Combinatorics of asymptotic representation theory

Part 3

joint work with Dan Romik

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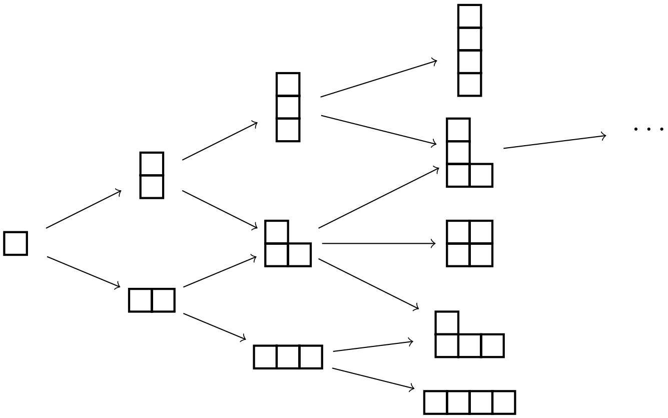
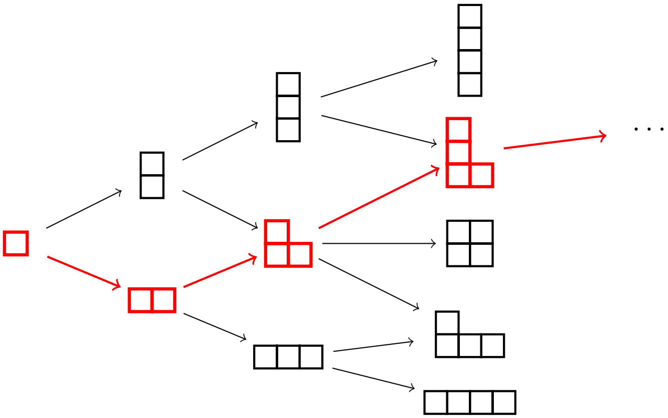
Technische Universität München
and
Polska Akademia Nauk
and
Uniwersytet Wrocławski

Young graph: irreducible representations of symmetric groups $S_1 \subset S_2 \subset S_3 \subset \dots$



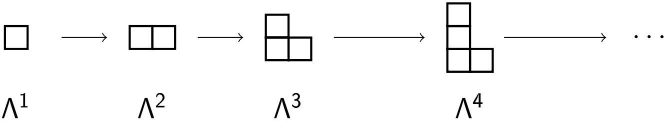
Tool for studying S_∞

Young graph: irreducible representations of symmetric groups $S_1 \subset S_2 \subset S_3 \subset \dots$

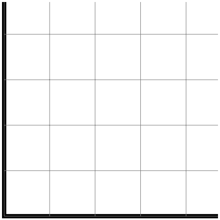


Tool for studying S_∞

paths in Young graph ⟷ tableaux



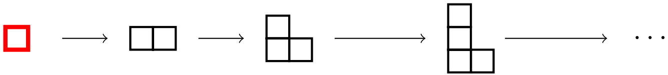
infinite path in Young graph



infinite tableau

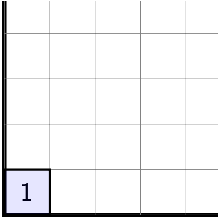
$\Omega :=$ set of infinite tableaux / set of infinite paths

paths in Young graph $\longleftrightarrow$ tableaux



Λ^1 Λ^2 Λ^3 Λ^4

infinite path in Young graph



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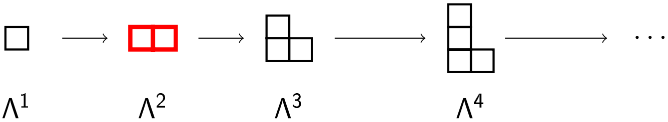
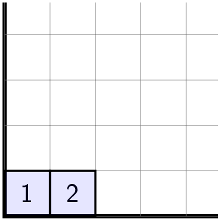


infinite path in Young graph



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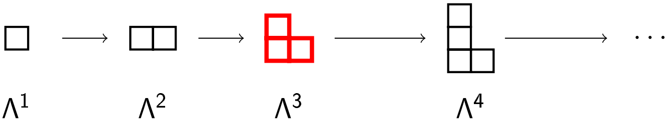

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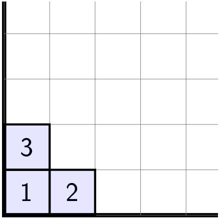

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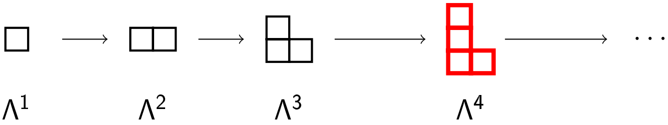

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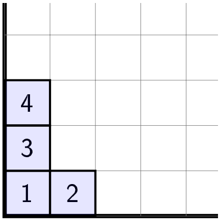

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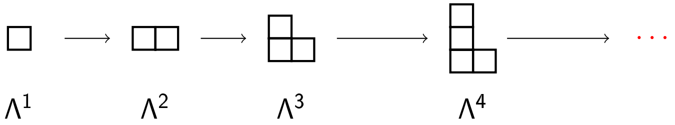


infinite path in Young graph

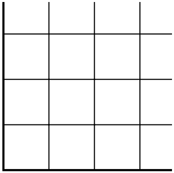
$\vdots$		$\vdots$		
6	15	21	24	
4	12	17	19	$\cdots$
3	5	8	11	
1	2	7	9	$\cdots$

infinite tableau

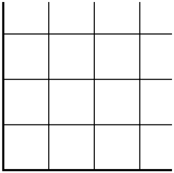
$\Omega :=$ set of infinite tableaux / set of infinite paths

infinite Robinson-Schensted-Knuth (RSK) map

infinite word $\overset{\text{RSK}}{\mapsto}$ recording tableau



insertion tableau



recording tableau

F O X D R P B Z U L G E A T W N S M Y V C J H Q I K

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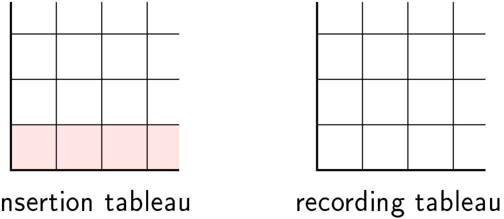

