# Dyson-Schwinger systems on rooted trees

# Loïc Foissy

# Bertinoro September 2013

Loïc Foissy Dyson-Schwinger systems on rooted trees

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Hopf algebra of decorated rooted trees  $H_R^l$  Dyson-Schwinger systems

### Let I be a set. Rooted trees decorated by I:

Coproduct:

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Hopf algebra of decorated rooted trees  $H_R^I$  Dyson-Schwinger systems

### Dyson-Schwinger system from QED:



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Hopf algebra of decorated rooted trees  $H_R^I$  Dyson-Schwinger systems

### Dyson-Schwinger system from QED:

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Hopf algebra of decorated rooted trees  $H_R^l$  Dyson-Schwinger systems

Dyson-Schwinger system from QED truncated at order 1:



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#### Recalls

Combinatorial Dyson-Schwinger systems Study of Dyson-Schwinger systems More realistic Dyson-Schwinger systems Hopf algebra of decorated rooted trees  $H_R^l$  Dyson-Schwinger systems

Lifting to decorated trees:

$$\begin{array}{rcl} X_1 & = & B_1 \left( \frac{(1+X_1)^3}{(1-X_3)^2(1-X_2)} \right), \\ X_2 & = & B_2 \left( \frac{(1+X_1)^2}{(1-X_3)^2} \right), \\ X_3 & = & B_3 \left( \frac{(1+X_1)^2}{(1-X_2)(1-X_3)} \right). \end{array}$$

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#### Recalls

Combinatorial Dyson-Schwinger systems Study of Dyson-Schwinger systems More realistic Dyson-Schwinger systems Hopf algebra of decorated rooted trees  $H'_R$ Dyson-Schwinger systems

$$\begin{aligned} X_1 &= \cdot_1 + 3\mathbf{1}_1^1 + \mathbf{1}_1^2 + 2\mathbf{1}_1^3 \\ &+ 9\mathbf{1}_1^1 + 3\mathbf{1}_1^2 + 6\mathbf{1}_1^3 + 2\mathbf{1}_1^1 + 2\mathbf{1}_1^3 + 4\mathbf{1}_1^1 + 2\mathbf{1}_1^2 + 2\mathbf{1}_1^3 \\ &+ 3^1V_1^1 + 3^1V_1^2 + 6^1V_1^2 + {}^2V_1^2 + 2^2V_1^3 + 3^3V_1^3 + \dots \end{aligned}$$

$$\begin{array}{rcl} X_2 & = & \cdot_2 + 2\mathfrak{l}_2^1 + \mathfrak{l}_2^3 \\ & & + 6\mathfrak{l}_2^1 + 2\mathfrak{l}_2^2 + 4\mathfrak{l}_2^3 + 4\mathfrak{l}_2^3 + 2\mathfrak{l}_2^3 + 2\mathfrak{l}_2^3 \\ & & + ^1 V_2^1 + 4^1 V_2^3 + 3^3 V_2^3 + \dots \end{array}$$

 $X_{3} = \cdot_{3} + 2\mathfrak{i}_{3}^{1} + \mathfrak{i}_{3}^{2} + \mathfrak{i}_{3}^{3}$  $+ 6\mathfrak{i}_{3}^{1} + 2\mathfrak{i}_{3}^{2} + 4\mathfrak{i}_{3}^{3} + 2\mathfrak{i}_{3}^{1} + 2\mathfrak{i}_{3}^{2} + 2\mathfrak{i}_{3}^{3} + 2\mathfrak{i}_{3}^{1} + \mathfrak{i}_{3}^{2} + \mathfrak{i}_{3}^{3}$  $+ {}^{1}\mathsf{V}_{3}^{1} + 2{}^{1}\mathsf{V}_{3}^{2} + 2{}^{1}\mathsf{V}_{3}^{3} + {}^{2}\mathsf{V}_{3}^{2} + {}^{2}\mathsf{V}_{3}^{3} + {}^{3}\mathsf{V}_{3}^{3} + \dots$ 

