A combinatorial non-commutative Hopf algebra of graphs

Nguyen Hoang-Nghia

Laboratoire d'Informatique de Paris Nord
Université Paris 13

71th Séminaire Lotharingien de Combinatoire
Bertinoro, July 10 2013

Outline

Introduction

Why discrete scales?

Non-commutative graph algebra structure

Hopf algebra structure

Introduction

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

- The Hopf algebra of rooted forests first appeared in the work of A. Dür [2]
- It has been rediscovered by D. Kreimer in the context of quantum field theory [10].
- A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [6] and R. Holtkamp [8]. This Hopf algebra is self-dual.
- Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [3, 4, 5], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

$\hookrightarrow$ Inspired by constructive quantum field theory [11, V. Rivasseau], we propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edge.
The decorations take values in the positive integers (or even in any totally ordered infinite set), and edges of a connected graph carry pairwise distinct decorations.

Why discrete scales?

- The idea of decorating the edges of a graph with discrete scales comes from quantum field theory
- This versatile technique of multi-scale analysis was successfully applied for scalar quantum field theory renormalization (see [11, V. Rivasseau]), renormalization of scalar quantum field theory on the non-commutative Moyal space (see, for example, [7, R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa]) and recently to the renormalization of quantum gravity tensor models [1, J. Ben Geloun and V. Rivasseau].
- The combinatorics of the multi-scale renormalization was encoded in a Hopf algebraic framework in [9, T. Krajewski, V. Rivasseau and A. Tanasa]

- The idea of decorating the edges of a graph with discrete scales comes from quantum field theory
- This versatile technique of multi-scale analysis was successfully applied for scalar quantum field theory renormalization (see [11, V. Rivasseau]), renormalization of scalar quantum field theory on the non-commutative Moyal space (see, for example, [7, R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa]) and recently to the renormalization of quantum gravity tensor models [1, J. Ben Geloun and V. Rivasseau].
- The combinatorics of the multi-scale renormalization was encoded in a Hopf algebraic framework in [9, T. Krajewski, V. Rivasseau and A. Tanasa]

- The idea of decorating the edges of a graph with discrete scales comes from quantum field theory
- This versatile technique of multi-scale analysis was successfully applied for scalar quantum field theory renormalization (see [11, V. Rivasseau]), renormalization of scalar quantum field theory on the non-commutative Moyal space (see, for example, [7, R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa]) and recently to the renormalization of quantum gravity tensor models [1, J. Ben Geloun and V. Rivasseau].
- The combinatorics of the multi-scale renormalization was encoded in a Hopf algebraic framework in [9, T. Krajewski, V. Rivasseau and A. Tanasa]

Non-commutative graph algebra structure

Totally assignment

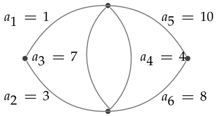
Definition 3.1.

A **total assignment** μ for a connected graph Γ is a list of distinct integers i_l, $l = 1, \ldots, E$ associated to the edges of the graph Γ, where E is the number of edges of the respective graph.

Definition 3.2.

A **connected total assigned graph** (connected TAG) is a pair (Γ, μ) formed by a connected graph Γ together with a totally ordered scale assignment μ.

Example



One can associate to a given graph an infinite number of totally ordered scale assignments.

THE VECTOR SPACE OF TAGS

Let $\mathbb{K}$ be a field of characteristic 0. Let $CTAG$ be a set of connected TAGs. One sets

$$\mathcal{H} = \mathbb{K}\left\langle CTAG \right\rangle . \tag{1}$$

The **product** m on $\mathcal{H}$ is given by the operation of **non-commutative** disjoint union of TAGs, i.e. the resulting graph is given by the ordered concatenation of graphs and each disjoint component keeps its scale assignment.

$$m((G_1, \mu_1), (G_2, \mu_2)) = (G_1 \underset{\rightarrow}{\sqcup} G_2, \mu_1 \sqcup \mu_2). \tag{2}$$

The empty TAG is the empty graph and the empty list assignment, denoted by $1_{\mathcal{H}}$.

EXAMPLE

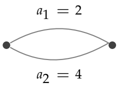


Figure: The TAG (Γ_1, μ_1).

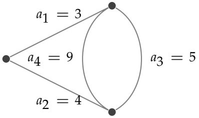


Figure: The TAG (Γ_2, μ_2).

