Dual Bayer-Billera relations and Kazhdan-Lusztig polynomials

FABRIZIO CASELLI
(with Francesco Brenti, U Roma)

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

0-1 sequences

$\{0,1\}^n$ set of binary sequences of length i and

$$\{0,1\}^* \stackrel{\mathrm{def}}{=} \bigcup_{n=0}^{\infty} \{0,1\}^n$$

$\{0,1\}^*$ has a monoid structure given by juxtaposition.

Bayer-Billera relations

$\alpha : \{0,1\}^n \to \mathbb{C}$ satisfies the *Bayer-Billera* (or, *generalized Dehn-Sommerville*) *relations* if

for all $T \in \{0,1\}^n$, $T = P \cdot 0^j \cdot S$, 0^j maximal sequence of consecutive 0's in T,

$$\sum_{i=0}^{j-1} (-1)^i \alpha(P \cdot 0^i 1 0^{j-i-1} \cdot S) = (1 + (-1)^{j-1}) \alpha(T)$$

The vector space of all functions $\alpha : \{0,1\}^n \to \mathbb{C}$ satisfying the Bayer-Billera relations has dimension F_n (where $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).

Eulerian posets

P a finite graded Eulerian poset of rank $n+1$, with $\hat{0}$, $\hat{1}$, and rank function ρ.
For any chain $\mathcal{C} : \hat{0} < x_1 < \cdots < x_k < \hat{1}$ let

$$supp(\mathcal{C}) = d_1 \cdots d_n \in \{0,1\}^n$$

given by $d_i = 1$ if and only if $\rho(x_j) = i$ for some j.
The *flag f-vector* of P is the function $f_P : \{0,1\}^n \to \mathbb{N}$ given by

$$f_P(S) = |\{\mathcal{C} : supp(\mathcal{C}) = S\}| \ \ \forall S \in \{0,1\}^n$$

Well-known that the function f_P satisfies Bayer-Billera relations.
Special cases if poset P is the face lattice of a convex polytope or a Bruhat interval.
Recent generalization by Ehrenborg-Goresky-Readdy for a Whitney stratification of a closed subset of a smooth manifold.

The Bayer-Billera relations also arise in another way in the theory of Coxeter groups.
Recall that a *Coxeter system* is a pair (W, S), where S is a finite set and W is a group defined by:

$$W \stackrel{\text{def}}{=} < S \mid (st)^{m(s,t)} = e, \forall s, t \in S >$$

where, for all $s, t \in S$:

$$m(s,s) = 1 \quad (\Rightarrow s^2 = e \quad \forall s)$$

$$m(s,t) = m(t,s)$$

$$m(s,t) \in \{2, 3, ..., \infty\}, \text{ if } s \neq t$$

Given $u \in W$ one lets

$$\ell(u) \stackrel{\text{def}}{=} min\{r : u = s_1 \cdots s_r \text{ for some } s_1, ..., s_r \in S\}$$

(*length* of u).

Reflections and Bruhat order

We let

$$T \stackrel{\text{def}}{=} \{usu^{-1} : u \in W,\ s \in S\}$$

(set of *reflections* of W).
The *Bruhat graph* of (W, S) is the directed graph $B(W, S)$ with

- W as vertex
- directed edges $x \xrightarrow{t} xt$ where $t \in T$ and $\ell(x) < \ell(xt)$.

Bruhat order on W is the partial order which is the transitive closure of the Bruhat graph.

The symmetric group

S_n is a Coxeter group with respect to the generating set

$$S \stackrel{\text{def}}{=} \{(1,2),(2,3),\ldots,(n-1,n)\}$$

The reflections are

$$T = \{(a,b) : 1 \leq a < b \leq n\}$$

and the length function is

$$\ell(u) = inv(u)$$

for all $u \in S_n$ (*number of inversions* of u).
If $u = u_1 \cdots u_n \in S_n$ and $t = (a,b) \in T$, (a¡b) then

$$u \xrightarrow{t} ut$$

if and only if

$$u_a < u_b$$

(ut is obtained from u by switching u_a and u_b.)

