Combinatorial Dyson-Schwinger equations and systems I

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In QFT, one studies the behaviour of particles in a quantum fields.

- Several types of particles: electrons, photons, bosons, etc.
- Several types of interactions: an electron can capture/eject a photon, etc.

One wants to predict certain physical constants: mass or charge of the electron,etc.

- Develop the constant in a formal series, indexed by certain combinatorial objects: the Feynman graphs.
- Attach to any Feynman graph a real/complex number: Feynman rules and Renormalization.

- The expansion as a formal series gives formal sums of Feynman graphs: the propagators (vertex functions, two-points functions).
- These formal sums are characterized by a set of equations: the Dyson-Schwinger equations.
- In order to be "physically meaningful", these functions should be compatible with the extraction/contraction Hopf algebra structure on Feynman graphs. This imposes strong constraints on the Dyson-Schwinger equations.
- Because of a 1-cocycle property, everything can be lifted and studied to the level of decorated rooted trees.

To a given QFT is attached a family of graphs.

Feynman graphs

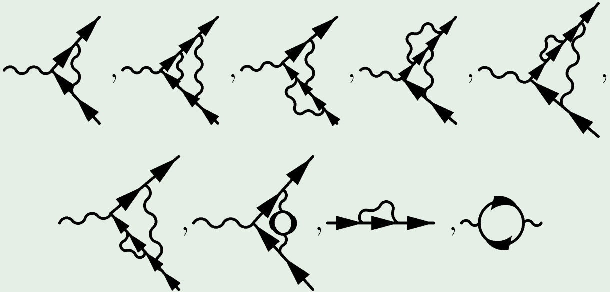
1. A finite number of possible half-edges.
2. A finite number of possible vertices.
3. A finite number of possible external half-edges (external structure).
4. The graph is connected and 1-PI.

To each external structure is associated a formal series in the Feynman graphs.

In QED

1. Half-edges:  (electron),  (photon).
2. Vertices:  .
3. External structures:  ,  ,  .

Examples in QED



Other examples

- Φ^3.
- Quantum Chromodynamics.

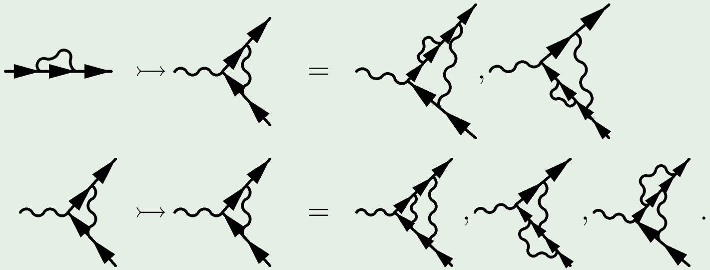
Subgraphs and contraction

1. A subgraph of a Feynman graph Γ is a subset γ of the set of half-edges Γ such that γ and the vertices of Γ with all half edges in γ is itself a Feynman graph.
2. If Γ is a Feynman graph and $\gamma_1, \ldots, \gamma_k$ are disjoint subgraphs of Γ, $\Gamma/\gamma_1 \ldots \gamma_k$ is the Feynman graph obtained by replacing $\gamma_1, \ldots, \gamma_k$ by vertices in Γ.

Insertion

Let Γ_1 and Γ_2 be two Feynman graphs. According to the external structure of Γ_1, you can replace a vertex or an edge of Γ_2 by Γ_1 in order to obtain a new Feynman graph.

Examples in QED



Let A and B be two vector spaces.

- The tensor product of A and B is a space $A \otimes B$ with a bilinear product $\otimes : A \times B \longrightarrow A \otimes B$ satisfying a universal property: if $f : A \times B \longrightarrow C$ is bilinear, there exists a unique linear map $F : A \otimes B \longrightarrow C$ such that $F(a \otimes b) = f(a, b)$ for all $(a, b) \in A \times B$.
- If $(e_i)_{i \in I}$ is a basis of A and $(f_j)_{j \in J}$ is a basis of B, then $(e_i \otimes f_j)_{i \in I, j \in J}$ is a basis $A \otimes B$.