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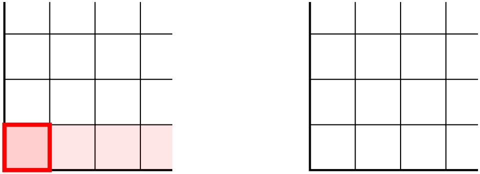

insertion tableau recording tableau

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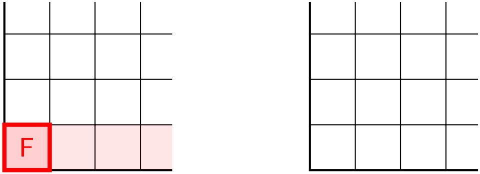

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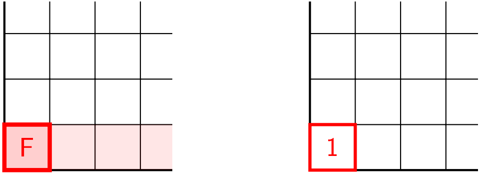

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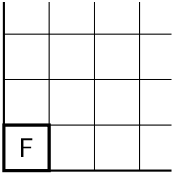

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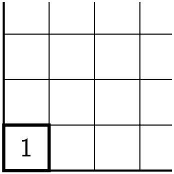

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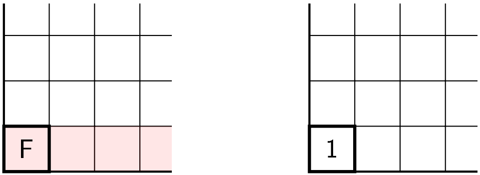

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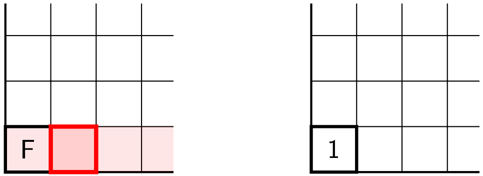

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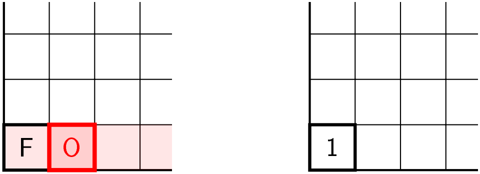

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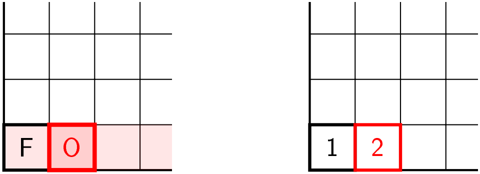

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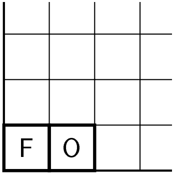

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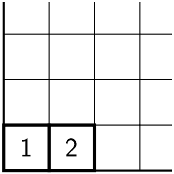

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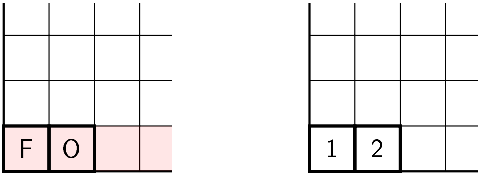

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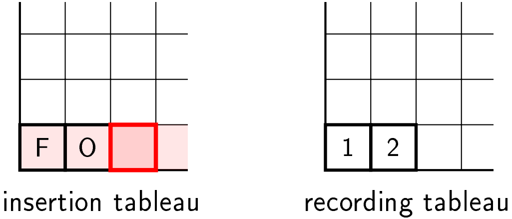

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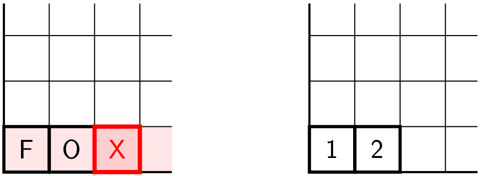

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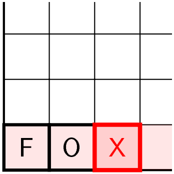
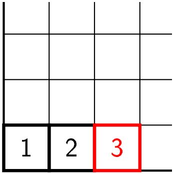


insertion tableau



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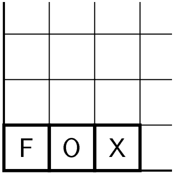

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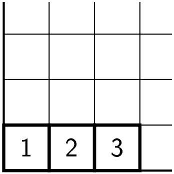

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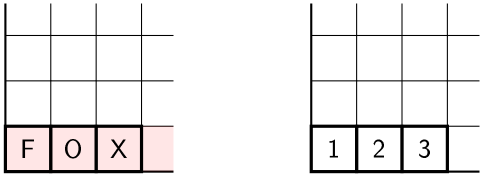

insertion tableau recording tableau

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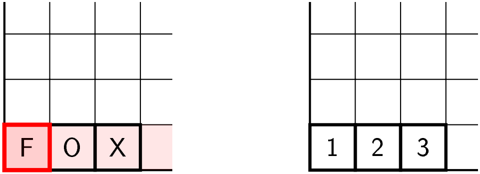

