Combinatorics of asymptotic representation theory

Part 2

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if you missed the first talk... normalized characters

for $\pi \in S_k$ and $\lambda \vdash n$ we define normalized character

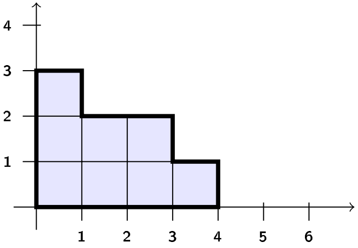
$$\mathrm{Ch}_{\pi}(\lambda) := \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\mathrm{Tr}\, \rho^{\lambda}(\pi)}{\dim \rho^{\lambda}}$$

$$\mathrm{Ch}_k(\lambda) := \mathrm{Ch}_{\underbrace{(1,2,\dots,k)}_{\in S_k}}(\lambda)$$

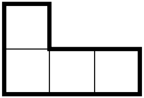
if you missed the first talk. . . continuous functionals of shape

for $k \geq 2$

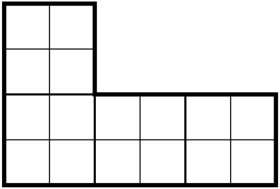
$$S_k(\lambda) := (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2} \, dx \, dy$$



if you missed the first talk. . . dilations



Young diagram λ

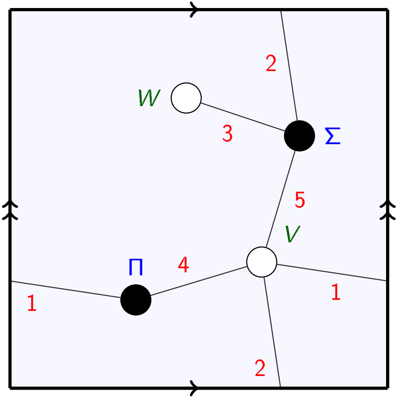


dilated diagram 2λ

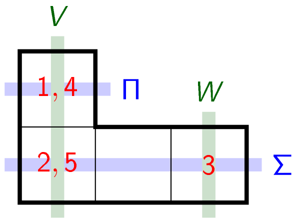
S_k is homogeneous of degree k:

$$S_k(r\lambda) = r^k S_k(\lambda)$$

if you missed. . . Stanley's character formula



→Stanley, Féray, Śniady

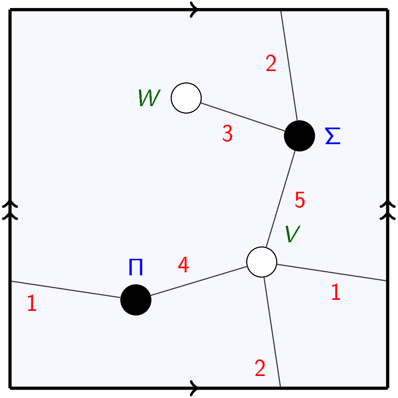


$N_M(\lambda) = \#$ embeddings of M to λ

$$\mathrm{Ch}_k(\lambda) = \sum_M (-1)^{k-\#\text{white vertices}} N_M(\lambda),$$

where the sum runs over maps M with k edges

if you missed. . . Stanley's character formula



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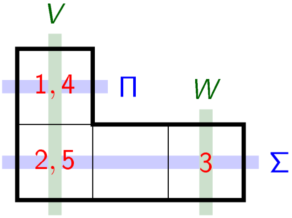


$N_M(\lambda) = \#$ embeddings of M to λ

$$\mathrm{Ch}_k(\lambda) = \sum_M (-1)^{k-\#\text{white vertices}} \; N_M(\lambda),$$

degree of $N_M = k + 1 - \mathrm{genus}(M)$

free cumulants 1

$$\mathrm{Ch}_1 = \underbrace{S_2}_{\text{degree } 2},$$

$$\mathrm{Ch}_2 = \underbrace{S_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_3 = \underbrace{S_4 - \frac{3}{2} S_2^2}_{\text{degree } 4} + \underbrace{S_2}_{\text{degree } 2},$$

