A generalization of Tamari order by means of interval orders

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Based on works with Filippo Disanto, Renzo Pinzani and Simone Rinaldi.

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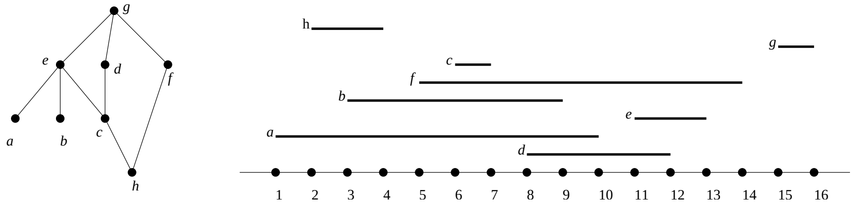
Interval orders

An *interval order* is a partially ordered set $(X, \leq)$ such that there is a mapping φ of the elements of X into subintervals of $\mathbf{R}$ such that, for $x < y$ in X, the right-hand endpoint if $\varphi(x)$ is less than the left-hand endpoint of $\varphi(y)$.



Why interval orders?

- Introduced by Fishburn.
- Useful in several applications, also in non strictly mathematical context, such as:
  - analysis of time spans over which animal species are found;
  - study of occurrences of styles of pottery in archaeological strata;
  - experimental psichology;
  - economic theories;
  - philosophical ontology;
  - computer science.
- Have some very nice purely abstract characterizations:
  - in terms of forbidden subposet: they are precisely $(2+2)$-free posets (Fishburn);
  - in terms of the sets of their principal upsets and principal downsets: they are those posets for which any two distinct upsets are comparable with respect to containment (and the same holds for downsets) (Rabinovich).
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Enumeration of finite interval orders

Generating function for (finite) interval orders:

$$F(t) = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1 - t)^i)$$

(Bousquet-Melou, Claesson, Dukes, Kitaev).

Found using a bijective approach, involving other interesting combinatorial structures, namely bivincular pattern avoiding permutations, regular linearized chord diagrams and ascent sequences.

Several works arose from these results, concerning in particular ascent sequences and the new notion of bivincular pattern in permutations (Aleksandrowicz, Asinowski, Barequet, Harmse, Jelinek, Khamis, Kubitzke, Parviainen, Pergola, Pinzani, Remmel, Steingrimsson,...).

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Our goal

Following Gian-Carlo Rota, "it often happens that a set of objects to be counted possesses a natural ordering".

Even if the problem of enumerating interval orders has already been solved, it could be interesting to investigate order-theoretic features of such a class of objects.

More specifically, is it possible to define some interesting partial order structure on the set of interval orders? Such a structure should be both "natural" and compatible with possible (already existing) orders on specific subsets of relevant interest.

We have been able to define a partial order structure on interval orders of the same size which is:

- easily described in terms of a very natural labelling;
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Series parallel orders

A *series parallel order* is a poset built up from single-element partial orders using two operations, called series composition and parallel composition (or sometime also ordinal sum and direct sum).

Series parallel orders also have a nice forbidden subposet characterization: they correspond to N-free posets, where N denote the *fence of order 4*.

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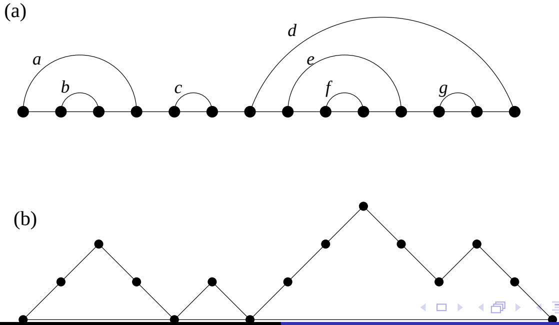
How many series parallel interval orders?

Theorem

Series parallel interval orders are counted by Catalan numbers.

They are in bijection with *perfect noncrossing matchings* or, equivalently, with Dyck paths.



Perfect noncrossing matchings

In a perfect noncrossing matching, there are two natural relations defined on the set of its arcs:

- the arc x lies below the arc y (xSy);
- the arc x lies on the left of the arc y (xRy)

Some properties of S and R:

1. S and R are (strict) partial orders;
2. given any two arcs, they belong precisely to one of the two relations (either in one direction or in the opposite one).
3. If x lies below y, and in turn y is on the left of z, then necessarily x is on the left of z.