- The tensor product of vector spaces is associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
- We shall identify $K \otimes A$, $A \otimes K$ and A via the identification of $1 \otimes a$, $a \otimes 1$ and a.

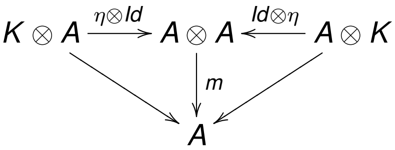
If A is an associative algebra, its (bilinear) product becomes a linear map $m : A \otimes A \longrightarrow A$, sending $a \otimes b$ on ab. The associativity is given by the following commuting square:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes Id} & A \otimes A \\ {\scriptstyle Id \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

If A is unitary, its unit 1_A induces a linear map

$$\eta : \left\{ \begin{array}{rcl} K & \longrightarrow & A \\ \lambda & \longrightarrow & \lambda 1_A. \end{array} \right.$$

The unit axiom becomes:



Dualizing these diagrams, we obtain the coalgebra axioms

Coalgebra

A coalgebra is a vector space C with a map $\Delta : C \longrightarrow C \otimes C$ such that:

- $$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle Id \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes Id]{} & C \otimes C \otimes C \end{array}$$

Coalgebra

- There exists a map $\varepsilon : C \longrightarrow K$, called the counit, such that:

$$\begin{array}{ccccc} K\otimes C & \xleftarrow{\varepsilon\otimes Id} & C\otimes C & \xrightarrow{Id\otimes\varepsilon} & C\otimes K \\ & \searrow & \uparrow{\scriptstyle\Delta} & \swarrow & \\ & & C & & \end{array}$$

If A is an algebra, then $A \otimes A$ is an algebra, with:

$$(a_1 \otimes b_1).(a_2 \otimes b_2) = (a_1.a_2) \otimes (b_1.b_2).$$

Bialgebra and Hopf algebra

- A bialgebra is both an algebra and a coalgebra, such that the coproduct and the counit are algebra morphisms.
- A Hopf algebra is a bialgebra with a technical condition of existence of an antipode.

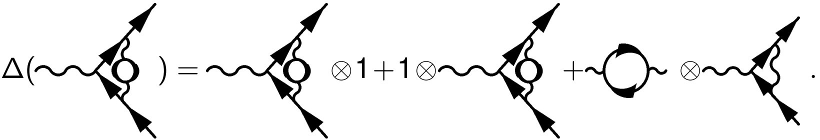
Examples

- If G is a group, KG is a Hopf algebra, with $\Delta(x) = x \otimes x$ for all $x \in G$.
- If $\mathfrak{g}$ is a Lie algebra, its enveloping algebra is a Hopf algebra, with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$.
- If H is a finite-dimensional Hopf algebra, then its dual is also a Hopf algebra.
- If H is a graded Hopf algebra, then its graded dual is also a Hopf algebra.

Construction

Let H_{FG} be a free commutative algebra generated by the set of Feynman graphs. It is given a coproduct: for all Feynman graph Γ,

$$\Delta(\Gamma) = \sum_{\gamma_1 \ldots \gamma_k \subseteq \Gamma} \gamma_1 \ldots \gamma_k \otimes \Gamma/\gamma_1 \ldots \gamma_k.$$



The Hopf algebra H_{FG} is graded by the number of loops:

$$|\Gamma| = \sharp E(\Gamma) - \sharp V(\Gamma) + 1.$$

Because of the 1-PI condition, it is connected, that is to say $(H_{FG})_0 = K 1_{H_{FG}}$. What is its dual?

Cartier-Quillen-Milnor-Moore theorem

Let H be a cocommutative, graded, connected Hopf algebra over a field of characteristic zero. Then it is the enveloping algebra of its primitive elements.

This theorem can be applied to the graded dual of H_{FG}.