$$\mathrm{Ch}_4 = \underbrace{S_5 - 4 S_2 S_3}_{\text{degree } 5} + \underbrace{5 S_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_5 = \underbrace{S_6 - 5 S_2 S_4 - \frac{5}{2} S_3^2 + \frac{25}{6} S_2^3}_{\text{degree } 6} + \underbrace{15 S_4 - \frac{35}{2} S_2^2}_{\text{degree } 4} + \underbrace{8 S_2}_{\text{degree } 2}.$$

free cumulants 1

$$\mathrm{Ch}_1 = \underbrace{S_2}_{R_2},$$

$$\mathrm{Ch}_2 = \underbrace{S_3}_{R_3},$$

$$\mathrm{Ch}_3 = \underbrace{S_4 - \frac{3}{2}S_2^2}_{R_4} + \underbrace{S_2}_{\text{degree } 2},$$

$$\mathrm{Ch}_4 = \underbrace{S_5 - 4S_2S_3}_{R_5} + \underbrace{5S_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_5 = \underbrace{S_6 - 5S_2S_4 - \frac{5}{2}S_3^2 + \frac{25}{6}S_2^3}_{R_6} + \underbrace{15S_4 - \frac{35}{2}S_2^2}_{\text{degree } 4} + \underbrace{8S_2}_{\text{degree } 2}.$$

free cumulants 2

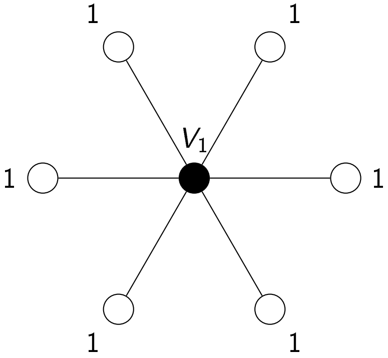
$s \mapsto \mathrm{Ch}_k(s\lambda)$ is a polynomial of degree $k+1$

$$\underbrace{R_{k+1}(\lambda)}_{\text{free cumulant}} := [s^{k+1}]\,\mathrm{Ch}_k(s\lambda) = \lim_{s\to\infty} \frac{\mathrm{Ch}_k(s\lambda)}{s^{k+1}}$$

$\longrightarrow$ Biane

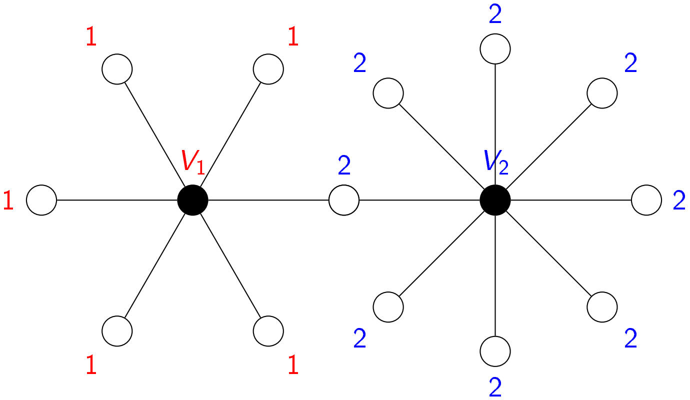
$$\mathrm{Ch}_k(\lambda) \approx R_{k+1}(\lambda)$$

free cumulants in terms of functionals of shape 1



$$R_{k+1} = S_{k+1} + \cdots$$

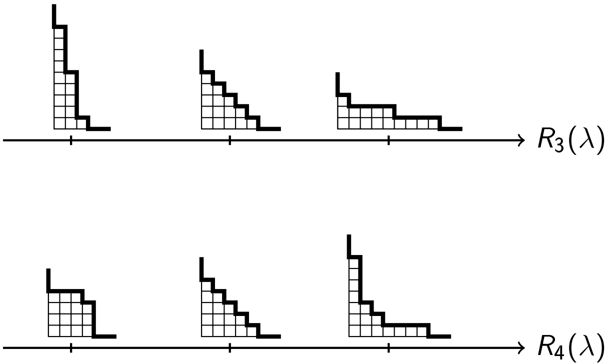
free cumulants in terms of functionals of shape 2



$$R_{k+1} = \cdots - \frac{1}{2} k \sum_{\substack{i_1, i_2 \geq 2, \\ i_1 + i_2 = k+1}} S_{i_1} S_{i_2} + \cdots$$

free cumulants in terms of functionals of shape 3

$$R_{k+1} = S_{k+1} - \frac{1}{2!} k \sum_{\substack{i_1, i_2 \geq 2, \\ i_1 + i_2 = k+1}} S_{i_1} S_{i_2} + \frac{1}{3!} k^2 \sum_{\substack{i_1, i_2, i_3 \geq 2, \\ i_1 + i_2 + i_3 = k+1}} S_{i_1} S_{i_2} S_{i_3} - \cdots$$