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Graph associated to a system Operations on Dyson-Schwinger systems Examples

# Definition

• Let  $f_1, \ldots, f_n \in K[[h_1, \ldots, h_n]] - K$ . The combinatorial Dyson-Schwinger systems attached to  $f = (f_1, \ldots, f_n)$  is:

$$(S): \begin{cases} X_1 = B_1^+(f_1(X_1, \dots, X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1, \dots, X_n)), \end{cases}$$

• Such a system has a unique solution

$$(X_1,\ldots,X_n)\in H_R^{\{1,\ldots,n\}}$$

- The subalgebra generated by the homogeneous components of the X(i)'s is denoted by  $H_{(S)}$ .
- If this subalgebra is Hopf, we shall say that the system is Hopf.

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### Graph associated to (S)

Let (*S*) be associated to  $(f_1, \ldots, f_n)$ . The oriented graph associated to (*S*) is defined by:

• The vertices are  $1, \ldots, n$ .

2 There is an edge from *i* to *j* if, and only if,  $\frac{\partial f_i}{\partial h_i} \neq 0$ .

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Graph associated to a system Operations on Dyson-Schwinger systems Examples

### Example coming from QED

$$\begin{array}{rcl} X_1 & = & B_1 \left( \frac{(1+X_1)^3}{(1-X_3)^2(1-X_2)} \right), \\ X_2 & = & B_2 \left( \frac{(1+X_1)^2}{(1-X_3)^2} \right), \\ X_3 & = & B_3 \left( \frac{(1+X_1)^2}{(1-X_2)(1-X_3)} \right). \end{array}$$

Graph:



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### Change of variables

Let (S) be the following system:

$$(S): \begin{cases} X_1 = B_1^+(f_1(X_1,...,X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1,...,X_n)). \end{cases}$$

If (*S*) is Hopf, then for all family  $(\lambda_1, \ldots, \lambda_n)$  of non-zero scalars, this system is Hopf:

$$(S): \begin{cases} X_1 = B_1^+(f_1(\lambda_1 X_1, \dots, \lambda_n X_n)) \\ \vdots \\ X_n = B_n^+(f_n(\lambda_1 X_1, \dots, \lambda_n X_n)). \end{cases}$$

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Graph associated to a system Operations on Dyson-Schwinger systems Examples

### Concatenation

Let (S) and (S') be the following systems:

$$(S): \begin{cases} X_1 = B_1^+(f_1(X_1, \dots, X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1, \dots, X_n)). \end{cases}$$
$$(S'): \begin{cases} X_1 = B_1^+(g_1(X_1, \dots, X_m)) \\ \vdots \\ X_m = B_m^+(g_m(X_1, \dots, X_m)). \end{cases}$$

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Graph associated to a system Operations on Dyson-Schwinger systems Examples

### Concatenation

The following system is Hopf if, and only if, the (S) and (S') are Hopf:

$$X_{1} = B_{1}^{+}(f_{1}(X_{1},...,X_{n}))$$

$$\vdots$$

$$X_{n} = B_{n}^{+}(f_{n}(X_{1},...,X_{n}))$$

$$X_{n+1} = B_{n+1}^{+}(g_{1}(X_{n+1},...,X_{n+m}))$$

$$\vdots$$

$$X_{n+m} = B_{n+m}^{+}(g_{m}(X_{n+1},...,X_{n+m}))$$

This property leads to the notion of connected (or indecomposable) system.

