TF-ISOMORPHISMS

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Raffaele Scapellato (Politecnico di Milano)

September 21, 2013

A famous graph and its less famous cousin

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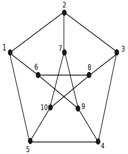

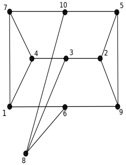

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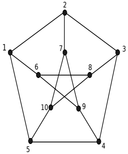

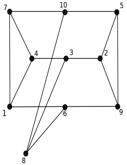

Neighbourhoods of Petersen:

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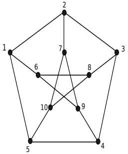

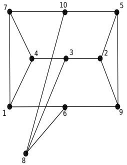

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An orbital (di)graph is an orbit of an arc (u, v) under this action.

Orbital (di)graphs

Some very simple and well-known facts.

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Theorem

Let G be an orbital (di)graph. Then $\Gamma \leq Aut(G)$. If Γ is vertex-transitive on V then G is vertex-transitive and arc-transitive and if G is disconnected then all components of G are isomorphic.

Two-fold orbitals

Let Γ be a subgroup of $S_V \times S_V$. Therefore the action now is

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(OFTEN we do NOT require that the actions of the projections $\pi_1, \pi_2 : \mathbf{\Gamma} \to S_n$, defined by $\pi_1((\alpha, \beta)) = \alpha$ and $\pi_2((\alpha, \beta)) = \beta$, are transitive on V.)

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The orbit of a pair (u, v) under this action is called a *two-fold orbital* or *TF-orbital*.

Two-fold orbitals (2)

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- They have loops.
- They are mixed: arcs and edges.
- If a TF-orbital is disconnected then its components are not necessarily isomorphic, even if the projections of $\mathbf{\Gamma}$ are transitive.

Conventions

An *mixed graph* is considered to be a finite set of vertices and a set of pairs of vertices which can be both ordered (arcs) and unordered (edges): if the ordered pairs (arcs) (u, v) and (v, u) both exist then we say that the arcs are *self-paired* and together they form the edge $\{a, b\}$.

Multiple arcs (repetition of the arc (x, y)) are not allowed, but loops (the arc (x, x)) are possible.

Conventions (2)

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If there is no loop and the set of arcs is self-paired then we get a *graph*.

Main concept: TF-isomorphism

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Let G and H be two (mixed) graphs. Suppose there are two bijections α, β from $V(G)$ to $V(H)$ such that

$$(u, v) \text{ is an arc of } G \text{ iff } (u^{\alpha}, v^{\beta}) \text{ is an arc of } H.$$

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If G and H are isomorphic under an isomorphism α then they are also TF-isomorphic via (α, α).
When $\alpha \neq \beta$ we say that the TF-isomorphism is *non-trivial*.

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If $G = H$ then we say that (α, β) is a *TF-automorphism* of G. The set of TF-automorphisms of G forms a group denoted by $\mathrm{Aut}^{\mathrm{TF}}(G)$. If we identify an automorphism α with (α, α) then we can consider $\mathrm{Aut}(G)$ to be a subgroup of $\mathrm{Aut}^{\mathrm{TF}}(G)$.

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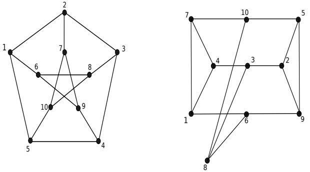

If $\alpha = \text{id}$ and $\beta = (1 \quad 9)(2 \quad 4)(5 \quad 7)$ then (α, β) is a TF-isomorphism from the Petersen graph to the second graph.

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So, for example, if $u \in V(G)$ is a source, then $\alpha(u)$ need not be a source but it is certainly not a sink.

What is preserved by a TF-isomorphism? (2)

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The canonical double cover!

Let G be a *graph*. Its *canonical double cover* $\mathbf{B}(G)$ is the graph whose vertex-set is

$$V(G) \times \mathbb{Z}_2 = \{v_i : v \in V(G), i \in \{0, 1\}\}$$

such that

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If G is bipartite then $\mathbf{B}(G)$ is disconnected: two components isomorphic to G.

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Theorem

Two graphs are TF-isomorphic iff they have isomorphic canonical double cover.

Petersen and cousin again

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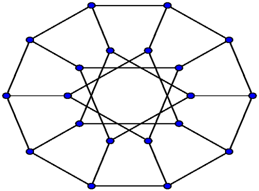

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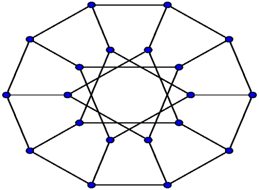

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A result of Scapellato and Pacco counts, for a given bipartite graph B, the number of graphs G such that $B = \mathbf{B}(G)$. For the Desargues graph this number is 2. Therefore Petersen's cousin is the only graph, not isomorphic to the Petersen Graph, which is TF-isomorphic to it.