Reflection orderings

A reflection ordering is a total ordering $<_T$ on T satisfying additional properties. In S_n the lex order is a reflection ordering. Given a path $\Delta = (a_0 \xrightarrow{t_1} a_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} a_r)$ in $B(W,S)$ from a_0 to a_r, we define its *length* to be

$$l(\Delta) \stackrel{\mathrm{def}}{=} r,$$

and its *descent string*, with respect to $<_T$, to be

$$D(\Delta) \stackrel{\mathrm{def}}{=} d_1 \cdots d_{r-1} \in \{0,1\}^{r-1}$$

where $d_i = 1$ if and only if $t_i >_T t_{i+1}$.

An example

If $<_T$ is the lex order on S_n and Δ is the path

$$123456 \xrightarrow{(2,4)} 143256 \xrightarrow{(1,2)} 413256 \xrightarrow{(2,5)} 453216$$

$$\xrightarrow{(5,6)} 453261 \xrightarrow{(2,5)} 463251$$

then $l(\Delta) = 5$ and $D(\Delta) = 1001$

Given $u, v \in W$, and $n \in \mathbb{N}$, we denote by $B_n(u, v)$ the set of all directed paths in $B(W, S)$ from u to v of length n, and we let $B(u, v) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} B_n(u, v)$.
For $u, v \in W$, $n \in \mathbb{P}$ and $S \in \{0, 1\}^{n-1}$ we let

$$c_{S,n}(u, v) \stackrel{\text{def}}{=} |\{\Delta \in B_n(u, v) : \; D(\Delta) \leq S\}|.$$

$c_{S,n}(u, v)$ does not depend on $<_T$ and

Theorem (Brenti '98)

Let (W, S) be a Coxeter system, $u, v \in W$, and $n > 0$. The function $S \mapsto c_{S,n}(u, v)$ satisfies the Bayer-Billera relations.

β-functions

It is often the case in combinatorics that whenever a function $\alpha : \{0,1\}^n \to \mathbb{C}$ satisfies the Bayer-Billera relations then the function $\beta : \{0,1\}^n \to \mathbb{C}$ defined by

$$\beta(S) = \sum_{T<S} (-1)^{|S|-|T|} \alpha(T) \tag{1}$$

is also of combinatorial interest. In fact, very often the function β is the one that is *most* interesting.
This was one of the motivations that led us to ask the following natural question.
If a function $\alpha : \{0,1\}^n \to \mathbb{C}$ satisfies the Bayer-Billera relations, then what relations are satisfied by the function $\beta : \{0,1\}^n \to \mathbb{C}$ defined by (1) ?

Dual Bayer-Billera relations

It is well known that the function β satisfies the relations

$$\beta(S) = \beta(\bar{S})$$

for all $S \in \{0,1\}^n$, where $\bar{S}$ is the complement string of S. However, these relations are not enough since they are 2^{n-1}, while at least $2^n - F_n$ are needed.

Theorem (Brenti-C, '13)

Let $\alpha, \beta : \{0,1\}^n \to \mathbb{C}$ be such that (1) holds. Then the following are equivalent:

1. *α satisfies the Bayer-Billera relations;*
2. *$\beta(S) = \beta(\bar{S})$ for all $S \in \{0,1\}^n$ and, for all $i \in [2, n-1]$, $S \in \{0,1\}^{i-1}$ and $T \in \{0,1\}^{n-i}$*

$$\beta(S \cdot 0 \cdot T) + \beta(S \cdot 1 \cdot T) = \beta(S \cdot 0 \cdot \bar{T}) + \beta(S \cdot 1 \cdot \bar{T})$$

Sparse strings

Call the relations (2) in the previous theorem the *dual Bayer-Billera relations*

Call the submonoid of $\{0,1\}^*$ generated by 0 and 01 the monoid of sparse strings.