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### Extension

Let (S) be the following system:

$$(S): \begin{cases} X_1 = B_1^+(f_1(X_1,...,X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1,...,X_n)). \end{cases}$$

Then (S') is an extension of (S):

$$(S'): \begin{cases} X_1 = B_1^+(f_1(X_1,\ldots,X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1,\ldots,X_n)) \\ X_{n+1} = B_{n+1}^+(1+a_1X_1). \end{cases}$$

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### Iterated extensions

$$(S): \begin{cases} X_1 = B_1 \left( (1 - \beta X_1)^{-\frac{1}{\beta}} \right), \\ X_2 = B_2 (1 + X_1), \\ X_3 = B_3 (1 + X_1), \\ X_4 = B_4 (1 + 2X_2 - X_3), \\ X_5 = B_5 (1 + X_4). \end{cases}$$

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## Dilatation

(S') is a dilatation of (S):

$$(S): \begin{cases} X_1 = B_1^+(f(X_1, X_2)), \\ X_2 = B_2^+(g(X_1, X_2)), \end{cases}$$

$$(S'): \begin{cases} X_1 = B_1^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_2 = B_2^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_3 = B_3^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_4 = B_4^+(g(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_5 = B_5^+(g(X_1 + X_2 + X_3, X_4 + X_5)). \end{cases}$$

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### Fundamental systems

Let  $\beta_1, \ldots, \beta_k \in K$ . The following system is an example of a *fundamental* system:

$$\begin{cases} X_i = B_i \left( (1 - \beta_i X_i) \prod_{j=1}^k (1 - \beta_j X_j)^{-\frac{1 + \beta_j}{\beta_j}} \prod_{j=k+1}^n (1 - X_j)^{-1} \right) \\ & \text{if } i \le k, \\ X_i = B_i \left( (1 - X_i) \prod_{j=1}^k (1 - \beta_j X_j)^{-\frac{1 + \beta_j}{\beta_j}} \prod_{j=k+1}^n (1 - X_j)^{-1} \right) \\ & \text{if } i > k. \end{cases}$$

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### Cyclic systems

The following systems are *cyclic*: if  $n \ge 2$ ,

$$\begin{cases} X_1 = B_1^+(1+X_2), \\ X_2 = B_2^+(1+X_3), \\ \vdots \\ X_n = B_n^+(1+X_1). \end{cases}$$

Graph on a cyclic system: an oriented cycle.

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#### Theorem

Let (*S*) be an SDSE. If it is Hopf, then, for all  $i, j \in I$ , for all  $n \ge 1$ , there exists a scalar  $\lambda_n^{(i,j)}$  such that for all tree t', which root is decorated by *i*:

$$\sum_t n_j(t,t') a_t = \lambda_{|t'|}^{(i,j)} a_{t'},$$

where  $n_j(t, t')$  is the number of leaves  $\ell$  of t decorated by j such that the cut of  $\ell$  gives t'.

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We shall denote by  $a_j^{(i)}$  the coefficient of  $h_j$  in  $f_i$  and by  $a_{j,k}^{(i)}$  the coefficient of  $h_j h_k$  in  $f_i$ .

### Lemma

$$rac{\partial f_i}{\partial h_i} 
eq 0$$
 if, and only if,  $a_j^{(i)} 
eq 0$ .

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#### Theorem

Let us assume that (S) is Hopf. Let us fix *i*.

• For all path  $i = i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_k$  in the graph of (*S*)

$$\lambda_k^{(i,j)} = a_j^{(i_k)} + \sum_{p=1}^{k-1} (1 + \delta_{j,i_{p+1}}) rac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}$$

In particular,  $\lambda_1^{(i,j)} = a_j^{(i)}$ .

**2** For all  $p_1, \cdots, p_n \in \mathbb{N}$ :

$$a_{(p_1,\cdots,p_j+1,\cdots,p_n)}^{(i)} = \frac{1}{p_j+1} \left( \lambda_{p_1+\cdots+p_n+1}^{(i,j)} - \sum_{l \in I} p_l a_j^{(l)} \right) a_{(p_1,\cdots,p_n)}^{(i)}.$$

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#### Lemma

Let (S) be a Hopf SDSE. In the graph associated to (S):



Let us assume that  $a_k^{(i)} = 0$ . As  $a_j^{(i)} \neq 0$ ,  $j \neq k$ . As  $a_k^{(i)} = 0$ ,

$$a_{j_{V_i}k} = a_{j,k}^{(i)} = 0.$$

Then:

$$\lambda_{2}^{(i,k)}a_{j}^{(i)} = \lambda_{2}^{(i,k)}a_{l_{i}}^{i} = a_{l_{i}}^{k} + a_{j}_{V_{i}} = a_{j}^{(i)}a_{k}^{(j)} + 0;$$

