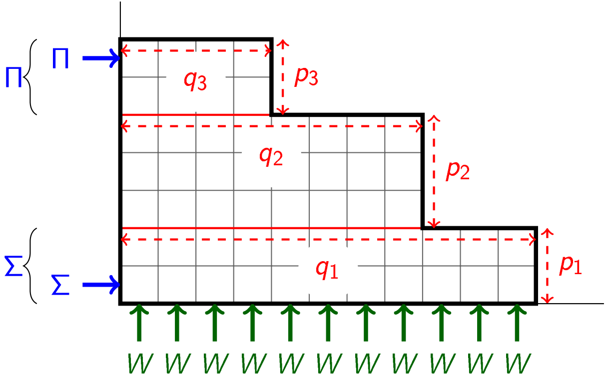
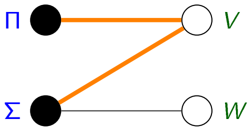
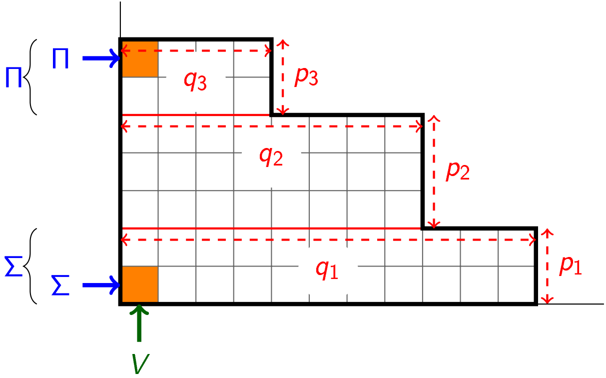




$p_1 \times p_3 \times q_1 \times$

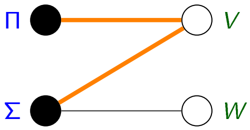
N_M in Stanley coordinates

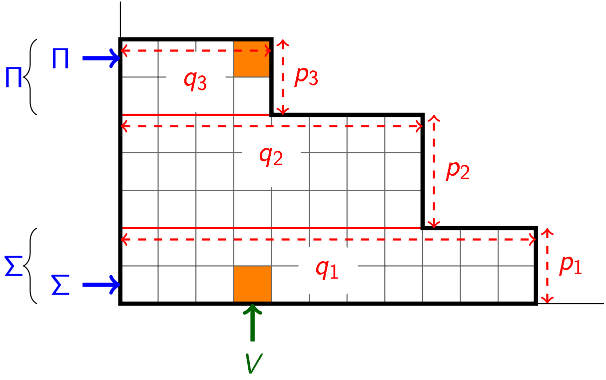




$p_1 \times p_3 \times q_1 \times$

N_M in Stanley coordinates





$p_1 \times p_3 \times q_1 \times$

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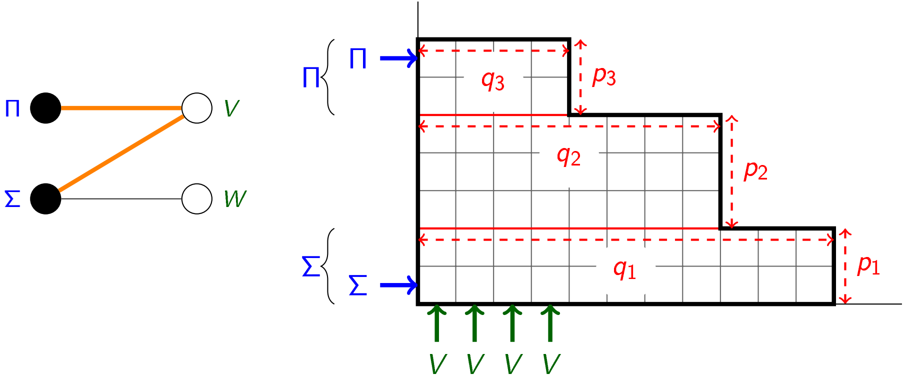
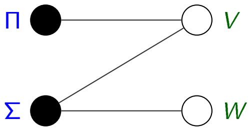
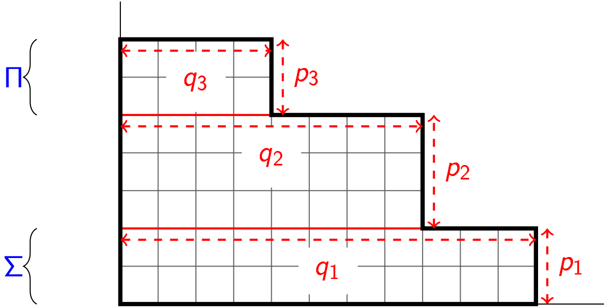


$p_1 \times p_3 \times q_1 \times q_3$

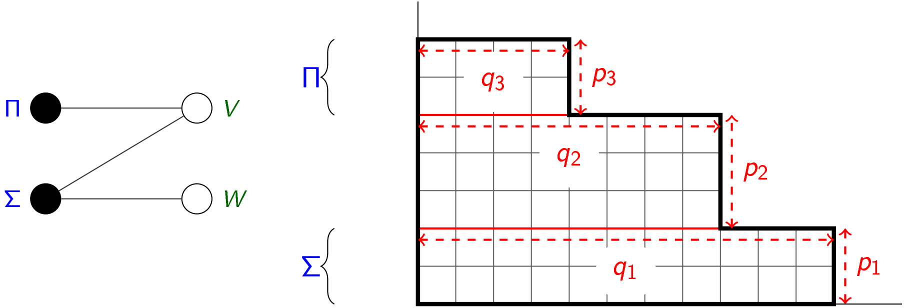
N_M in Stanley coordinates





$$p_1 \times p_3 \times q_1 \times q_3$$

N_M in Stanley coordinates



$$N_M(\mathbf{p}\times\mathbf{q}) = \sum_{F:V_\bullet\to\mathbb{N}} \left(\prod_{v\in V_\bullet} p_{F(v)}\right)\left(\prod_{w\in V_\circ} q_{G(w)}\right)$$

where $G(w) := \max_{\substack{v\in V_\bullet \\ v \text{ adjacent to } w}} F(v)$

Corollary:

$$(-1)^{\ell-1} \frac{\partial}{\partial S_{i_1}} \cdots \frac{\partial}{\partial S_{i_\ell}} \operatorname{Ch}_k \Big|_{S_2=S_3=\cdots=0} = (-1)^{\ell-1} [p_1 \cdots p_\ell q_1^{i_1-1} \cdots q_\ell^{i_\ell-1}] \operatorname{Ch}_k(\mathbf{p} \times \mathbf{q}) = \dots$$

number of maps

- with k edges,
- which have ℓ black vertices, labeled $V_1, \dots, V_\ell$,
- and there are $i_\ell - 1$ white vertices attached to V_ℓ,
- there are $i_{\ell-1} - 1$ white vertices which are attached to $V_{\ell-1}$ but not attached to V_ℓ,
- there are $i_{\ell-2} - 1$ white vertices which are attached to $V_{\ell-2}$ but not attached to $V_{\ell-1}, V_r$,
- ...
- there are $i_1 - 1$ white vertices which are attached to V_1 but not attached to $V_2, \dots, V_\ell$,