One has

$m((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)) =$. (3)

EXAMPLE

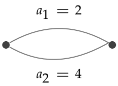


Figure: The TAG (Γ_1, μ_1).

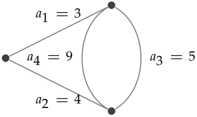
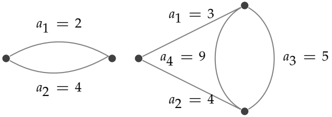


Figure: The TAG (Γ_2, μ_2).

One has

$$m((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)) =$$



. (3)

Algebra structure

One has:

Proposition 3.3.

$(\mathcal{H}, m, 1_{\mathcal{H}})$ *is an associative unitary algebra.*

Hopf algebra structure

TOTALLY ASSIGNED SUBGRAPH

Definition 4.1.

A subgraph γ of a graph Γ is the graph formed by a given subset of edges e of the set of edges of the graph Γ together with the vertices that the edges of e hook to in Γ.

Totally assigned subgraph

Definition 4.2.

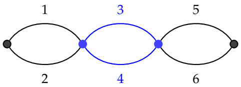
A **totally assigned subgraph** (γ, ν) of a given TAG (Γ, μ) is constructed in the following way.

- A graph γ is a subgraph of the graph Γ, in the usual way in 4.1.
- The totally ordered scale assignment ν of γ is given by the restriction of the totally ordered scale assignment μ to the edges of γ.

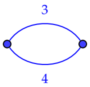
Remark 4.3.

- *If* Γ *is connected, then the order between various (possible) connected components of subgraph* γ*: the first connected component contains the edge with the smallest scale in the list of scales of the subgraph* γ*, the second connected component of the subgraph contains the edge with the smallest scale in the list of scales of* γ *after deleting from this list the scales corresponding to the edges of the first connected component.*
- *If* Γ *is not connected, we follow the algorithm for the first component of* Γ*, then for the second, etc., up to the last one.*

EXAMPLE



(a) A TAG (Γ,μ).

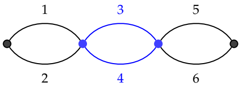


(b) The sub-TAG (γ,ν).

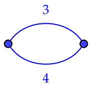
(c) A TAG (Γ,μ).

(d) The sub-TAG (γ',ν').

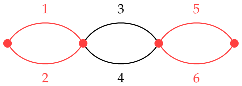
Example



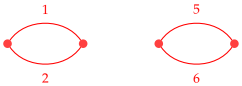
(a) A TAG (Γ,μ).



(b) The sub-TAG (γ,ν).



(c) A TAG (Γ,μ).



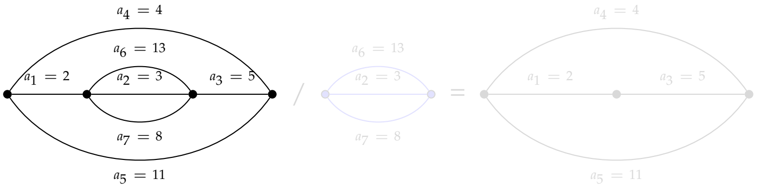
(d) The sub-TAG (γ',ν').

Shrinking

Definition 4.4.

The **shrinking** of a totally assigned subgraph (γ, ν) of a given TAG (Γ, μ) is defined in the following way: the subgraph γ is defined in the usual way (deleting the internal structure of γ), leading to the cograph Γ/γ. The totally ordered scale assignment μ/ν of the cograph Γ/γ is given by deleting the totally assignment ν from the initial totally assignment μ. The TAG $(\Gamma/\gamma, \mu/\nu)$ is called a totally assigned cograph.

For example, one has

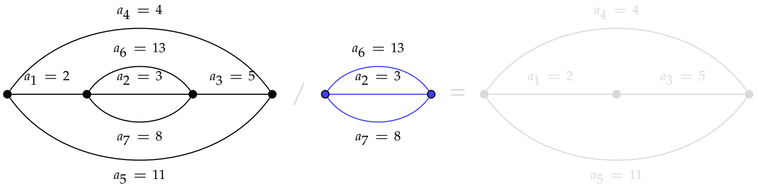


SHRINKING

Definition 4.4.