Kerov polynomials

characters $\longleftrightarrow$ shape of the Young diagram

$$\begin{aligned}
\mathrm{Ch}_1 &= R_2, \\
\mathrm{Ch}_2 &= R_3, \\
\mathrm{Ch}_3 &= R_4 + R_2, \\
\mathrm{Ch}_4 &= R_5 + 5R_3, \\
\mathrm{Ch}_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\mathrm{Ch}_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

positivity?

what Kerov polynomials count? transportation problem

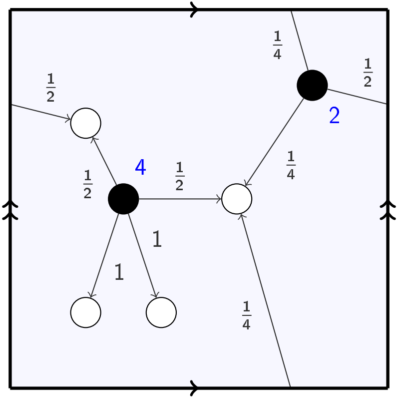
coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $i_1, \dots, i_\ell$,

each black vertex i produces $i-1$ units of liquid,

each white vertex demands 1 unit of the liquid,

each edge transports strictly positive amout of liquid from black to white vertex



→ Féray, Dołęga & Śniady

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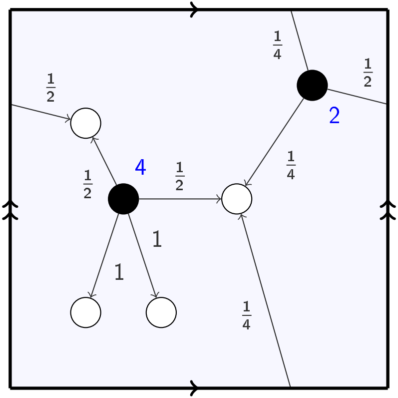

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strong restriction on the map: no disconnecting edges (except for white leaves)

what Kerov polynomials count? Hall marriage theorem

coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $i_1, \dots, i_\ell$,

...

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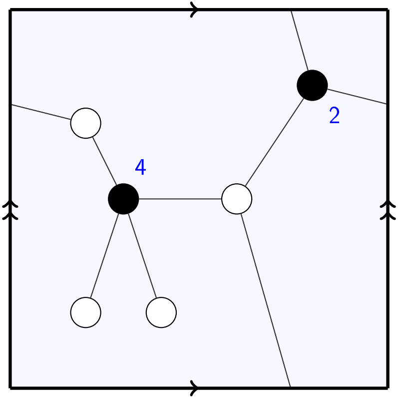


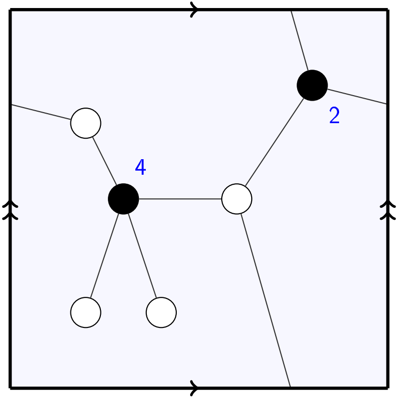
→ Féray, Dołęga & Śniady

what Kerov polynomials count? Hall marriage theorem

coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $i_1, \dots, i_\ell$,

each black vertex i wants to be married to $i-1$ white vertices,



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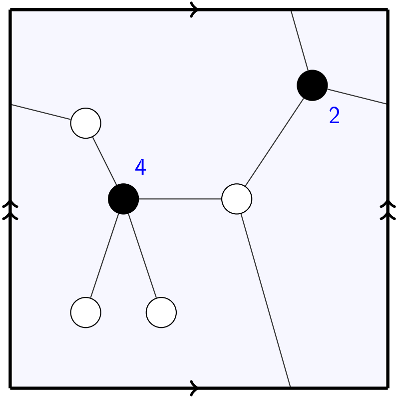