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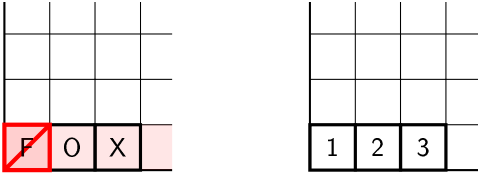

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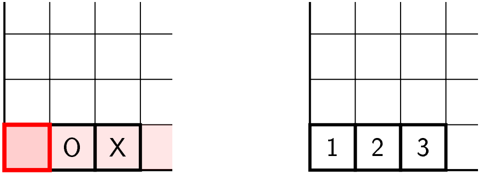

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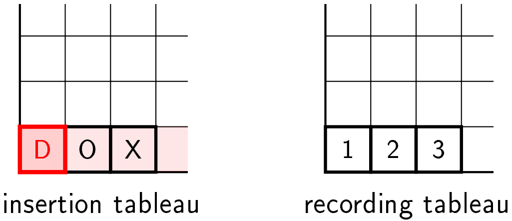

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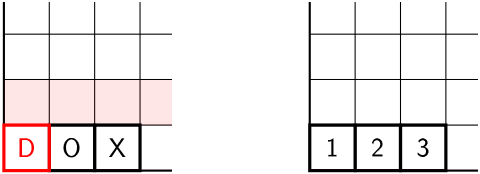

insertion tableau recording tableau

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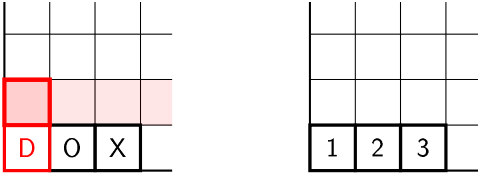

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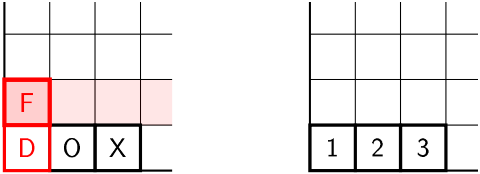

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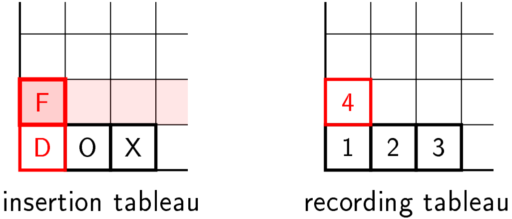

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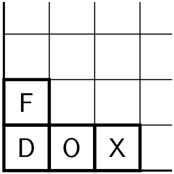
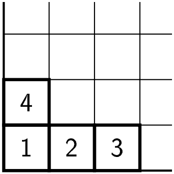


insertion tableau



recording tableau

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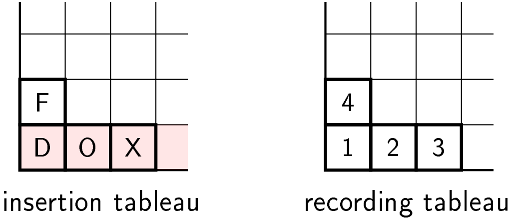

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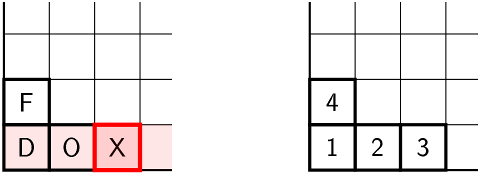

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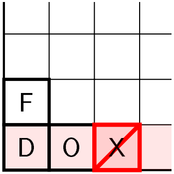

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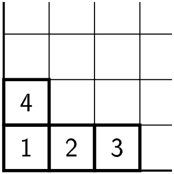

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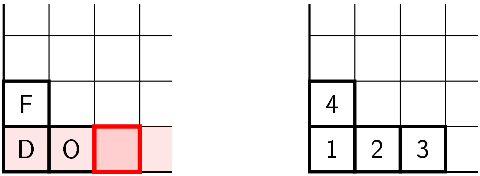

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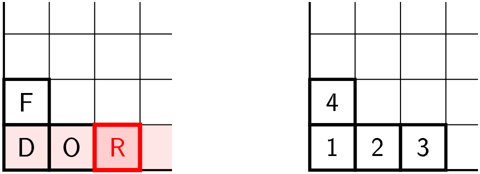

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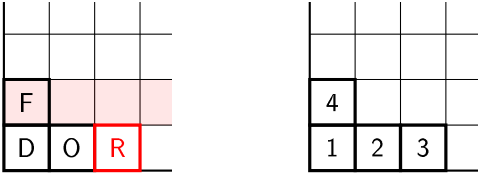

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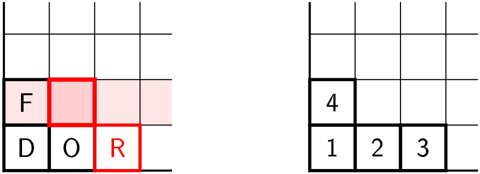

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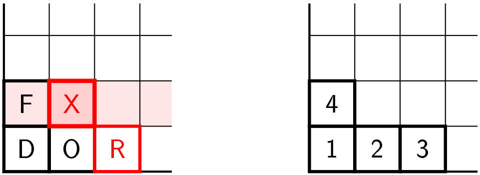

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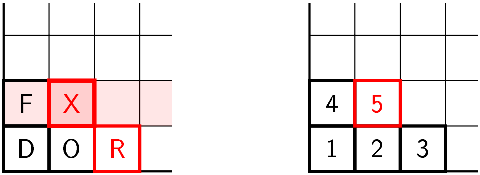

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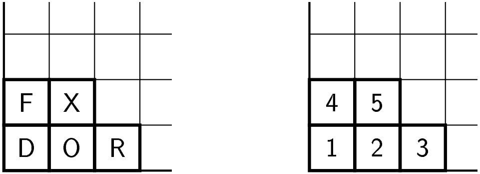