The **shrinking** of a totally assigned subgraph (γ, ν) of a given TAG (Γ, μ) is defined in the following way: the subgraph γ is defined in the usual way (deleting the internal structure of γ), leading to the cograph Γ/γ. The totally ordered scale assignment μ/ν of the cograph Γ/γ is given by deleting the totally assignment ν from the initial totally assignment μ. The TAG $(\Gamma/\gamma, \mu/\nu)$ is called a totally assigned cograph.

For example, one has

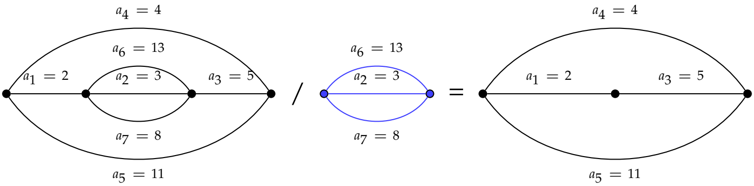


SHRINKING

Definition 4.4.

The **shrinking** of a totally assigned subgraph (γ, ν) of a given TAG (Γ, μ) is defined in the following way: the subgraph γ is defined in the usual way (deleting the internal structure of γ), leading to the cograph Γ/γ. The totally ordered scale assignment μ/ν of the cograph Γ/γ is given by deleting the totally assignment ν from the initial totally assignment μ. The TAG $(\Gamma/\gamma, \mu/\nu)$ is called a totally assigned cograph.

For example, one has



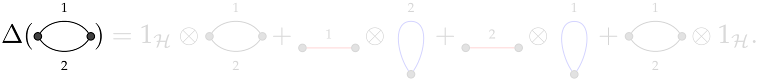
THE COPRODUCT

Let us now define the coproduct on the space $\mathcal{H}$, $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ as

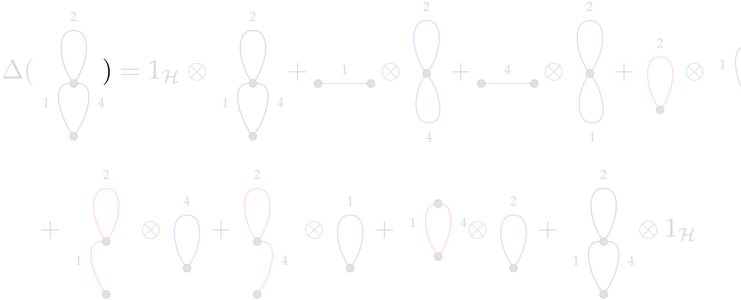
$$\Delta((\Gamma,\mu)) = (\Gamma,\mu)\otimes 1_{\mathcal{H}} + 1_{\mathcal{H}}\otimes(\Gamma,\mu) + \sum_{(\gamma,\nu)\subsetneq(\Gamma,\mu)} (\gamma,\nu)\otimes(\Gamma/\gamma,\mu/\nu). \quad (4)$$

for any TAG (Γ,μ).

EXAMPLE

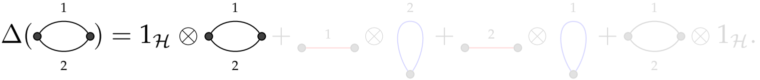


(5)

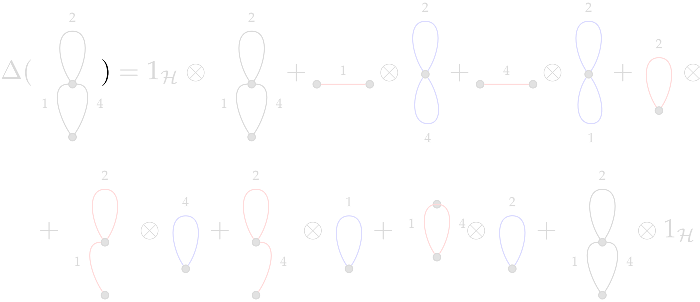


(6)

EXAMPLE

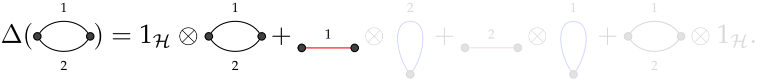


(5)