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Catalan pairs

A pair of relations (S, R) on a set X satisfying the three above properties (suitably formalized) is called a *Catalan pair*.

This name is justified by the following results.

Theorem

The set $C(n)$ of nonisomorphic Catalan pairs on a set having n elements has cardinality C_n, the n-th Catalan number.

Proposition

There is a bijection between $C(n)$ and the set of perfect noncrossing matchings of a set having $2n$ elements.

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Catalan pairs and Catalan structures

Apart from being one further combinatorial interpretation of Catalan numbers, Catalan pairs were introduced in order to simplify the solution of the famous exercise 6.19 in the second "Enumerative Combinatorics" book by Stanley...

Exercise. Pick your favorite Catalan structure and determine "its" Catalan pair.

Examples include noncrossing partitions, 312-avoiding permutations, 321-avoiding permutations, restricted sequences of integers, tilings of a staircase shape, parallelogram polyominoes.

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The first component of a Catalan pair

$$\mathbf{S}(n) = \{(X, S) \mid (\exists R)(S, R) \in \mathcal{C}(n)\}.$$

Proposition

If $(X, S) \in \mathbf{S}(n)$, then the Hasse diagram of (X, S) is a forest of rooted trees, where the roots of the trees are the maximal elements of S.

Corollary

There is a bijection between $\mathbf{S}(n)$ and the set of rooted trees with $n+1$ nodes.

Thus, for obvious cardinality reasons, the first component of a Catalan pair does not uniquely determine the pair.

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The second component of a Catalan pair

$$\mathbf{R}(n) = \{(X, R) \mid (\exists S)(S, R) \in \mathcal{C}(n)\}.$$

Theorem

If $(S_1, R), (S_2, R)$ are two Catalan pairs on X, then they are isomorphic.

Theorem

$(X, R) \in \mathbf{R}(n)$ if and only if it is a series parallel interval order, i.e. it does not contain any subposet isomorphic to $\mathbf{2} + \mathbf{2}$ or Z_4.

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The admissible labelling of an interval order

- $(P, \leq)$ interval order;
- $D(x) = \{y \in P \mid y < x\}$; $U(x) = \{y \in P \mid y > x\}$.
- $x \sim y$ when $D(x) = D(y)$ and $U(x) = U(y)$.

A linear extension λ of P is called an *admissible labelling* of P whenever, for all $x, y \in P$, if $\lambda(x) < \lambda(y)$ then either:

- $D(x) \subset D(y)$, or
- $D(x) = D(y)$ and $U(x) \subset U(y)$, or
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Let λ_1, λ_2 be two admissible labellings of P. Then there exists an automorphism f of P such that, for all $x \in P$, $\lambda_1(x) = \lambda_2(f(x))$.

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In the sequel we will consider *interval orders endowed with their admissible labelling*, and we will identify an element x with its label $\lambda(x)$.

The admissible labelling of an interval order

- $(P, \leq)$ interval order;
- $D(x) = \{y \in P \mid y < x\}$; $U(x) = \{y \in P \mid y > x\}$.
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A linear extension λ of P is called an *admissible labelling* of P whenever, for all $x, y \in P$, if $\lambda(x) < \lambda(y)$ then either:

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A partial order on interval orders

$AV_n(\mathbf{2}+\mathbf{2})$: set of interval orders having n elements.

Notation for $P \in AV_n(\mathbf{2}+\mathbf{2})$:

- $x \leq y$: x is less than or equal to y with respect to the partial order of P;
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A characterization of $\leq_T$

There is an easy characterization of $\leq_T$ which immediately follows from notations and previous results.

Proposition

Let P_1, P_2 be two interval orders on $X = \{1, 2, \dots, n\}$. The following are equivalent:

- i) *for each $x \in X$, $Up_{P_1}(x) \supseteq Up_{P_2}(x)$;*
- ii) *for each $x \in X$, $D_{P_1}(x) \supseteq D_{P_2}(x)$;*
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Dual closure systems and complete lattices

A *dual closure system* on a set Γ is a family of subsets $\mathcal{L}$ of Γ which is closed under arbitrary unions and contains the empty set.

A classical result connecting dual closure system and complete lattices is the following.