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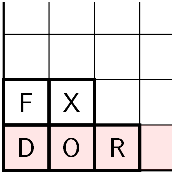
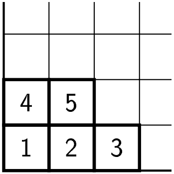


insertion tableau



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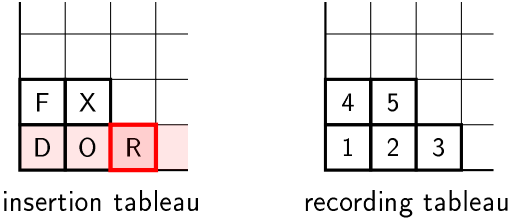

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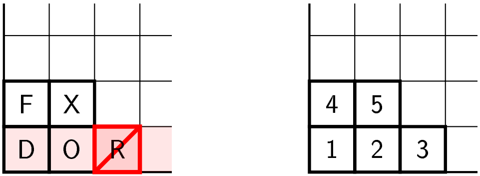

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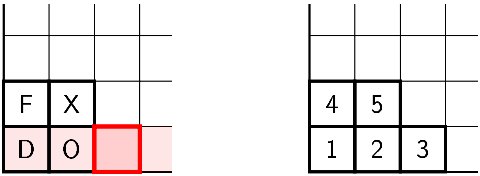

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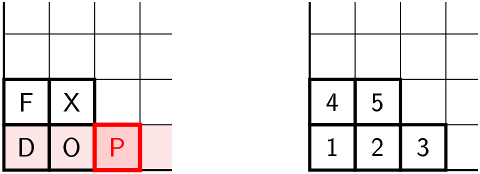

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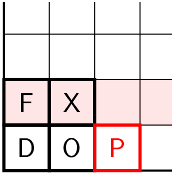

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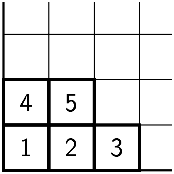

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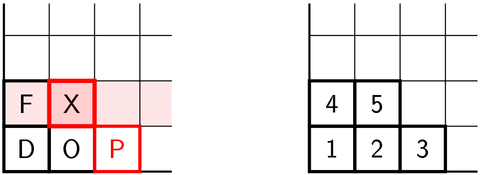

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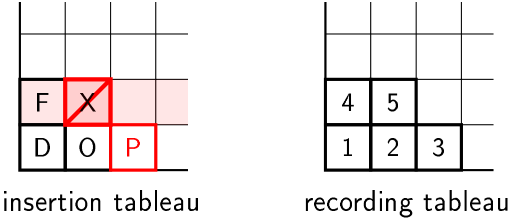

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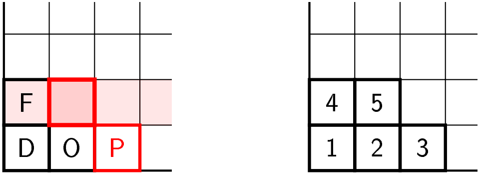

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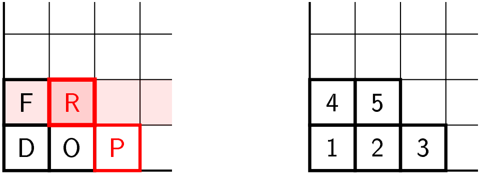

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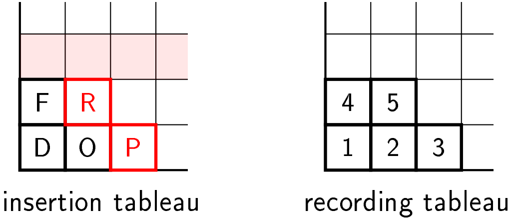

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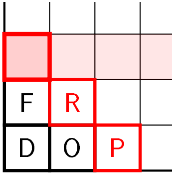

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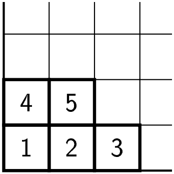

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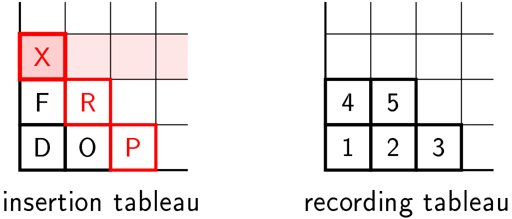

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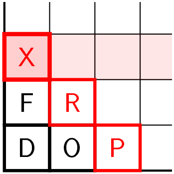

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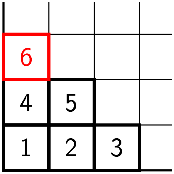

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X		
F	R	
D	O	P

insertion tableau

6		
4	5	
1	2	3

recording tableau

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insertion tableau

F	L	N	T
D	G	M	S
B	E	J	Q
A	C	H	I

recording tableau

7	16	22	25
6	10	14	24
4	5	9	17
1	2	3	8

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insertion tableau — recording tableau