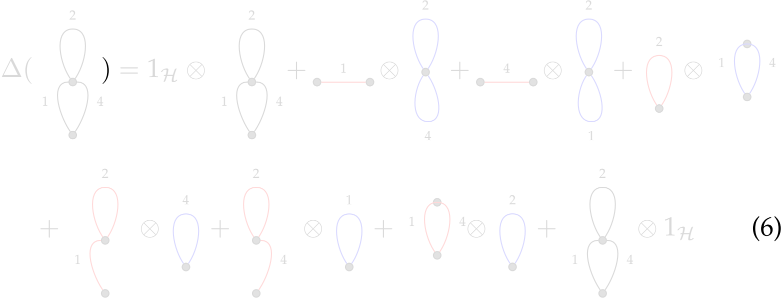


(6)

EXAMPLE

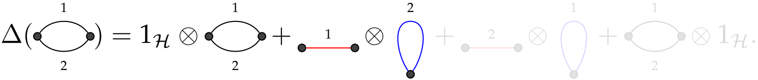


(5)

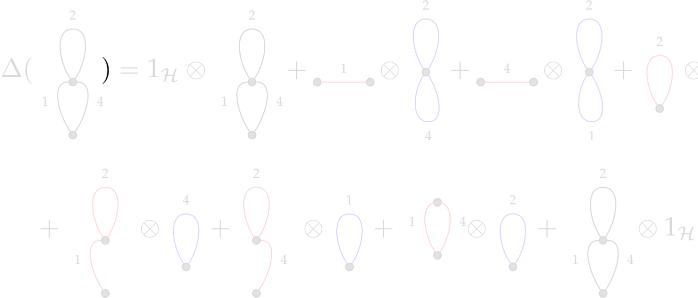


(6)

EXAMPLE

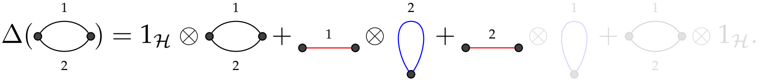


(5)

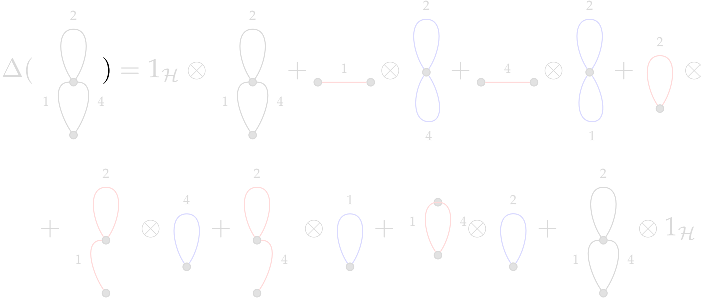


(6)

EXAMPLE

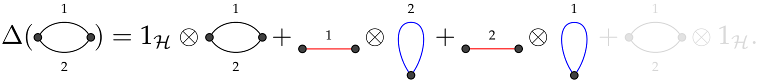


(5)

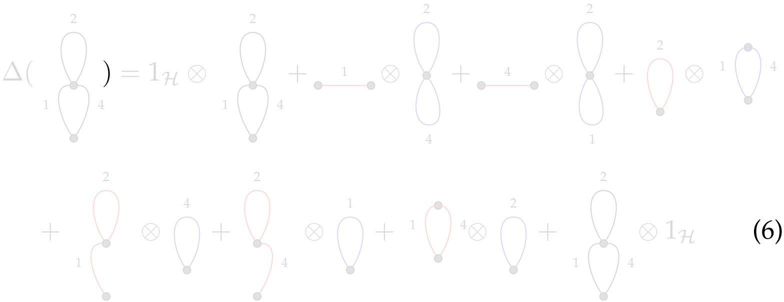


(6)

EXAMPLE

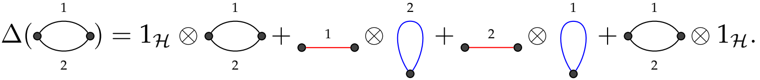


(5)