Theorem

Let $\mathcal{L}$ be a family of subsets of a set Γ. Suppose that there exists $A \in \mathcal{L}$ such that $A \subseteq B$ for all $B \in \mathcal{L}$ and $\mathcal{L}$ is closed under arbitrary nonempty unions. Then $\mathcal{L}$ is a complete lattice, in which

$$\begin{aligned} \bigwedge_{i \in I} A_i &= \bigcup_{i \in I} A_i, \\ \bigvee_{i \in I} A_i &= \bigcup \{B \in \mathcal{L} \mid B \subseteq \bigcap_{i \in I} A_i\}, \end{aligned} \tag{1}$$

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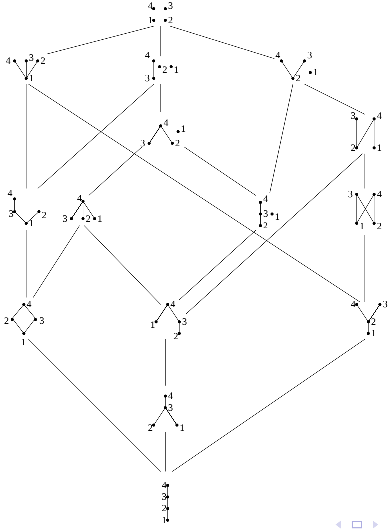
Theorem

$(AV_n(\mathbf{2}+\mathbf{2}), \leq_T)$ is a (complete) lattice, in which the meet and join operations are expressed as follows:

$$\begin{aligned} P_1 \wedge P_2 &= P_1 \cup P_2, \\ P_1 \vee P_2 &= \bigcup \{P \in AV_n(\mathbf{2}+\mathbf{2}) \mid P \subseteq P_1 \cap P_2\}. \end{aligned}$$

Proof. Let $\Gamma = \{(i,j) \in [n] \times [n] \mid i \leq j\}$. Then the set of interval orders on $[n]$ is clearly a family of subsets of Γ. So it will be enough to show that $\mathcal{L}$ satisfies the hypotheses of the previous theorem. ∎

$(AV_4(2+2), \leq_T)$



A map from trees to posets

Let T be a planar tree having $n+1$ nodes endowed with its preorder labelling. We will identify each node of T with its label.

Notations:

- $u_T(x)$: set of *descendants* of a node x of T,
- $x \prec y$ means that the label of the node x is less than the label of the node y.

Define a partial order relation R on $[n]$ by setting xRy whenever:

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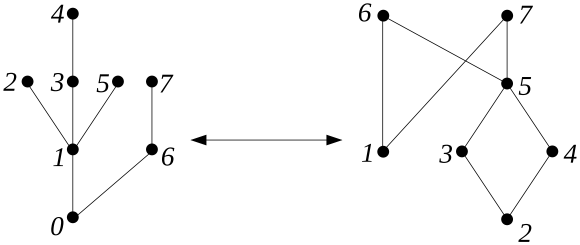
Example



Some properties

The map $T \longmapsto P_T$ from planar trees with their preoder labelling to posets has some interesting properties.

Proposition

1. *The structure $P_T = ([n], R)$ is a series parallel interval order, and the labelling of its elements (inherited from the labelling of T) is a linear extension of R.*
2. *Every series parallel interval order of size n is isomorphic to P_T, for some planar tree T.*
3. *Set $P_1 = P_{T_1}$ and $P_2 = P_{T_2}$, if we define $P_1 \leq_t P_2$ when, $\forall x \in [n]$, $D_{P_1}(x) \supseteq D_{P_2}(x)$, then $(AV_n(\mathbf{2}+\mathbf{2}, N), \leq_t)$ is the Tamari lattice of order n.*

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Preorder and admissible labellings

The labelling of P_T induced by the above map is a very special one.

Proposition

The labelling of the poset P_T determined by the preorder visit on the associated tree T coincides with the admissible labelling of R.

Combining the last results gives our final theorem.

Theorem

The Tamari lattice of order n is the restriction of the lattice $(AV_n(\mathbf{2}+\mathbf{2}), \leq_T)$ to the set of series parallel interval orders $AV_n(\mathbf{2}+\mathbf{2}, N)$.

We have thus obtained a very natural extension of the Tamari lattice. There are several others in the literature (Reading, Simion, Thomas), however the present one does not match any of the preceding ones.

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