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what Kerov polynomials count? Hall marriage theorem

coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

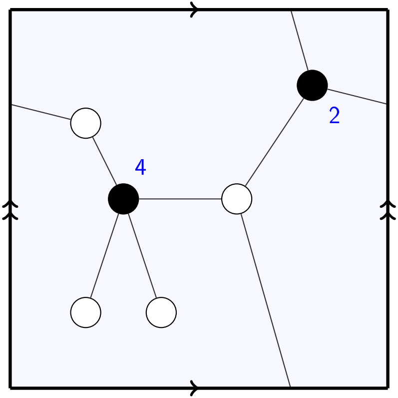
with black vertices labelled by $i_1, \dots, i_\ell$,

with $i_1 + \cdots + i_\ell$ vertices,

each nontrivial set B of black vertices has more than

$$\sum_{v \in B} (\text{label of vertex}) - 1$$

white neighbors,



→ Féray, Dołęga & Śniady

toy example: $[R_{k_1} R_{k_2}]F$

Theorem

if $F = F(\lambda)$ *is a polynomial in* $R_2, R_3, \ldots$ *then*

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} F \Big|_{R_2 = R_3 = \cdots = 0} = [p_1 p_2 q_1^{k_1 - 1} q_2^{k_2 - 1}] F(\mathbf{p} \times \mathbf{q}) - [p_1 p_2 q_2^{k_1 + k_2 - 2}] F(\mathbf{p} \times \mathbf{q})$$

toy example: $[R_{k_1} R_{k_2}]\,\mathrm{Ch}_n$

We are interested in maps with $k_1 + k_2 - 2$ white and two black vertices V_1, V_2.

$$\#(\text{maps such that } V_1 \text{ has} \geq k_1 \text{ friends, } V_2 \text{ has} \geq k_2 \text{ friends}) =$$

$$\#(\text{all maps}) - \#(\text{maps such that } V_1 \text{ has} \leq k_1 - 1 \text{ friends})$$

$$-\#(\text{maps such that } V_2 \text{ has} \leq k_2 - 1 \text{ friends}) =$$

$$(-1) \sum_{\substack{i+j=k_1+k_2-2,\\ 1 \leq j}} \left[p_1 p_2 q_1^i q_2^j\right] \mathrm{Ch}_k^{\mathbf{p}\times\mathbf{q}} + \sum_{\substack{i+j=k_1+k_2-2,\\ 1 \leq i \leq k_1-1}} \left[p_1 p_2 q_1^j q_2^i\right] \mathrm{Ch}_k^{\mathbf{p}\times\mathbf{q}}$$

$$+ \sum_{\substack{i+j=k_1+k_2-2,\\ 1 \leq j \leq k_2-1}} \left[p_1 p_2 q_1^i q_2^j\right] \mathrm{Ch}_k^{\mathbf{p}\times\mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}]\,\mathrm{Ch}_n^{\mathbf{p}\times\mathbf{q}} - [p_1 p_2 q_2^{k_1+k_2-2}]\,\mathrm{Ch}_n^{\mathbf{p}\times\mathbf{q}}$$

characters on two cycles

the normalized character $\mathrm{Ch}_{k,l}(\lambda)$

$$(1, 2, \dots, k)(k+1, k+2, \dots, k+l) \in \mathfrak{S}(k+l)$$

Kerov polynomials

$$\mathrm{Ch}_{3,2} = R_3 R_4 - 5 R_2 R_3 - 6 R_5 - 18 R_3$$

not nice!

(abstract) covariance

$$\mathrm{Cov}(\mathrm{Ch}_k, \mathrm{Ch}_l) := \mathrm{Ch}_{k,l} - \mathrm{Ch}_k \mathrm{Ch}_l$$

$$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2) = -\big(6 R_2 R_3 + 6 R_5 + 18 R_3\big)$$

is nice!

surprising cancellations

$$\mathrm{Ch}_2 = \underbrace{R_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_3 = \underbrace{R_4}_{\text{degree } 4} + R_2,$$

$$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2) = -\Big(6 \underbrace{R_2 R_3}_{\text{degree only } 5} + 6 \underbrace{R_5}_{\text{degree only } 5} + 18 R_3\Big)$$

explanation by Kerov polynomials:
$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2)$ counts connected maps with two cells, such that...