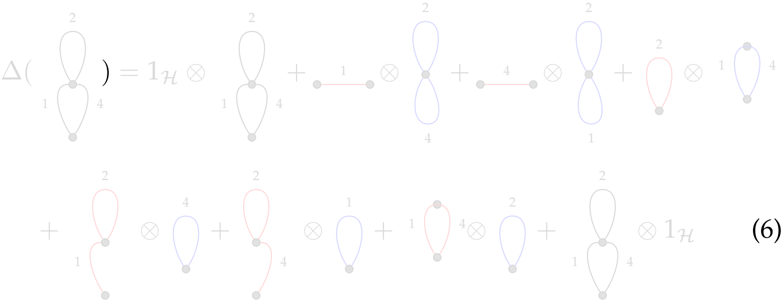


(6)

Example

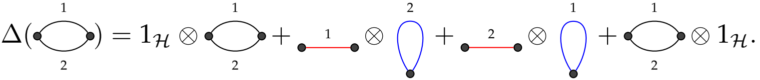


(5)

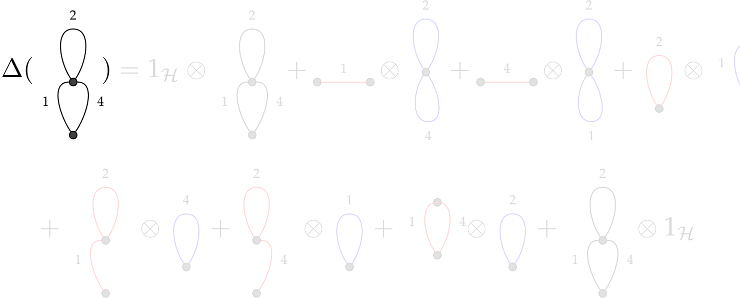


(6)

EXAMPLE

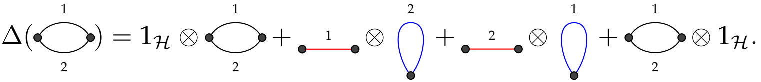


(5)

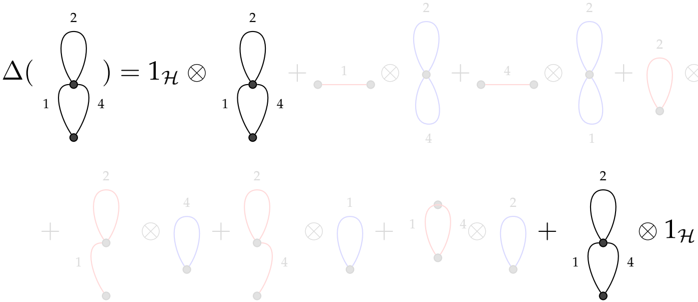


(6)

Example

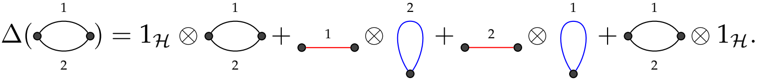


(5)

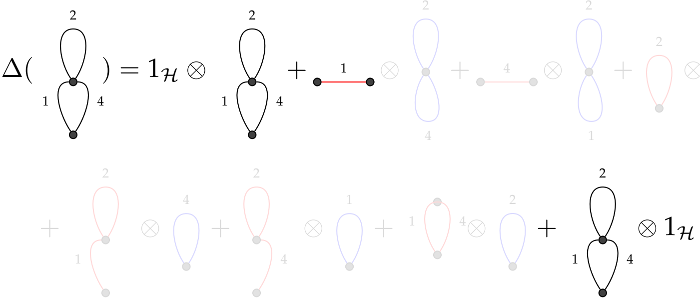


(6)

EXAMPLE

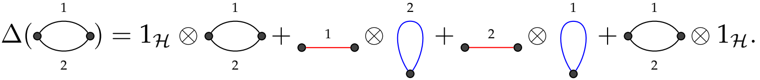


(5)

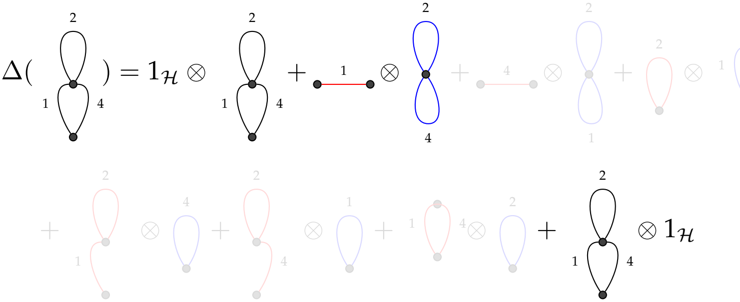


(6)