F O X D R P B Z U L G E A T W N S M Y V C J H Q I K

infinite Robinson-Schensted-Knuth (RSK) map

$$\text{infinite word} \overset{\text{RSK}}{\mapsto} \text{recording tableau}$$

7	16	22	25
6	10	14	24
4	5	9	17
1	2	3	8

~~insertion tableau~~ recording tableau

F O X D R P B Z U L G E A T W N S M Y V C J H Q I K

if $X_0, X_1, \ldots$ are independent $U(0,1)$ random variables then

$$\text{RSK}(X_0, X_1, \ldots) \overset{\text{distribution}}{=} \text{Plancherel measure}$$

infinite Robinson-Schensted-Knuth (RSK) map

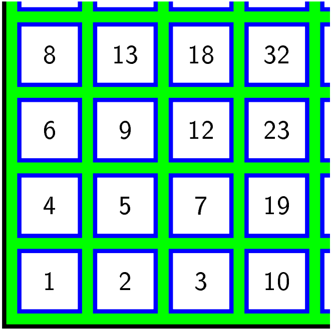
$$\begin{array}{c} (x_0, x_1, \dots) \\ \Big\downarrow{\scriptstyle \mathsf{RSK}} \\ t \end{array}$$

infinite Robinson-Schensted-Knuth (RSK) map

$$(x_0, x_1, \dots)$$

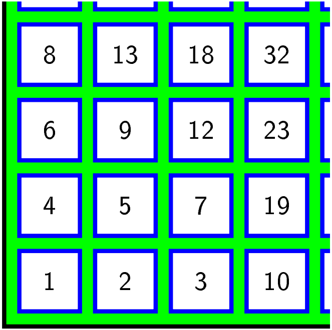
RSK $\downarrow$ $\uparrow$ inverse?

$$t$$



jeu de taquin

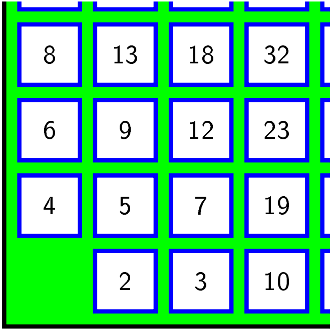
① start with $t \in \Omega$,



jeu de taquin

① start with $t \in \Omega$,

② remove corner box,



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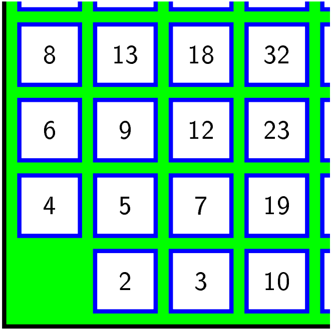
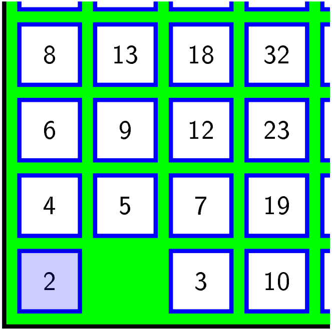


jeu de taquin

1. start with $t \in \Omega$,
2. remove corner box,
3. sliding,



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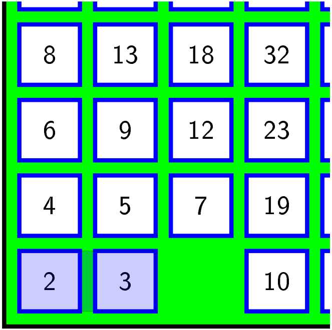

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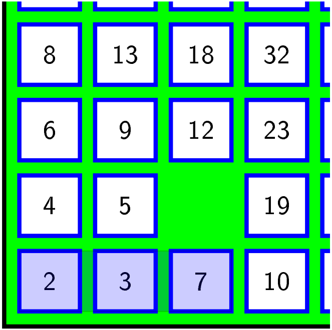

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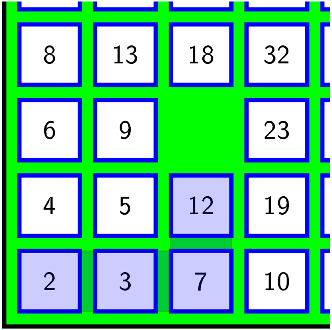

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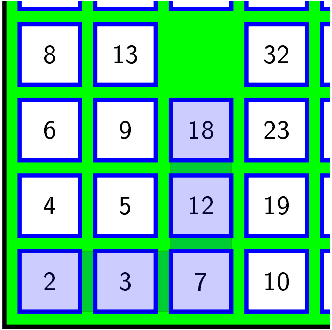

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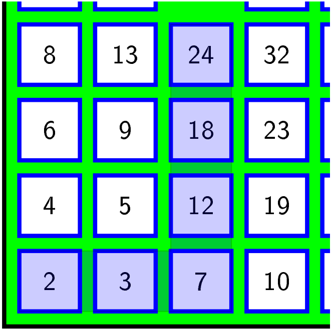

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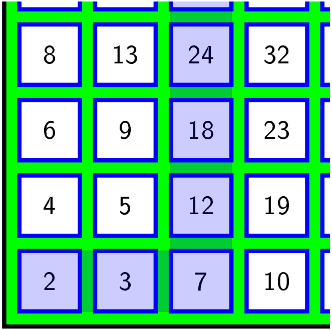

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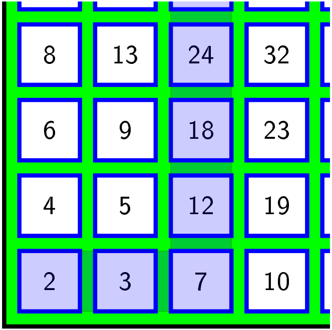

jeu de taquin

① start with $t \in \Omega$,

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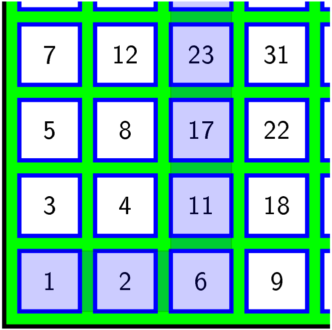
③ sliding,

④ subtract 1 from all boxes



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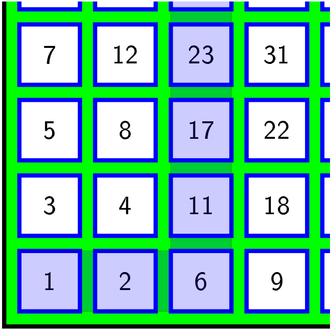