Primitive elements of H_{FG}^*

- Basis of primitive elements: for any Feynman graph Γ,

$$f_\Gamma(\gamma_1 \ldots \gamma_k) = \sharp Aut(\Gamma)\delta_{\gamma_1 \ldots \gamma_k, \Gamma}.$$

- The Lie bracket is given by:

$$[f_{\Gamma_1}, f_{\Gamma_2}] = \sum_{\Gamma = \Gamma_1 \rightarrowtail \Gamma_2} f_\Gamma - \sum_{\Gamma = \Gamma_2 \rightarrowtail \Gamma_1} f_\Gamma.$$

We define:

$$f_{\Gamma_1} \circ f_{\Gamma_2} = \sum_{\Gamma = \Gamma_1 \rightarrowtail \Gamma_2} f_\Gamma.$$

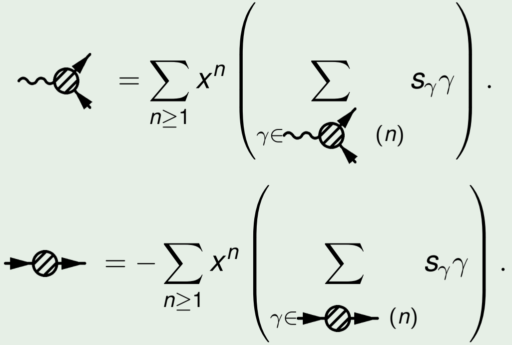
The product $\circ$ is not associative, but satisfies:

$$f_1 \circ (f_2 \circ f_3) - (f_1 \circ f_2) \circ f_3 = f_2 \circ (f_1 \circ f_3) - (f_2 \circ f_1) \circ f_3.$$

It is (left) prelie.

In the context of QFT, we shall consider some special infinite sums of Feynman graphs:

Propagators in QED



Propagators in QED

 $= -\sum_{n\geq 1} x^n \left(\sum_{\gamma \in} \right.$  $\left. (n) \quad s_\gamma \gamma \right).$

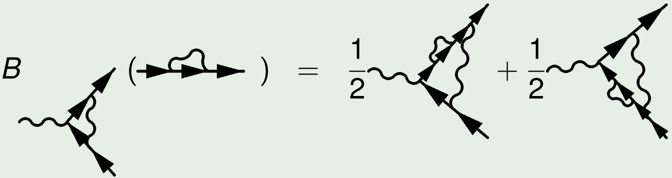
They live in the completion of H_{FG}.

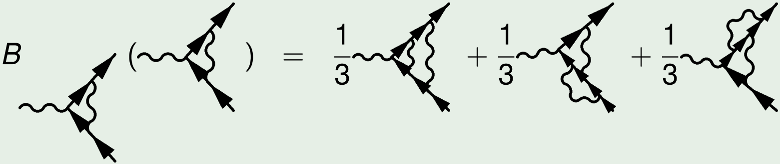
How to describe the propagators?

- For any primitive Feynman graph γ, one defines the insertion operator B_γ over H_{FG}. This operator associates to a graph G the sum (with symmetry coefficients) of the insertions of G into γ.
- The propagators then satisfy a system of equations involving the insertion operators, called systems of Dyson-Schwinger equations.

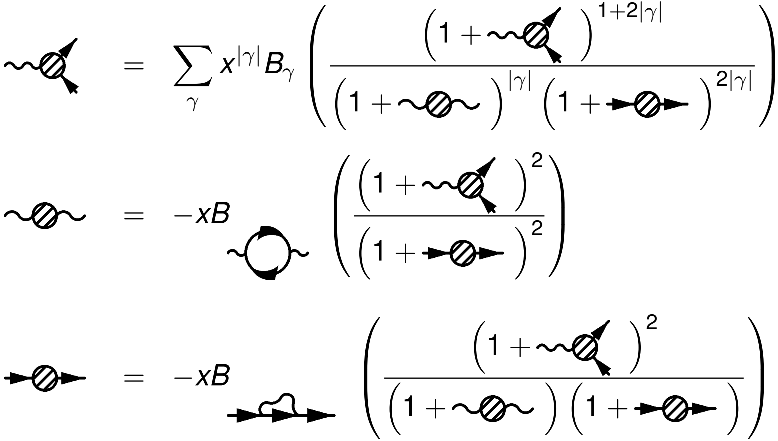
Example

In QED :