EXAMPLE

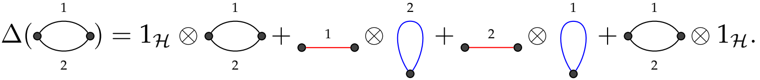


(5)

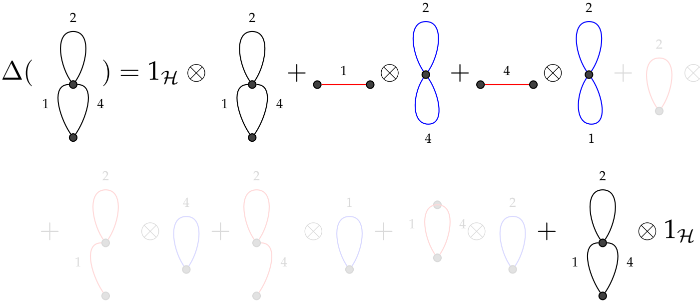


(6)

EXAMPLE

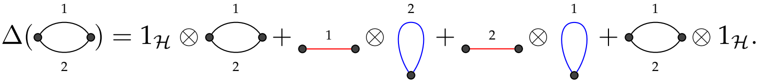


(5)

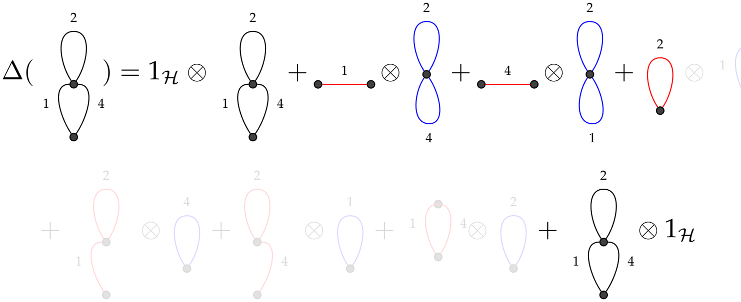


(6)

Example

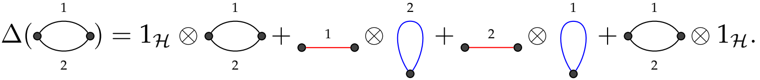


(5)

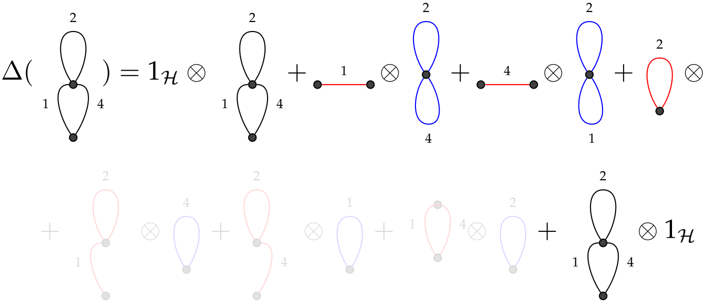


(6)

EXAMPLE

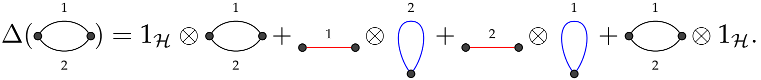


(5)

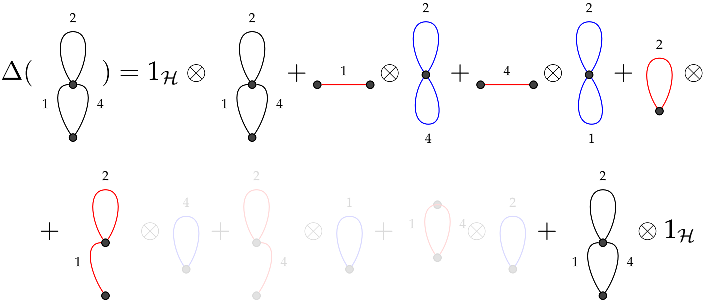


(6)

EXAMPLE

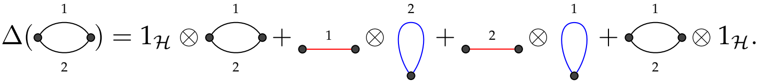


(5)