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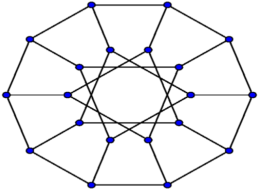

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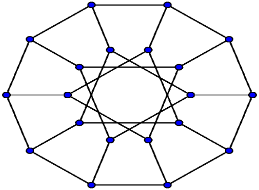

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Disconnected TF-orbital graphs (TOGs)

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Theorem

Let G be a disconnected TOG with no isolated vertices and let its connected components be $G_1, ..., G_k$ and suppose

$$|V(G_1)| \geq |V(G_2)| \geq, ... \geq |V(G_k)|.$$

Then each $G_i (i = 1, ..., k)$ is still a TOG. Moreover:

(i) *if $|V(G_1) = |V(G_k)|$, then $G_1, G_2, ..., G_k$ are pairwise TF-isomorphic (which could include "isomoprhic")*

(ii) *otherwise, there exists a unique index $r \in \{1, ...k-1\}$ such that $G_1, \ldots, G_r$ are isomorphic, $G_{r+1}, \ldots, G_k$ are mutually TF-isomorphic / isomorphic, and $G = G_1$ is the CDC of each of $G_{r+1}, \ldots, G_k$.*

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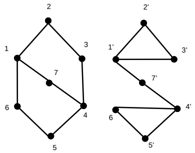

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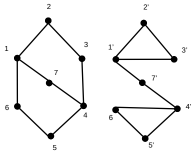

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For example, path $1 \to 2 \to 3$ is transformed into $1' \to 2' \cup * \to *$.

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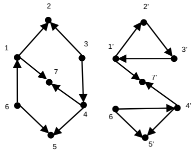

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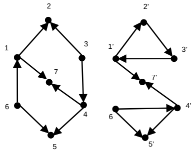

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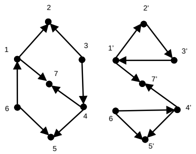

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Note that open alternating trails can be mapped into "closed" alternating trails: $1 \rightarrow 2 \leftarrow 3 \rightarrow 4$ is mapped onto $1' \rightarrow 2' \leftarrow 3' \rightarrow 1'$

Various types of Z-trails

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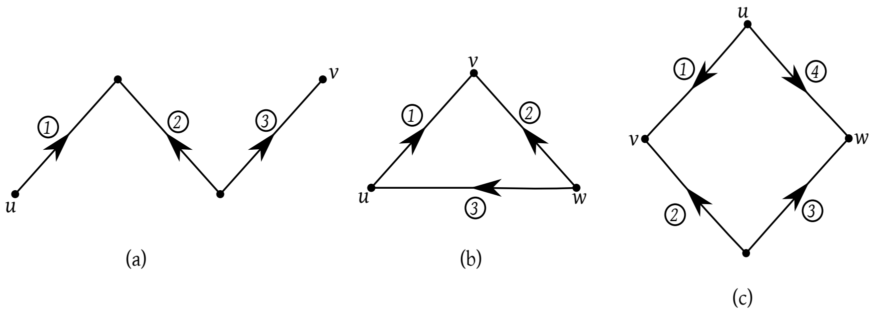

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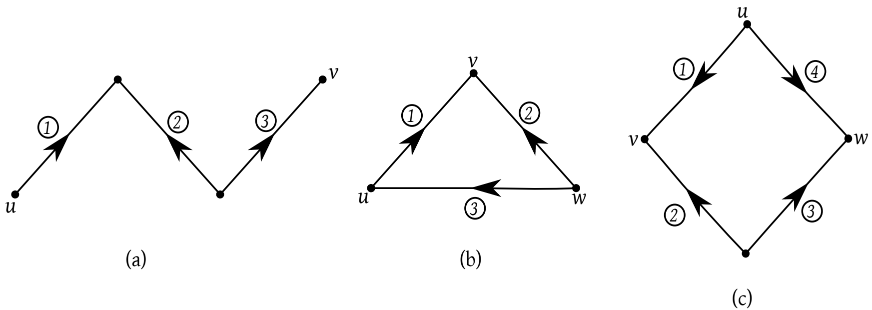

Illustrating the difference between (a) open Z-trail, (b) semi-closed Z-trail and (c) closed Z-trail.