In QED:



Other example (Bergbauer, Kreimer)

$$X = \sum_{\gamma \text{ primitive}} B_\gamma \left((1 + X)^{|\gamma|+1} \right).$$

Question

For a given system of Dyson-Schwinger equations (S), is the subalgebra generated by the homogeneous components of (S) a Hopf subalgebra?

Proposition

The operators B_γ satisfy: for all $x \in H_{FG}$,

$$\Delta \circ B_\gamma(x) = B_\gamma(x) \otimes 1 + (Id \otimes B_\gamma) \circ \Delta(x).$$

This relation allows to lift any system of Dyson-Schwinger equation to the Hopf algebra of decorated rooted trees.

Cartier-Quillen cohomology

let C be a coalgebra and let (B, δ_G, δ_D) be a C-bicomodule.

- $D_n = \mathcal{L}(B, C^{\otimes n})$.
- For all $l \in D_n$:

$$b_n(L) = \sum_{i=1}^{n} (-1)^i (Id^{\otimes(i-1)} \otimes \Delta \otimes Id^{\otimes(n-i)}) \circ L + (Id \otimes L) \circ \delta_G + (-1)^{n+1} (L \otimes Id) \circ \delta_D.$$

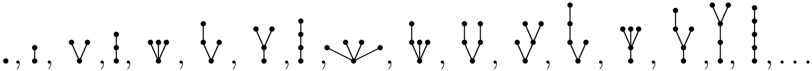
A particular case

We take $B = C$, $\delta_G(b) = \Delta(b)$ and $\delta_D(b) = b \otimes 1$. A 1-cocycle of C is a linear map $L : C \longrightarrow C$, such that for all $b \in C$:

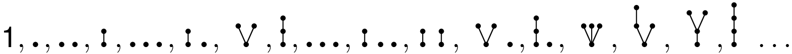
$$(Id \otimes L) \circ \Delta(b) - \Delta \circ L(b) + b \otimes 1 = 0.$$

So B_γ is a 1-cocycle of H_{FG} for all primitive Feynman graph.

The Hopf algebra of rooted trees H_R (or Connes-Kreimer Hopf algebra) is the free commutative algebra generated by the set of rooted trees.

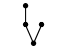
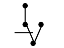
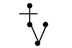
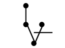
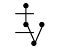
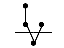
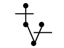
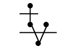


The set of rooted forests is a linear basis of H_R:



The coproduct is given by admissible cuts:

$$\Delta(t) = \sum_{c \text{ admissible cut}} P^c(t) \otimes R^c(t).$$

cut c									total
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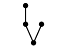
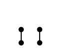
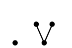
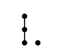
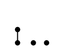
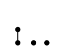
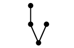
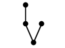
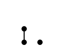
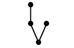
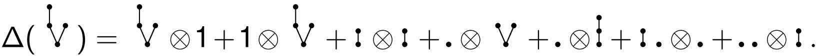
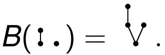
$W^c(t)$									
$R^c(t)$					$\times$			$\times$	1
$P^c(t)$	1				$\times$			$\times$	



The grafting operator of H_R is the map $B : H_R \longrightarrow H_R$, associating to a forest $t_1 \dots t_n$ the tree obtained by grafting $t_1, \dots, t_n$ on a common root. For example:



Proposition

For all $x \in H_R$:

$$\Delta \circ B(x) = B(x) \otimes 1 + (Id \otimes B) \circ \Delta(x).$$

So B is a 1-cocycle of H_R.