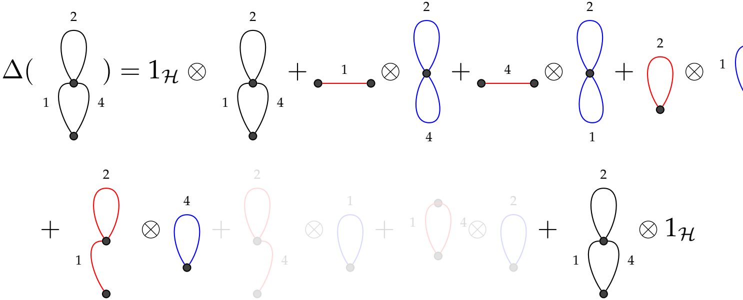


(6)

EXAMPLE

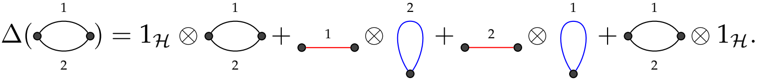

Δ( ) = $1_{\mathcal{H}}$ ⊗ + ⊗ + ⊗ + ⊗ $1_{\mathcal{H}}$.
1
2
1
2
1
2
2
1
1
2

(5)

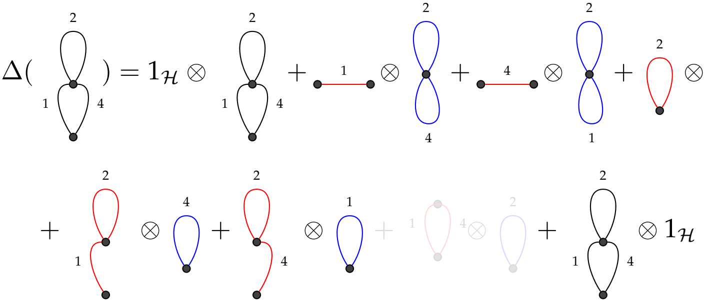

Δ( ) = $1_{\mathcal{H}}$ ⊗ + ⊗ + ⊗ + ⊗
+ ⊗ + ⊗ + ⊗ + ⊗ $1_{\mathcal{H}}$
2
1
4
2
1
4
1
2
4
4
2
1
2
1
4
2
1
4
2
1
4
1
1
4
2
2
1
4

(6)

EXAMPLE

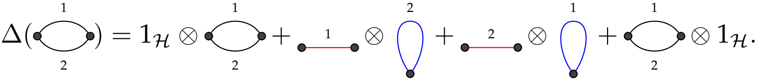


(5)

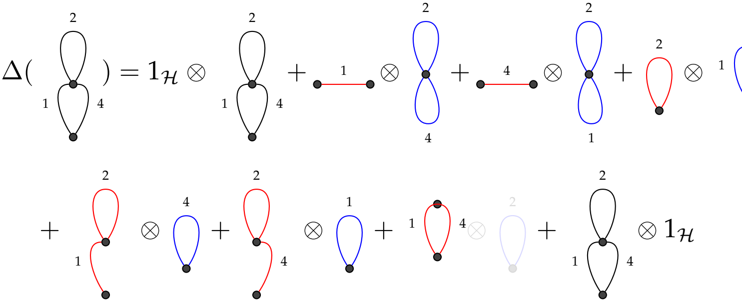


(6)

Example

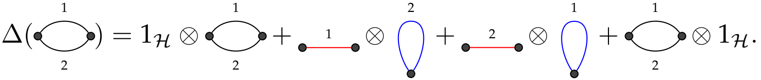


(5)

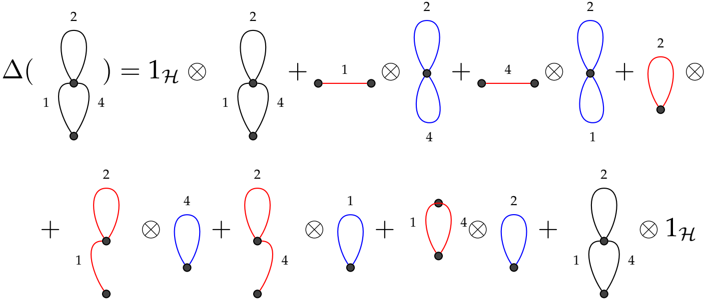


(6)

EXAMPLE



(5)



(6)

Bialgebra structure

Proposition 4.5.