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① start with $t \in \Omega$,

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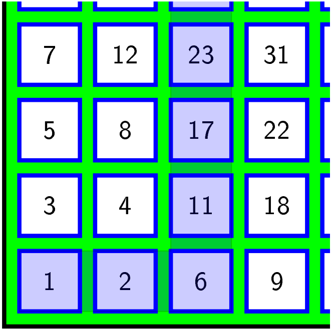
output:

- new tableau $J(t)$,
- blue trajectory $\mathbf{c}(t) = (c_1, c_2, \dots)$



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'how representation of $S_{\{1,2,3,\ldots\}}$ is related to its restriction to $S_{\{2,3,\ldots\}}$?'

jeu de taquin - overview

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

original tableau t

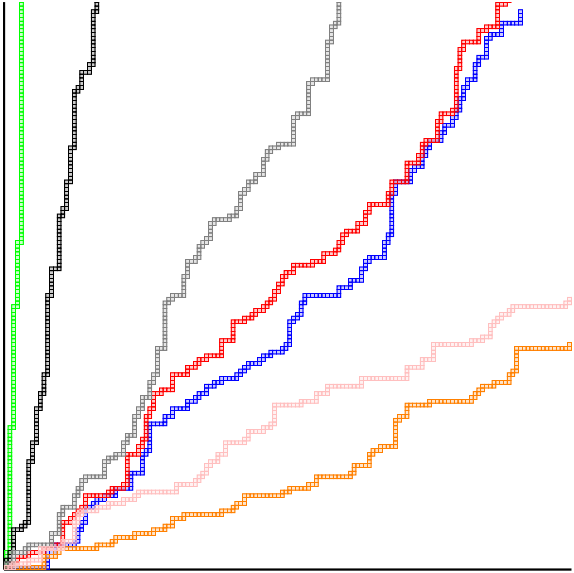
8	13	24	32
6	9	18	23
4	5	12	19
2	3	7	10

outcome of slidings

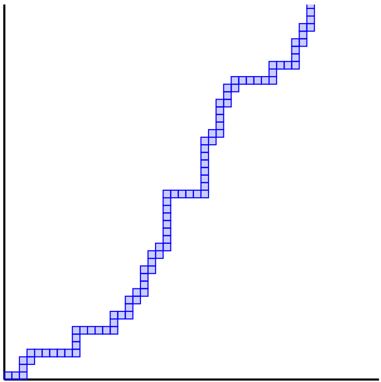
7	12	23	31
5	8	17	22
3	4	11	18
1	2	6	9

new tableau $J(t)$

trajectories of jeu de taquin



trajectories of jeu de taquin



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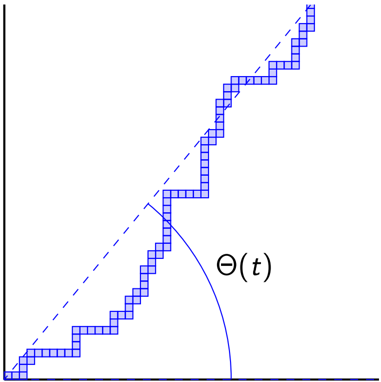
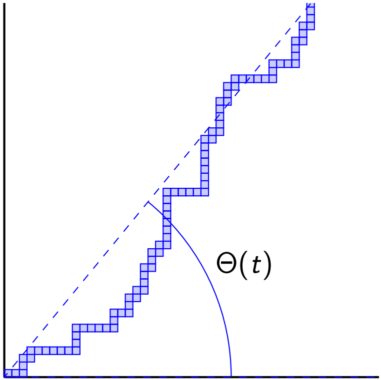


trajectories of jeu de taquin



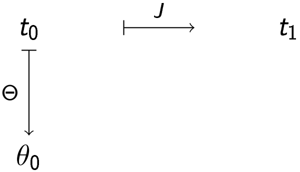
if $t = \mathrm{RSK}(X_0, X_1, \dots) \in \Omega$ is random, Plancherel distributed

then its jdt trajectory $\mathbf{c}(t)$ is almost surely asymptotically a straight line,

i.e.

$$\lim_{k\to\infty} \frac{c_k}{\|c_k\|} = \big(\cos\Theta(t), \sin\Theta(t)\big)$$

exists almost surely


t_0
J
t_1
Θ
θ_0

$$
\begin{array}{ccccc}
t_0 & \overset{J}{\longmapsto} & t_1 & \overset{J}{\longmapsto} & \cdots \\
\Theta \downarrow & & \Theta \downarrow & & \\
\theta_0 & & \theta_1 & &
\end{array}
$$

$$\begin{array}{ccccc} t_0 & \stackrel{J}{\longmapsto} & t_1 & \stackrel{J}{\longmapsto} & \cdots \\ \Theta \big\downarrow & & \Theta \big\downarrow & & \\ \theta_0 & & \theta_1 & & \end{array}$$

jeu de taquin dynamical system $(\Omega, \text{Plancherel}, J)$

$$
\begin{array}{ccccc}
(x_0, x_1, \ldots) & & & & \\
\Big\downarrow{\scriptstyle \mathrm{RSK}} & & & & \\
t_0 & \overset{J}{\longmapsto} & t_1 & \overset{J}{\longmapsto} & \cdots \\
\Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Theta} & & \\
\theta_0 & & \theta_1 & &
\end{array}
$$