Hence:

$$\lambda_2^{(i,k)} = a_k^{(j)} \neq 0.$$

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Moreover, As  $a_l^{(i)} \neq 0$ ,  $l \neq k$ . Then:

$$a_{l}^{(i)}\lambda_{2}^{(i,k)} = \lambda_{2}^{(i,k)}a_{l_{i}} = a_{l_{i}}^{*} + a_{l_{i}} = a_{l_{i}}^{(i)}a_{k}^{(l)} + 0.$$

SO:

$$\lambda_2^{(i,k)} = a_k^{(l)}.$$

Hence:

$$a_k^{(l)}=a_k^{(j)}\neq 0.$$

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• A first special case is given by i = k:

2 A second special case is given by i = l:



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### Proposition

Let (S) be a Hopf Dyson-Schwinger system with the following graph:

1 ←→ 2 .

Up to a change of variables, two cases can occur:

$$(S): \begin{cases} X_1 = B_1(1+X_2), \\ X_2 = B_2(1+X_1). \end{cases}$$

$$(S): \begin{cases} X_1 = B_1((1-X_2)^{-1}), \\ X_2 = B_2((1-X_1)^{-1}). \end{cases}$$

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# We put:

$$f_1(h_2) = \sum_{i=0}^{\infty} a_i h_2^i, \qquad f_2(h_1) = \sum_{i=0}^{\infty} b_i h_1^i.$$

Up to a change of variables, assume that  $a_1 = b_1 = 1$ . Then:

$$\lambda_3^{(1,1)} = \lambda_3^{(1,1)} a_{\frac{1}{2}\frac{1}{1}} = 2a_1 \gamma_1^1 = 2b_2$$

On the other hand:

$$2a_2b_2 = \lambda_3^{(1,1)}a_2 v_1^2 = a_2 v_1^1 = 2a_2.$$

So  $2a_2b_2 = 2a_2$  and  $a_2 = 0$  or  $b_2 = 1$ . Similarly,  $b_2 = 0$  or  $a_2 = 1$ . Finally:

$$a_2 = b_2 = 0$$
 or 1.

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In the first case,  $f_1(h_2) = 1 + h_2$  and  $f_2(h_1) = 1 + h_1$ . In the second case, consider the path  $1 \rightarrow 2 \rightarrow 1 \rightarrow ...$  of length *n*.

• If n = 2k is even:

$$\lambda_n^{(1,2)} = 2 + 2(k-1) = 2k = n.$$

• If n = 2k + 1 is odd:

$$\lambda_n^{(1,2)} = 1 + 2k = n.$$

So:

$$\lambda_n^{(1,2)} = n$$
 for all  $n \ge 1$ .

Hence, for all  $n \ge 1$ ,  $a_{n+1} = a_n$  and finally  $f_1(h_2) = (1 - h_2)^{-1}$ . Similarly,  $f_2(h_1) = (1 - h_1)^{-1}$ .

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### Main theorem

Let (S) be Hopf combinatorial Dyson-Schwinger system. Then (S) is obtained from the concatenation of fundamental or cyclic systems with the help of a change of variables, a dilatation and a finite number of extensions.

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If (*S*) is a Hopf, the dual of  $H_{(S)}$  is the enveloping algebra of a prelie algebra  $\mathfrak{g}_{(S)}$ .

# Description of $\mathfrak{g}_{(S)}$

It has a basis  $(e_i(p))_{1 \le i \le n, p \ge 1}$ . The prelie product is given by:

$$e_i(p) \circ e_j(q) = \lambda_q^{(j,i)} e_j(p+q).$$

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As a consequence,  $g_i = Vect(e_i(p), p \ge 1)$  is a prelie subalgebra. In the fundamental case, there are three possibilities:

•  $i \le k$ , with  $\beta_i = -1$ . Then  $e_i(p) \circ e_i(q) = e_i(p+q)$ :  $g_i$  is an associative, commutative algebra.