It is easy to see that there are F_n sparse strings in $\{0,1\}^n$.

If $\beta : \{0,1\}^n \to \mathbb{C}$ satisfies the dual Bayer-Billera relations then it is uniquely determined by its values on the sparse strings, the other values being linear combinations of these.

We look for such a linear expansion explicitly.

A sparse expansion

Let $a_1, \ldots, a_k \in \mathbb{P}$

Define $\mathcal{F}(a_1, \ldots, a_k)$ to be the set of all integer sequences $(i_1, \ldots, i_k)$ such that:

i) $0 \leq i_j \leq a_j$ for all j;

ii) if $i_{j-1} = a_{j-1}$ then $i_j = 0$;

iii) if $i_j = 0$ and $a_j \equiv 1 \pmod 2$ then $i_{j-1} = a_{j-1}$;

for all $j = 1, \ldots, k$.

Example

If $(a_1, a_2, a_3) = (2, 1, 3)$ then

$$\mathcal{F}(2,1,3) = \{(2,0,1), (2,0,2), (2,0,3), (0,1,0), (1,1,0)\}$$

A sparse expansion

We let $E_{0,a} = 0^a$ and

$$E_{i,a} = 0^{i-1}10^{a-i} \ \ \forall i = 1, \ldots, a.$$

Note that the definition of $\mathcal{F}(a_1, \ldots, a_k)$ implies that

$$E_{i_1,a_1} \cdots E_{i_k,a_k}$$

does not have two consecutive 1's.
$E \in \{0,1\}^n$; its *exponent composition* $(a_1, a_2, \ldots)$ is given by

$$E = \begin{cases} 1^{a_1}\,0^{a_2}\,1^{a_3}\cdots, & \text{if } E_1 = 1, \\ 0^{a_1}\,1^{a_2}\,0^{a_3}\cdots, & \text{if } E_1 = 0. \end{cases}$$

For example, the exponent composition of 00110 is $(2, 2, 1)$.

A sparse expansion

Theorem (Brenti-C, '13)

Let $\beta : \{0,1\}^n \to \mathbb{C}$ *satisfy the dual Bayer-Billera relations,* $T \in \{0,1\}^n$ *and* $(a_1, \dots, a_k)$ *be its exponent composition. Then*

$$\beta(T) = \sum_{(i_2,\dots,i_k) \in \mathcal{F}(a_2,\dots,a_k)} (-1)^{\sum_{j : i_j \neq 0} i_j - 1} \beta(0^{a_1} E_{i_2,a_2} \cdots E_{i_k,a_k}),$$

Application: new non-recursive formula for the Kazhdan-Lusztig polynomials of a Coxeter system (W, S) which holds in complete generality.
Recall $S \mapsto c_{S,n}(u, v)$ satisfies the Bayer-Billera relations.
In this case the function given by (1) is

$$b_{S,n}(u, v) = |\{\Delta \in B_n(u, v) : \ D(\Delta) = S\}|,$$

for all $u, v \in W$, $n \in \mathbb{P}$, and $S \subseteq [n-1]$.
Hence, the function $S \mapsto b_{S,n}(u, v)$ satisfies the conclusion of Theorem 3 for all $u, v \in W$ and $n \in \mathbb{P}$.

A *lattice path* on $[n]$ is a function $\Gamma : [n] \to \mathbb{Z}$ such that $\Gamma(0) = 0$ and

$$|\Gamma(i+1) - \Gamma(i)| = 1$$

for all $i \in [n-1]$.
We let

$$N(\Gamma) \stackrel{\text{def}}{=} \eta_1 \cdots \eta_{n-1} \in \{0,1\}^{n-1},$$

where $\eta_i = 1$ if and only if $\Gamma(i) < 0$,
and $d_+(\Gamma) \stackrel{\text{def}}{=}$ number of up-steps of Γ.

