Directed graphs related to association schemes

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A d-class association scheme consists of (relations with adjacency) $\{0,1\}$ matrices $A_0 = I, A_1, \ldots, A_d$ satisfying

- $A_0 + A_1 + \ldots + A_d = J$
- for each i there exists i' so that $A_i^{\mathsf{t}} = A_{i'}$
- there exist intersection numbers p_{ij}^h so that

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$$

Each A_i, $i > 0$ is adjacency matrix of a directed or undirected graph.

The association scheme is said to primitive if all these graphs are connected.
Otherwise it is imprimitive.

In a symmetric 2-class association scheme
A_1 is adjacency matrix of a strongly regular graph satisfying

$$A_1^2 = p_{11}^0 A_0 + p_{11}^1 A_1 + p_{11}^2 A_2 = kI + \lambda A_1 + \mu(J - I - A_1),$$

and A_2 is adjacency matrix of complementary graph (also strongly regular).

Strongly regular graph means:

- every vertex has degree k
- two adjacent vertices have λ common neighbours
- two non-adjacent vertices have μ common neighbours

In a non-symmetric 2-class association scheme A_1 is the adjacency matrix of doubly regular tournament:

A tournament (orientation of complete graph) of order n is called doubly regular if every vertex has in-degree and out-degree k, and The number of directed paths of length 2 from x to y is

$$\begin{cases} \lambda & \text{if } x \to y, \\ \lambda + 1 & \text{if } x \leftarrow y. \end{cases}$$

$n = 2k + 1 = 4\lambda + 3.$

Paley (1933). If $n = 4\lambda + 3$ is a prime power then there is a doubly regular tournament on vertices.

An Hadamard matrix of order n is an $n \times n$ $\{1, -1\}$ matrix H such that $HH^{\mathrm{t}} = nI$.

An Hadamard matrix is called skew if $H + H^{\mathrm{t}} = 2I$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ - & 1 & - & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \end{pmatrix}$$

Reid and Brown (1972)

There exists a doubly regular tournament of order n if and only if there exists a skew Hadamard matrix of order $n + 1$.

Duval 1988 considered what he called directed strongly regular graphs.

These graphs have adjacency matrix A satisfying

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

But other directed graph variations of strongly regular graphs could be natural.
In particular we equations involving A^{t}.

Non-symmetric 3-class association schemes.

Matrices:

$A_0 = I$

$A_1 = A$

$A_2 = A^{\mathrm{t}}$

$A_3 = J - I - A - A^{\mathrm{t}}$

A_3 is adjacency matrix of a strongly regular graph.
A_1 and A_2 are adjacency matrices of opposite orientations of the complementary strongly regular graph.

$$
\begin{aligned}
AJ = JA = \kappa J && (1)\\
AA^{\mathsf{t}} = \kappa I + \lambda(A + A^{\mathsf{t}}) + \mu(J - I - A - A^{\mathsf{t}}) && (2)\\
A^{\mathsf{t}}A = \kappa I + \lambda(A + A^{\mathsf{t}}) + \mu(J - I - A - A^{\mathsf{t}}) && (3)\\
A^2 = \alpha A + \beta A^{\mathsf{t}} + \gamma(J - I - A - A^{\mathsf{t}}), && (4)
\end{aligned}
$$

where $\kappa = p^0_{12}$,$\lambda = p^1_{12} = p^1_{21}$, $\mu = p^3_{12} = p^3_{21}$, $\alpha = p^1_{11}$, $\beta = p^2_{11}$, $\gamma = p^3_{11}$.

(1): every vertex has in-degree and out-degree κ.
(2/3): the number of common out-/in- neighbours of x and y is exactly

$$\begin{cases} \lambda & \text{if } x \to y \text{ or } x \leftarrow y \\ \mu & \text{otherwise.} \end{cases}$$

(4): the number of directed paths of length 2 from x to y is exactly

$$\begin{cases} \alpha & \text{if } x \to y, \\ \beta & \text{if } x \leftarrow y, \\ \gamma & \text{otherwise.} \end{cases}$$

Imprimitive case

Assume:
(Directed) graph with matrix A_1 is connected,
Graph with matrix A_3 disconnected.

Vertices partitioned in m blocks of size r.
Directed graph with matrix A_1 is an orientation the complete m-partite graph $K_{r,\ldots,r}$.

Conjecture. Every orientation of the complete m-partite graph $K_{r,\dots,r}$ with adjacency matrix A satisfying

$$AJ = JA = \kappa J$$

and

$$AA^{\mathrm{t}} = A^{\mathrm{t}}A = \kappa I + \lambda(A + A^{\mathrm{t}}) + \mu(J - I - A - A^{\mathrm{t}})$$

is a relation of a non-symmetric 3-class association scheme.

Such a matrix has 4 or 5 distinct eigenvalues.
Conjecture is true for 4 eigenvalues.

Definition (JJKS)

A directed graph Γ with adjacency matrix A is said to be a doubly regular (m, r)-team tournament if

- Γ is an orientation of the complete m-partite graph $K_{r,\ldots,r}$.
- $AJ = JA = \kappa J$
- $A^2 = \alpha A + \beta A^{\mathsf{t}} + \gamma(J - I - A - A^{\mathsf{t}})$.

Theorem (JJKS)

Every doubly regular (m, r)-team tournament is of one of the following three types.