Gaussian fluctuations

(abstract) cumulant

$$k(\mathrm{Ch}_{i_1}, \dots, \mathrm{Ch}_{i_\ell}) = \mathrm{Ch}_{i_1,\dots,i_\ell} - \cdots$$

surprising cancellation:

$$\deg k(\mathrm{Ch}_{i_1}, \dots, \mathrm{Ch}_{i_\ell}) = \deg \mathrm{Ch}_{i_1} + \cdots + \deg \mathrm{Ch}_{i_\ell} - 2(\ell - 1)$$

$\mathrm{Ch}_1, \mathrm{Ch}_2, \mathrm{Ch}_3, \dots$ behave asymptotically as (abstract) Gaussian random variables

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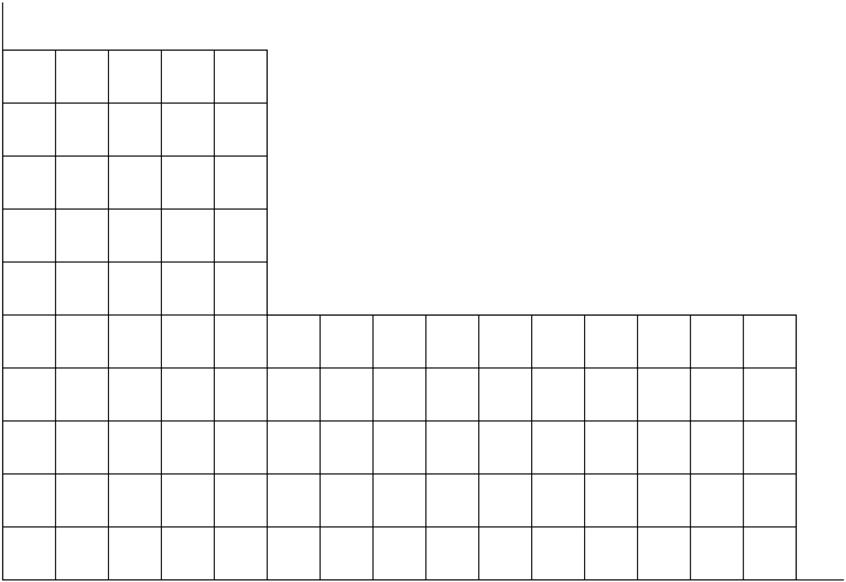
Theorem

for a large class of reducible representations of $\mathfrak{S}(n)$,
if we randomly select an irreducible component ρ^λ, for $n \to \infty$
λ will concentrate around some limit shape →BIANE
and the fluctuations are Gaussian →KEROV, ŚNIADY

random Young tableaux 1



random Young tableaux 1

75	81	89	98	100										
58	60	72	94	99										
51	56	62	93	95										
26	38	54	79	92										
18	33	37	59	87										
12	20	35	36	42	46	67	68	70	78	82	84	88	90	97
11	17	19	22	30	43	52	55	64	65	66	74	83	85	96
8	10	13	21	23	29	34	45	47	49	63	71	76	80	91
2	7	9	15	16	24	27	39	41	44	48	57	69	77	86
1	3	4	5	6	14	25	28	31	32	40	50	53	61	73