Upsilon polynomials

Let $\mathcal{L}(n)$ the set of all the lattice paths Γ on $[0, n]$ such that $\Gamma(n) < 0$.
Given $E \in \{0,1\}^{n-1}$ we define a polynomial $\Upsilon_E(q)$ by

$$\Upsilon_E(q) \stackrel{\text{def}}{=} (-1)^{m_0(E)} \sum_{\{\Gamma \in \mathcal{L}(n) : N(\Gamma) = E\}} (-q)^{d_+(\Gamma)}$$

where $m_0(E) \stackrel{\text{def}}{=} |\{j \in [n-1] : E_j = 0\}|$.

The set $\mathcal{G}(T)$

For $E \in \{0,1\}^{n-1}$ let

$$\partial(E) \stackrel{\text{def}}{=} \{i \in [n-2] : E_i \neq E_{i+1}\}.$$

Let $T \in \{0,1\}^{n-1}$ and $s_1 < \cdots < s_t$ be the positions of the 1's in T, $s_0 \stackrel{\text{def}}{=} 0$, $s_{t+1} \stackrel{\text{def}}{=} n$. We define $\mathcal{G}(T)$ to be the set of all $E \in \{0,1\}^{n-1}$ such that:

i) $|\partial(E) \cap (s_j, s_{j+1})| = 1$ for all $j \in [0, t-1]$;

ii) $|\partial(E) \cap (s_t, s_{t+1})| \leq 1$;

iii) if $\partial(E) \cap (s_t, s_{t+1}) = \{x\}$ then $x \equiv n-1 \pmod 2$.

Note that $\mathcal{G}(T)$ is empty if T is not sparse.

The polynomials Ω_T

Given such an $E \in \mathcal{G}(T)$ we define

$$sgn(E, T) \stackrel{\text{def}}{=} (-1)^{\sum_{i=1}^{t}(s_i - x_i - 1)}$$

where $\{x_i\} \stackrel{\text{def}}{=} \partial(E) \cap (s_{i-1}, s_i)$ for $i \in [t]$, and let

$$\Omega_T(q) \stackrel{\text{def}}{=} \sum_{E \in \mathcal{G}(T)} sgn(E, T)\, \Upsilon_E(q).$$

Recall that $\mathcal{G}(T) = \emptyset$ (and hence $\Omega_T(q) = 0$) if T is not sparse.

The main result

Given a Coxeter system (W, S), $u, v \in W$, $u < v$, let $P_{u,v}(q)$ be the Kazhdan-Lusztig polynomial of u, v.
Then we have

Theorem (Brenti-C, '13)

Let (W, S) be a Coxeter system and $u, v \in W$, $u < v$. Then

$$P_{u,v}(q) = \sum_{T \in \{0,1\}^*} q^{\frac{\ell(u,v)-1-\ell(T)}{2}} b_T(u, v)\Omega_T(q).$$

Other relations?

All combinatorial nonrecursive formulas known for the Kazhdan-Lusztig polynomials which hold in complete generality express each coefficient of the Kazhdan-Lusztig polynomial of u, v as a linear combination of the numbers $b_T(u, v)$.
This was one of the motivations that led us to ask the following natural question
Are there any other linear relations that hold among the numbers $b_S(u, v)$*, besides those implied by Theorem 2 (i.e., by the dual Bayer-Billera relations)?*
Are there any non-trivial linear relations that hold among the numbers

$$\{b_T(u, v) : T \in \{0, 1\}^*, Tsparse\}?$$

No other linear relations

The answer to the last question is “no”.

Theorem

Let $\{a_T\}_{\{T\in\{0,1\}^*, Tsparse\}} \subseteq \mathbb{C}$ *be such that*

$$\sum_{\{T\in\{0,1\}^*, Tsparse\}} a_T \; b_T(u,v) = 0$$

for all Coxeter systems (W,S) *and all* $u, v \in W$*. Then* $a_T = 0$ *for all* $T \in \{0,1\}^*$*,* T *sparse.*