Type 1 A cocliqueextension of a doubly regular tournament.
Association scheme.

Type 2 For every vertex $x \in V_i$, exactly half of the vertices in V_j $(j \neq i)$ are out-neighbours of x, and $\alpha = \beta = \frac{(m-2)r}{4}$, and $\gamma = \frac{(m-1)r^2}{4(r-1)}$.
These graphs are association schemes.
Necessary condition:
$r - 1$ divides $m - 1$. m and r are even.

Type 3 For every pair $\{i, j\}$ either V_i is partitioned in two sets V_i' and V_i'' of size $\frac{r}{2}$ so that all edges between V_i and V_j are directed from V_i' to V_j and from V_j to V_i'', or similarly with i and j interchanged.

The parameters are
$\alpha = \frac{(m-1)r}{4} - \frac{3r}{8}$, $\beta = \frac{(m-1)r}{4} + \frac{r}{8}$, $\gamma = \frac{(m-1)r^2}{8(r-1)}$.

Not association scheme.

No examples are known.

8 divides r and $4(r-1)$ divides $m-1$.

Smallest possible case: $r = 8$, $m = 29$.

The results for association schemes was first proved by Goldbach and Claasen (1996).

Type 2

Three cases:

Type 2a: $r = m$ even

Type 2b: $r = 2$, $m \equiv 0 \mod 4$

Type 2c: $4 \leq r < m$, $r - 1$ divides $m - 1$, m and r are even.
There exist examples with $r = 4, m = 16$.

Type 2a: $r = m$

An Hadamard matrix H of order m^2 (m even) is said to be Bush-type if H is block matrix with $m \times m$ blocks H_{ij} of size $m \times m$ such that

$$H_{ii} = J_m$$

and

$$H_{ij} J_m = J_m H_{ij} = 0, \quad \text{for } i \neq j.$$

$$\begin{pmatrix} 1 & 1 & & 1 & - \\ 1 & 1 & & - & 1 \\ & & & & \\ - & 1 & & 1 & 1 \\ 1 & - & & 1 & 1 \end{pmatrix}$$

Theorem

A doubly regular (m, m)-team tournament of type 2
is equivalent to
a Bush-type Hadamard matrix of order m^2 with the property that
$H_{ij} = -H_{ji}^{\mathrm{t}}$ for all pairs i, j with $i \neq j$.

$m \equiv 0 \mod 4$:

Kharaghani (2000)
If there exists a Hadamard matrix of order m
then there exists
a Bush-type Hadamard matrix of order m^2.

Ionin and Kharaghani (2003) This construction can be modified so that the Bush-type Hadamard matrix is "skew".

$m \equiv 2 \mod 4$:

Janko (2001) constructed a Bush-type Hadamard matrix of order 36.

Theorem (J 2009) There exists "skew" Bush-type Hadamard matrices of order 36.

In an incomplete computer search I found 4 association schemes with $r = m = 6$. These four schemes all have trivial automorphism groups.

Doubly regular (m,m)-team tournament of type 2 are also known to be DRAD (doubly regular asymmetric digraphs) as

$$AA^{\mathrm{t}} = aI + bJ$$

N. Ito 1987 claimed that imprimitive DRAD's can not have rank 4 automorphism group, except for $m = 4$.

J. 2010 found a counterexample with $m = 8$

Davis and Polhill 2010 There exist such DRAD's (doubly regular (m,m)-team tournament of type 2) with a rank 4 automorphism group when m is power of 2.

Type 2b

$r = 2$: The condition $r - 1$ divides $m - 1$ is satisfied for every (even) m.
It can be proved that m is multiple of 4.

Let T be a tournament with adjacency matrix A. Then graph with adjacency matrix

$$\begin{pmatrix} 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & & & & 1 & & & \\ \vdots & & A & & \vdots & & A^{\mathrm{t}} & \\ 0 & & & & 1 & & & \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 1 & & & & 0 & & & \\ \vdots & & A^{\mathrm{t}} & & \vdots & & A & \\ 1 & & & & 0 & & & \end{pmatrix}$$

is denoted by $\mathcal{D}(T)$.

Theorem

Let Γ be a doubly regular $(m, 2)$-team tournament of type 2. Then Γ is isomorphic to $\mathcal{D}(T)$ for some doubly regular tournament T.

$\mathcal{D}(T)$ has a unique automorphism of order 2. It interchanges pairs of non-adjacent vertices.

Theorem

Suppose that for some doubly regular tournament T, the graph $\Gamma = \mathcal{D}(T)$ is vertex transitive and has automorphism group G. Then the Sylow 2-subgroup S of G is the generalized quarternion group of order 2^n, where 2^n is the highest power of 2 that divides the order of Γ.

Theorem

Suppose that for some doubly regular tournament T, the graph $\Gamma = \mathcal{D}(T)$ is vertex transitive and has order 2^n. Then Γ is a Cayley graph of the generalized quarternion group of order 2^n.

Theorem

Let $q \equiv 3 \mod 4$ be a prime power and P_q be the Paley tournament of order q. Let $\Gamma = \mathcal{D}(P_q)$.

Then $SL(2,q)$ acts as a group of automorphisms on Γ and this group contains a dicyclic subgroup acting regularly on $V(\Gamma)$.

The dicyclic group of order $4n$:

$$\langle x, y \mid x^n = y^2, y^4 = 1, yxy^{-1} = x^{-1} \rangle$$