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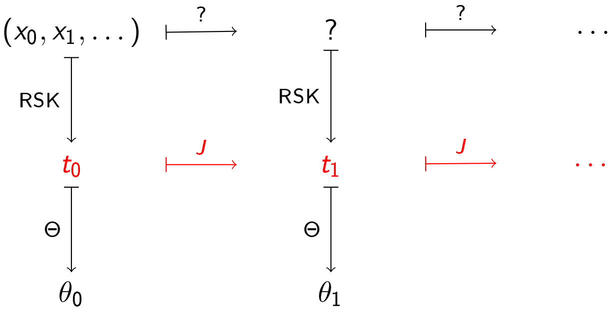

jeu de taquin dynamical system $(\Omega, \text{Plancherel}, J)$

$$
\begin{array}{ccccc}
(x_0, x_1, \dots) & \xmapsto{\;s\;} & (x_1, x_2, \dots) & \xmapsto{\;s\;} & \cdots \\
\Big\downarrow{\scriptstyle \mathsf{RSK}} & & \Big\downarrow{\scriptstyle \mathsf{RSK}} & & \\
t_0 & \xmapsto{\;J\;} & t_1 & \xmapsto{\;J\;} & \cdots \\
\Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Theta} & & \\
\theta_0 & & \theta_1 & &
\end{array}
$$

jeu de taquin dynamical system $(\Omega, \text{Plancherel}, J)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$

$$
\begin{array}{ccccc}
(x_0, x_1, \dots) & \stackrel{s}{\longmapsto} & (x_1, x_2, \dots) & \stackrel{s}{\longmapsto} & \cdots \\
\downarrow{\scriptstyle \text{RSK}} & & \downarrow{\scriptstyle \text{RSK}} & & \\
{\color{red} t_0} & {\color{red} \stackrel{J}{\longmapsto}} & {\color{red} t_1} & {\color{red} \stackrel{J}{\longmapsto}} & {\color{red} \cdots} \\
\downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} & & \\
\theta_0 & & \theta_1 & &
\end{array}
$$

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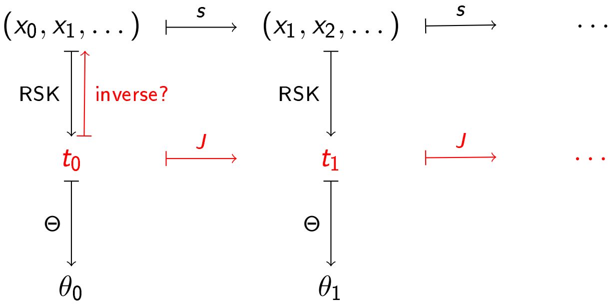

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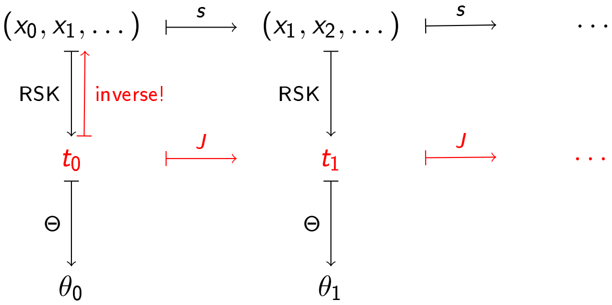

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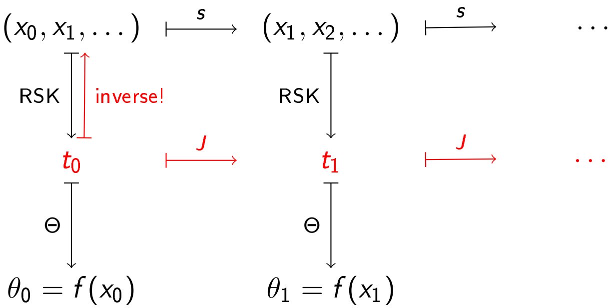
i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$

$$
\begin{array}{ccccc}
(x_0, x_1, \dots) & \xmapsto{s} & (x_1, x_2, \dots) & \xmapsto{s} & \cdots \\
\text{RSK} \downarrow \uparrow \text{inverse!} & & \text{RSK} \downarrow & & \\
t_0 & \xmapsto{J} & t_1 & \xmapsto{J} & \cdots \\
\Theta \downarrow & & \Theta \downarrow & & \\
\theta_0 = f(x_0) & & \theta_1 = f(x_1) & &
\end{array}
$$

jeu de taquin dynamical system $\left(\Omega, \text{Plancherel}, J\right)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$



jeu de taquin dynamical system $\left(\Omega, \text{Plancherel}, J\right)$

the jeu de taquin dynamical system is isomorphic to i.i.d. shift

the inverse map is given by $x_i = f^{-1}(\theta_i)$

some consequences of the isomorphism:

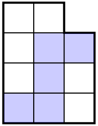
- jdt is a measure-preserving transformation,
- jdt is ergodic,
- slope angles $\theta_0, \theta_1, \ldots$ are independent random variables (put paths $\mathbf{c}(t_0), \mathbf{c}(t_1), \ldots$ are not independent),
- generalizations to other probability measures on Ω / other representations of S_∞,

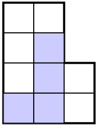
why Θ exists and is a function of x_0?

x_0 is fixed

$x_1, x_2, \ldots$ are random, independent $U(0,1)$

$$\begin{array}{ccc} (x_0, x_1, \ldots) & \overset{s}{\longmapsto} & (x_1, x_2, \ldots) \\ \big\downarrow \text{RSK} & & \big\downarrow \text{RSK} \\ \Lambda^1 \nearrow \Lambda^2 \nearrow \ldots & \overset{J}{\longmapsto} & \overline{\Lambda}^1 \nearrow \overline{\Lambda}^2 \nearrow \ldots \end{array}$$