Universal property

Let A be a commutative Hopf algebra and let $L : A \longrightarrow A$ be a 1-cocycle of A. Then there exists a unique Hopf algebra morphism $\phi : H_R \longrightarrow A$ with $\phi \circ B = L \circ \phi$.

This will be generalized to the case of several 1-cocycles with the help of decorated rooted trees.

- H_R is graded by the number of vertices and B is homogeneous of degree 1.
- Let $Y = B_\gamma(f(Y))$ be a Dyson-Schwinger equation in a suitable Hopf algebra of Feynman graphs H_{FG}, such that $|\gamma| = 1$.
- There exists a Hopf algebra morphism $\phi : H_R \longrightarrow H_{FG}$, such that $\phi \circ B = B_\gamma \circ \phi$. This morphism is homogeneous of degree 0.
- Let X be the solution of $X = B(f(X))$. Then $\phi(X) = Y$ and for all $n \geq 1$, $\phi(X(n)) = Y(n)$.
- Consequently, if the subalgebra generated by the $X(n)$'s is Hopf, so is the subalgebra generated by the $Y(n)$'s.

Definition

Let $f(h) \in K[[h]]$.

- The combinatorial Dyson-Schwinger equations associated to $f(h)$ is:
$$X = B(f(X)),$$
where X lives in the completion of H_R.
- This equation has a unique solution $X = \sum X(n)$, with:
$$\left\{ \begin{array}{rcl} X(1) & = & p_0 \bullet, \\ X(n+1) & = & \displaystyle\sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} p_k B(X(a_1)\ldots X(a_k)), \end{array} \right.$$
where $f(h) = p_0 + p_1 h + p_2 h^2 + \ldots$

$X(1) = p_0 \bullet,$

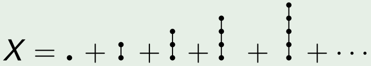
$X(2) = p_0 p_1$,

$X(3) = p_0 p_1^2$ $+ p_0^2 p_2$  ,

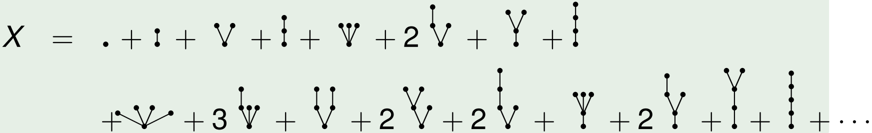
$X(4) = p_0 p_1^3$ $+ p_0^2 p_1 p_2$  $+ 2 p_0^2 p_1 p_2$  $+ p_0^3 p_3$  .

Examples

- If $f(h) = 1 + h$:



- If $f(h) = (1 - h)^{-1}$:



Let $f(h) \in K[[h]]$. The homogeneous components of the unique solution of the combinatorial Dyson-Schwinger equation associated to $f(h)$ generate a subalgebra of H_R denoted by H_f.

H_f is not always a Hopf subalgebra

For example, for $f(h) = 1 + h + h^2 + 2h^3 + \cdots$, then:

So:

$$\Delta(X(4)) = X(4) \otimes 1 + 1 \otimes X(4) + (10X(1)^2 + 3X(2)) \otimes X(2)$$
$$+(X(1)^3 + 2X(1)X(2) + X(3)) \otimes X(1)$$

$+X(1) \otimes (8$  $+ 5$ $)$.

If $f(0) = 0$, the unique solution of $X = B(f(X))$ is 0. From now, up to a normalization we shall assume that $f(0) = 1$.

Theorem

Let $f(h) \in K[[h]]$, with $f(0) = 1$. The following assertions are equivalent:

1. H_f is a Hopf subalgebra of H_R.
2. There exists $(\alpha, \beta) \in K^2$ such that $(1 - \alpha\beta h)f'(h) = \alpha f(h)$.
3. There exists $(\alpha, \beta) \in K^2$ such that $f(h) = 1$ if $\alpha = 0$ or $f(h) = e^{\alpha h}$ if $\beta = 0$ or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.