② *i* > *k*. Then  $e_i(p) \circ e_i(q) = 0$ :  $g_i$  is a trivial prelie algebra.

● *i* ≤ *k* and  $\beta_i \neq -1$ . Then  $b_j \neq 0$ , and  $\mathfrak{g}_i$  is a Faà di Bruno prelie algebra with parameter given by:

$$\lambda_i = \frac{-\beta_i}{\mathbf{1} + \beta_i}.$$

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## Proposition

Let (*S*) be a fundamental SDSE. If k < n or if there exists  $i \le k$ , such that  $\beta_i \ne -1$ , then the Lie algebra  $\mathfrak{g}_{(S)}$  can be decomposed in a semi-direct product:

$$\mathfrak{g}_{(S)} = (M_1 \oplus \ldots \oplus M_k) \rtimes \mathfrak{g}_0,$$

where:

•  $\mathfrak{g}_0$  is a Lie subalgebra of  $\mathfrak{g}_{(S)}$ , isomorphic to the Faà di Bruno Lie algebra, with basis  $(f_n^0)_{n\geq 1}$  such that for all  $m, n \geq 1$ :

$$[f_m^0, f_n^0] = (n-m)f_{n+m}^0.$$

For all 1 ≤ *i* ≤ *k*, *M<sub>i</sub>* is an abelian Lie subalgebra of g(S), with basis (*f<sup>i</sup><sub>n</sub>*)<sub>n≥1</sub>.

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### Proposition

## • For all $1 \le i \le k$ , $M_i$ is a left $\mathfrak{g}_0$ -module in the following way:

$$f_m^0.f_n^i=nf_{m+n}^i.$$

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### Proposition

Let (*S*) be a cyclic SDSE, possibly with dilatations and extensions. The prelie  $g_{(S)}$  admits a basis  $(e_i(k))_{1 \le i \le n, k \ge 1}$  such that:

 $e_i(k) \circ e_j(l) = \begin{cases} e_j(k+l) \text{ if there exists a path from } j \text{ to } i \text{ of length } l, \\ 0 \text{ otherwise.} \end{cases}$ 

This prelie product is associative.

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Definition and main result Examples

### We now consider systems of the form :

$$(S): \begin{cases} X_1 = \sum_{i \in J_1} B_{1,i}^+(f_{1,i}(X_1, \dots, X_n)) \\ \vdots \\ X_n = \sum_{i \in J_n} B_{n,i}^+(f_{n,i}(X_1, \dots, X_n)), \end{cases}$$

where for all  $k, i, B_{k,i}$  is a 1-cocycle of degree *i*.

#### Theorem

We assume that  $1 \in J_k$  for all k. Then (*S*) is entirely determined by  $f_{1,1}, \ldots, f_{n,1}$ .

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Definition and main result Examples

### Fundamental system

$$X_{i} = \sum_{\substack{q \in J_{i} \\ \text{if } i \leq k,}} B_{i,q} \left( (1 - \beta_{i}X_{i}) \prod_{j=1}^{k} (1 - \beta_{j}X_{j})^{-\frac{1 + \beta_{j}}{\beta_{j}}q} \prod_{j=k+1}^{n} (1 - X_{j})^{-q} \right)$$
$$X_{i} = \sum_{\substack{q \in J_{i} \\ \text{if } i > k,}} B_{i,q} \left( (1 - X_{i}) \prod_{j=1}^{k} (1 - \beta_{j}X_{j})^{-\frac{1 + \beta_{j}}{\beta_{j}}q} \prod_{j=k+1}^{n} (1 - X_{j})^{-q} \right)$$

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Definition and main result Examples