random Young tableaux 1

75	81	89	98	100										
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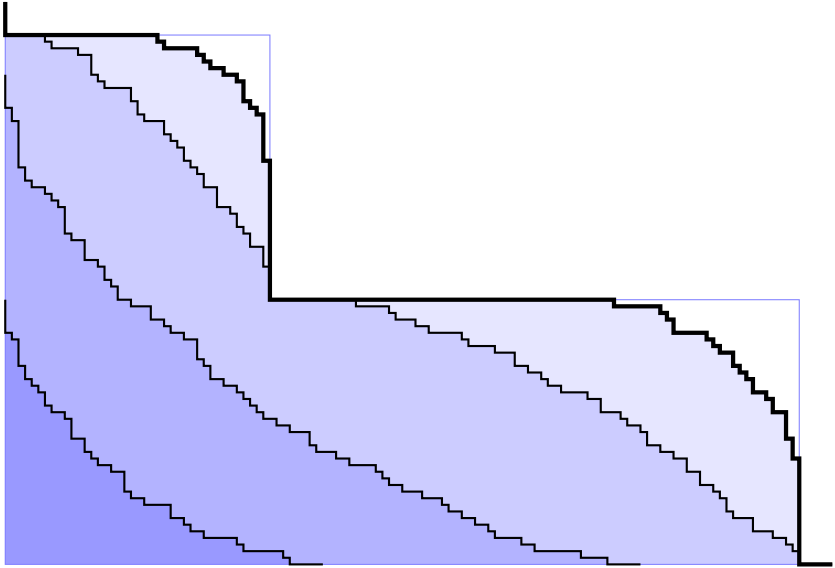
8	10	13	21	23	29	34	45	47	49	63	71	76	80	91
2	7	9	15	16	24	27	39	41	44	48	57	69	77	86
1	3	4	5	6	14	25	28	31	32	40	50	53	61	73

restriction $\rho^\lambda \downarrow^{\mathfrak{S}(n)}_{\mathfrak{S}(m)}$ to a subgroup

random Young tableaux 2



random Young tableaux 3: explanation

characters $\longleftrightarrow$ shape of the Young diagram

$$\mathrm{Ch}_k \approx R_{k+1}$$

random Young tableaux 4: explanation

we decompose $\rho^\lambda \big\downarrow^{\mathfrak{S}(n)}_{\mathfrak{S}(m)}$ into irreducible components and randomly select one of them, say ρ^μ

$$R_{k+1}(\lambda) \approx \mathrm{Ch}_k(\lambda) \approx n^k \frac{\operatorname{Tr} \chi^\lambda([k])}{\operatorname{Tr} \chi^\lambda(e)},$$

$$\mathbb{E} R_{k+1}(\mu) \approx \mathbb{E}\, \mathrm{Ch}_k(\mu) \approx m^k \mathbb{E} \frac{\operatorname{Tr} \chi^\mu([k])}{\operatorname{Tr} \chi^\mu(e)} = m^k \frac{\operatorname{Tr} \chi^\lambda([k])}{\operatorname{Tr} \chi^\lambda(e)}$$

thus for a *typical* random Young diagram μ we can expect that

$$R_{k+1}(\mu) \approx \left(\frac{m}{n}\right)^k R_{k+1}(\lambda).$$

Goulden-Rattan polynomials 1

$$\mathrm{Ch}_k - \underbrace{R_{k+1}}_{\text{degree } k+1} = \frac{(k+1)k(k-1)}{24} \underbrace{C_{k-1}}_{\text{degree } k-1} + \cdots$$

$$\vdots$$

$$\mathrm{Ch}_6 - R_7 = \frac{35}{4} C_5 + 42 C_3,$$

$$\mathrm{Ch}_7 - R_8 = 14 C_6 + \frac{469}{3} C_4 + \frac{203}{3} C_2^2 + 180 C_2.$$

→Goulden & Rattan

positivity?

Goulden-Rattan polynomials 2

$$C_k = \sum_{i_1+\cdots+i_\ell=k} \prod_{1\leq s\leq \ell} (i_s - 1)R_{i_s}$$

Jack polynomials

$$\begin{aligned}
\mathrm{Ch}_1^{(\gamma)} &= R_2, \\
\mathrm{Ch}_2^{(\gamma)} &= R_3 + \gamma R_2, \\
\mathrm{Ch}_3^{(\gamma)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\
\mathrm{Ch}_4^{(\gamma)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2, \\
\mathrm{Ch}_5^{(\gamma)} &= R_6 + 10\gamma R_5 + 5\gamma R_3 R_2 + 15 R_4 + 5 R_2^2 + \gamma^2 (35 R_4 + 10 R_2^2) + \\
&\quad (55\gamma + 50\gamma^3) R_3 + (8 + 46\gamma^2 + 24\gamma^4) R_2
\end{aligned}$$

→LASSALLE

positivity? integer coefficients?