$\Lambda^{n+1} =$ 

$\overline{\Lambda}^n =$ 

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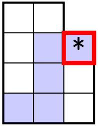
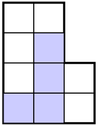
(x_0, x_1, \dots) & \overset{s}{\longmapsto} & (x_1, x_2, \dots) \\
\downarrow \text{RSK} & & \downarrow \text{RSK} \\
\Lambda^1 \nearrow \Lambda^2 \nearrow \dots & \overset{J}{\longmapsto} & \overline{\Lambda}^1 \nearrow \overline{\Lambda}^2 \nearrow \dots
\end{array}$$

$\Lambda^{n+1} =$ 

$\overline{\Lambda}^n =$ 

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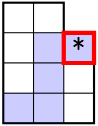
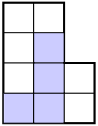
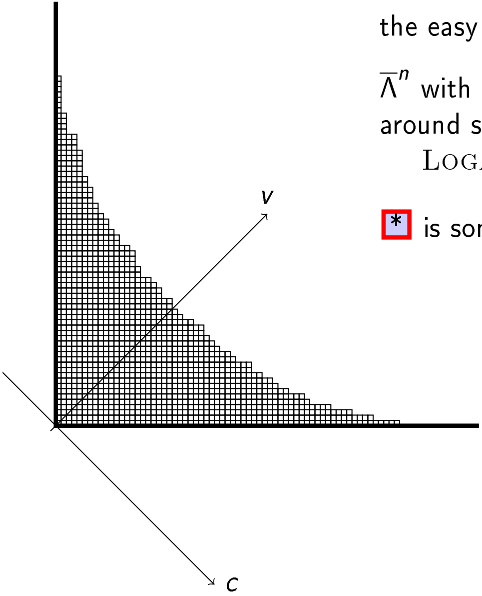
$\Lambda^{n+1} =$ 

$\overline{\Lambda}^n =$ 

$$\boxed{*} = \Lambda^{n+1} \setminus \overline{\Lambda}^n = \text{RSK}(x_0, \dots, x_n) \setminus \text{RSK}(x_1, \dots, x_n)$$

Is it true that asymptotically position of $\boxed{*}$ depends only on x_0?



the easy part:

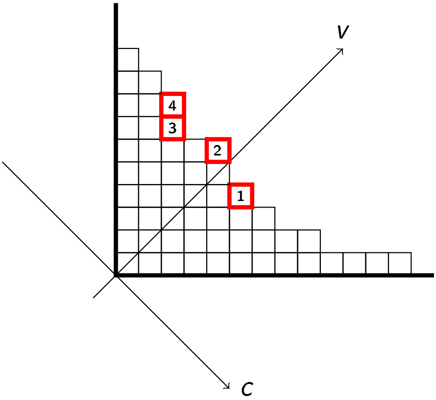
$\overline{\Lambda}^n$ with high probability concentrates around some limit shape

LOGAN, SHEPP, VERSHIK, KEROV

* is somewhere on the boundary of $\overline{\Lambda}^n$

instead of (for deterministic x_0) studying

$$\mathrm{RSK}(x_0, \dots, x_n) \setminus \mathrm{RSK}(x_1, \dots, x_n) = \boxed{*}$$



we study (for random $0 < t_1 < \cdots < t_k < 1$)

$$\mathrm{RSK}(t_1, \dots, t_k, x_1, \dots, x_n) \setminus \mathrm{RSK}(x_1, \dots, x_n) = \{\boxed{1}, \dots, \boxed{k}\}$$

plactic Littlewood-Richarson rule

if $0 \leq x_1, \dots, x_n \leq 1$ is a random sequence, conditioned in such a way that

$$\text{shape of } \mathrm{RSK}(x_1, \dots, x_n) = \lambda;$$

and $0 \leq t_1, \dots, t_k \leq 1$ is a random sequence, conditioned in such a way that

$$\text{shape of } \mathrm{RSK}(t_1, \dots, t_k) = \mu;$$

then the random Young diagram

$$\text{shape of } \mathrm{RSK}(t_1, \dots, t_k, x_1, \dots, x_n)$$

has the same distribution as random irreducible component of

$$V^\lambda \otimes V^\mu \big\uparrow_{S_n \times S_k}^{S_{n+k}}$$

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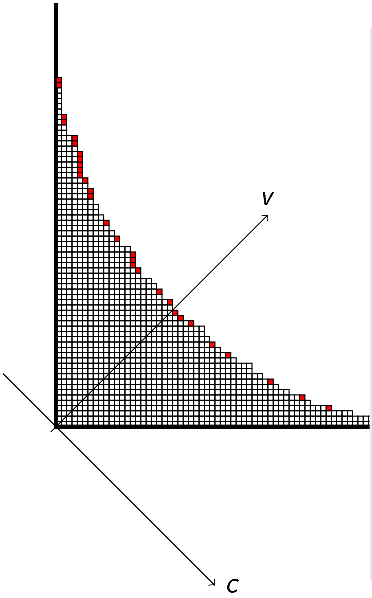
has the same distribution as random irreducible component of

$$V^\lambda \otimes V^{(k)} \uparrow_{S_n \times S_k}^{S_{n+k}}$$

growth of Young diagrams and Jucys-Murphy elements



let $\lambda \vdash n$, $\mu \vdash k$ be fixed Young diagrams

let Γ be a random irreducible component of $V^\lambda \otimes V^\mu \big\uparrow_{S_n \times S_k}^{S_{n+k}}$

let $c_{n+1}, \dots, c_{n+k}$ be the contents of boxes of $\Gamma \setminus \lambda$

then for any symmetric polynomial $P(x_{n+1}, \dots, x_{n+k})$ we have

$$\mathbb{E} P(c_{n+1}, \dots, c_{n+k}) = \left(\chi^\lambda \otimes \chi^\mu \right) \left(P(X_{n+1}, \dots, X_{n+k}) \big\downarrow_{S_n \times S_k}^{S_{n+k}} \right).$$