The coproduct defined in (4) *is coassociative.*

$\hookrightarrow$ One has

Theorem 4.6.

The triple $(\mathcal{H}, \Delta, \epsilon)$ *is a coassociative coalgebra with counit.*

Hopf algebra structure

Proposition 4.7.

Let (G_1, μ_1) *and* (G_2, μ_2) *be two TAGs in* $\mathcal{H}$*. One has*

$$\Delta(m((G_1, \mu_1), (G_2, \mu_2))) = m^{\otimes 2} \circ \tau_{23}(\Delta(G_1, \mu_1), \Delta(G_2, \mu_2)) \quad (7)$$

where τ_{23} *is the flip of the two middle factors in* $\mathcal{H}^{\otimes 4}$*.*

Theorem 4.8.

$(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ *is a bialgebra.*

Hopf algebra

For all $n \in \mathbb{N}$, one calls $\mathcal{H}(n)$ the vector space generated by the TAGs with n edges. Then one has $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}(n)$. Moreover, one has:

1. For all $m, n \in \mathbb{N}$, $\mathcal{H}(m)\mathcal{H}(n) \subseteq \mathcal{H}(m+n)$.
2. For all $n \in \mathbb{N}$, $\Delta(\mathcal{H}(n)) \subseteq \sum_{i+j=n} \mathcal{H}(i) \otimes \mathcal{H}(j)$.

One thus concludes that $\mathcal{H}$ is a *graded bialgebra*.
Note that $\mathcal{H}$ is *connected*, i.e. $\mathcal{H}(0) = \mathbb{K}1_{\mathcal{H}}$.
Then, the antipode $S : \mathcal{H} \longrightarrow \mathcal{H}$ is given by the recursive formula:

$$S(1_{\mathcal{H}}) = -1_{\mathcal{H}};$$
$$S((\Gamma,\mu)) = -(\Gamma,\mu) - \sum_{(\gamma,\nu)\subsetneq(\Gamma,\mu)} (\gamma,\nu) \otimes (\Gamma/\gamma, \mu/\nu), \text{ if } (\Gamma,\mu) \neq 1_{\mathcal{H}}. \quad (8)$$

We can now state the main result:

Theorem 4.9.

The bialgebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon, S)$ is a Hopf algebra.

Conclusion

We propose a noncommutative version of a Hopf algebra of graphs, by putting decorations on each edges.
This work is based on arXiv:1307.3928 [math.CO]

BIBLIOGRAPHY

- J. Ben Geloun and V. Rivasseau, *A Renormalizable 4-Dimensional Tensor Field Theory*, Comm. Math. Phys. **318** (2013), 69-109.
- A. D ür, *Möbius functions, incidence algebras and power series representations*, Lecture Notes in Mathematics **1202**, Springer-Verlag, Berlin (1986).
- A. Connes, D. Kreimer, *Hopf algebras, Renormalisation and Noncommutative Geometry*, Comm. Math. Phys. **199** (1998), 203-242.
- A. Connes, D. Kreimer, *Renormalization in Quantum Field Theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. **210** (2000), 249-273.
- A. Connes, D. Kreimer, *Renormalization in Quantum Field Theory and the Riemann-Hilbert problem II: the β-function, diffeomorphisms and the renormalization group*, Comm. Math. Phys. **216** (2001), 215-241.
- L. Foissy, *Les algèbres de Hopf des arbres enracinés décorés, I*, Bull. Sci. math. 126 (2002) 193-239.
- R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa, *A Translation-invariant renormalizable non-commutative scalar model*, Commun. Math. Phys. **287** (2009) 275 [arXiv:0802.0791 [math-ph]].
- R. Holtkamp, *Comparison of Hopf algebras on trees*, Archiv der Mathematik **80** (2003), 368-383.
- T. Krajewski, V. Rivasseau and A. Tanasa, *Combinatorial Hopf algebraic description of the multiscale renormalization in quantum field theory*, arXiv:1211.4429.
- D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, Adv. Theor. Math. Phys. **2** (1998).
- V. Rivasseau, *From perturbative to constructive renormalization*, Princeton University Press (1991).

Thank you for your attention!