$1 \Longrightarrow 2$. We put $f(h) = 1 + p_1 h + p_2 h^2 + \cdots$. Then $X(1) = \bullet$. Let us write:

$$\Delta(X(n+1)) = X(n+1) \otimes 1 + 1 \otimes X(n+1) + X(1) \otimes Y(n) + \ldots.$$

1. By definition of the coproduct, $Y(n)$ is obtained by cutting a leaf in all possible ways in $X(n+1)$. So it is a linear span of trees of degree n.
2. As H_f is a Hopf subalgebra, $Y(n)$ belongs to H_f.

Hence, there exists a scalar λ_n such that $Y(n) = \lambda_n X_n$.

lemma

Let us write:

$$X = \sum_t a_t t.$$

For any rooted tree t:

$$\lambda_{|t|} a_t = \sum_{t'} n(t, t') a_{t'},$$

where $n(t, t')$ is the number of leaves of t' such that the cut of this leaf gives t.

We here assume that f is not constant. We can prove that $p_1 \neq 0$.

For t the ladder $(B)^n(1)$, we obtain:

$$p_1^{n-1}\lambda_n = 2(n-1)p_1^{n-2}p_2 + p_1^n.$$

Hence:

$$\lambda_n = 2\frac{p_2}{p_1}(n-1) + p_1.$$

We put $\alpha = p_1$ and $\beta = 2\frac{p_2}{p_1^2} - 1$, then:

$$\lambda_n = \alpha(1 + (n-1)(1+\beta)).$$

For t the corolla $B(\bullet^{n-1})$, we obtain:

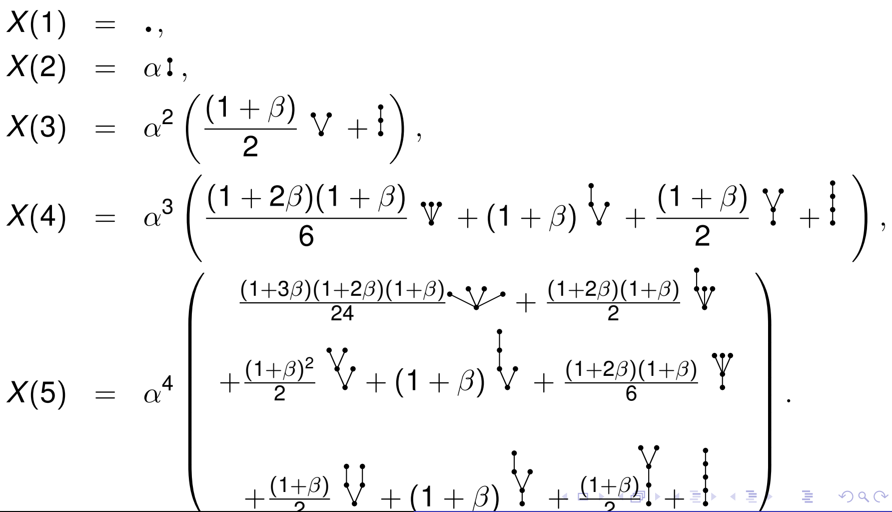
$$\lambda_n p_{n-1} = np_n + (n-1)p_{n-1}p_1.$$

Hence:

$$\alpha(1+(n-1)\beta)p_{n-1} = np_n.$$

Summing:

$$(1-\alpha\beta h)f'(h) = \alpha f(h).$$



Particular cases

- If $(\alpha, \beta) = (1, -1)$, $f = 1 + h$ and $X(n) = (B)^n(1)$ for all n.
- If $(\alpha, \beta) = (1, 1)$, $f = (1 - h)^{-1}$ and:

$$X(n) = \sum_{|t|=n} \sharp\{\text{embeddings of } t \text{ in the plane}\} t.$$

- Si $(\alpha, \beta) = (1, 0)$, $f = e^h$ and:

$$X(n) = \sum_{|t|=n} \frac{1}{\sharp\{\text{symmetries of } t\}} t.$$