For example, we choose n = 3, k = 2,  $\beta_1 = -1/3$   $\beta_2 = 1$ ,  $J_1 = \mathbb{N}^*$ ,  $J_2 = J_3 = \{1\}$ . After a change of variables  $h_1 \longrightarrow 3h_1$ , we obtain:

$$(S): \begin{cases} X_1 = \sum_{k \ge 1} B_{1,k} \left( \frac{(1+X_1)^{1+2k}}{(1-X_2)^{2k}(1-X_3)^k} \right), \\ X_2 = B_2 \left( \frac{(1+X_1)^2}{(1-X_2)(1-X_3)} \right), \\ X_3 = B_3 \left( \frac{(1+X_1)^2}{(1-X_2)} \right). \end{cases}$$

This is the example of the introduction, with  $X_1 = -\infty$ ,

$$X_2 = -$$
 ,  $X_3 = \sim$   $\sim$  .

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Definition and main result Examples

 $\begin{aligned} X_{2} &= \cdot_{2} + 2\mathfrak{l}_{2}^{(1,1)} + \mathfrak{l}_{3}^{2} \\ &+ 6\mathfrak{l}_{2}^{(1,1)} + 2\mathfrak{l}_{2}^{(1,1)} + 4\mathfrak{l}_{2}^{(1,1)} + 4\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{2} + 2\mathfrak{l}_{3}^{3} \\ &+ (1,1) V_{2}^{(1,1)} + 4^{(1,1)} V_{2}^{3} + 3^{3}V_{2}^{3} + 2\mathfrak{l}_{2}^{(1,2)} + \dots \end{aligned} \\ X_{3} &= \cdot_{3} + 2\mathfrak{l}_{3}^{(1,1)} + \mathfrak{l}_{3}^{2} + \mathfrak{l}_{3}^{3} + 6\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{(1,1)} \\ &+ 4\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{2} + 2\mathfrak{l}_{3}^{3} + 2\mathfrak{l}_{3}^{(1,1)} + \mathfrak{l}_{3}^{2} + \mathfrak{l}_{3}^{3} + \mathfrak{l}_{3}^{(1,1)} \\ &+ 4\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{(1,1)} + 2\mathfrak{l}_{3}^{2} + 2\mathfrak{l}_{3}^{3} + \mathfrak{l}_{3}^{3} + \mathfrak{l}_{3}^{3} + \mathfrak{l}_{3}^{(1,2)} + \dots \end{aligned}$ 

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Definition and main result Examples

# Cyclic systems

$$(S): \left\{ \begin{array}{rcl} X_{\overline{1}} & = & \sum_{j \in I_1} B_{1,j} \left( 1 + X_{\overline{1+j}} \right), \\ & \vdots \\ X_{\overline{n}} & = & \sum_{j \in I_1} B_{n,j} \left( 1 + X_{\overline{n+j}} \right). \end{array} \right.$$

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Definition and main result Examples

*n* = 3:

$$\begin{split} X_{\overline{1}} &= \cdot_{(\overline{1},1)} + \cdot_{(\overline{1},2)} + \ddagger_{(\overline{1},1]}^{[\overline{2},1]} + \cdot_{(\overline{1},3)} + \ddagger_{(\overline{1},2]}^{[\overline{3},1]} + \ddagger_{(\overline{1},1]}^{[\overline{3},1]} , \\ X_{\overline{2}} &= \cdot_{(\overline{2},1)} + \cdot_{(\overline{2},2)} + \ddagger_{(\overline{2},1]}^{[\overline{3},1]} + \cdot_{(\overline{2},3)} + \ddagger_{(\overline{2},2]}^{[\overline{1},1]} + \ddagger_{(\overline{2},1]}^{[\overline{1},1]} , \\ X_{\overline{3}} &= \cdot_{(\overline{3},1)} + \cdot_{(\overline{3},2)} + \ddagger_{(\overline{3},1]}^{[\overline{1},1]} + \cdot_{(\overline{3},3)} + \ddagger_{(\overline{3},2]}^{[\overline{3},1]} + \ddagger_{(\overline{3},1]}^{[\overline{2},1]} , \end{split}$$

Loïc Foissy Dyson-Schwinger systems on rooted trees