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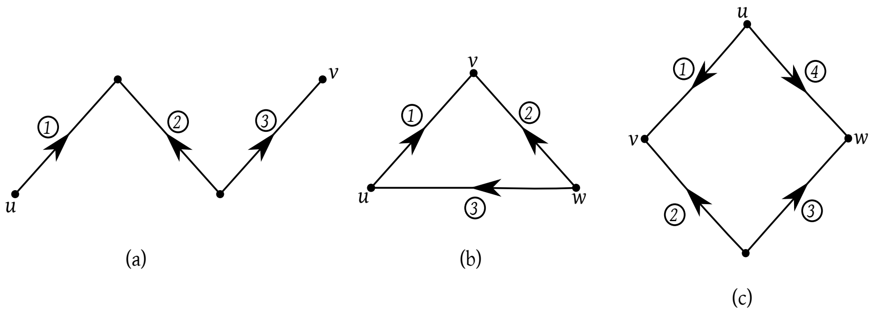


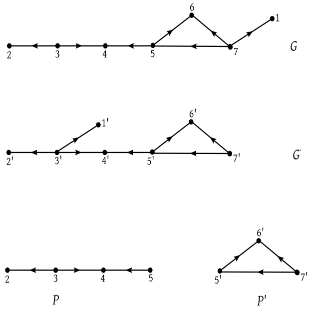
Illustrating the difference between (a) open Z-trail, (b) semi-closed Z-trail and (c) closed Z-trail.
An open Z-trail can be mapped into a semi-closed Z-trail.

Typical results on Z-trails

Theorem

Let G and G' be mixed graphs and suppose that P is a Z-trail in the graph G. Let (α, β) be a TF-isomorphism from G to G'. Then there exists a Z-trail P' in G' such that (α, β) restricted to P maps P to P'.

Typical results on Z-trails (2)



G and G' are TF-isomorphic digraphs.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ * & * & 7' & * & 5' & * & 3' \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1' & 5' & * & 6' & 2' & 4' \end{pmatrix}$$

P of G is mapped into the semi-closed P' of G'. The trails P and P' are not TF-isomorphic.

Typical results on Z-trails (3)

Theorem

A TF-isomorphism takes closed Z-trails into closed Z-trails.

Asymmetric graphs admitting non-trivial TF-automorphism

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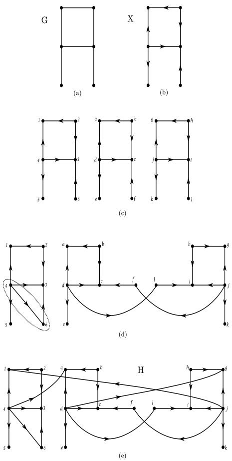

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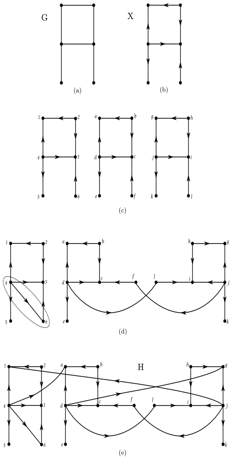

$$\alpha = (1 * *)(2\ b\ h)(3 * *)(4\ d\ j)(5 * *)(6\ f\ l)$$
$$\beta = (1\ a\ g)(2 * *)(3\ c\ i)(4 * *)(5\ e\ k)(6 * *)$$

Stability

Let G be a graph. Then it is clear that $\mathrm{Aut}(\mathbf{B}(G))$ contains $\mathrm{Aut}(G) \times \mathbb{Z}_2$. But $\mathrm{Aut}(\mathbf{B}(G)$ can be larger.

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Definition

A graph is said to be unstable if $\mathrm{Aut}(G) \times \mathbb{Z}_2$ is a proper subgroup of $\mathrm{Aut}(\mathbf{B}(G))$. In other words, a graph G is unstable if at least one element of $\mathrm{Aut}(\mathbf{B}(G))$ is not a lifting of some element of $\mathrm{Aut}(G)$.

Stability (2)

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Theorem

Let G be a graph. Then $Aut(\mathbf{B}(G)) = Aut^{TF}(G) \rtimes \mathbb{Z}_2$. Therefore G is unstable if and only if it has a non-trivial TF-automorphism.

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3. Can use the technique of graph invariants under the action of TF-isomorphisms, such as Z-trails;

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3. Can use the technique of graph invariants under the action of TF-isomorphisms, such as Z-trails;
4. In the same vein, can use knowledge of how TF-isomorphisms act on special types of subgraphs such as triangles.

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That is, the rows (vertices) of the matrix B_1 can be permuted such that the columns become a permutation of the columns of B_2. In the case of automorphisms, we say that a permutation of the vertices of V is an automorphism if the permutation applied to the rows of B gives B with columns permuted.

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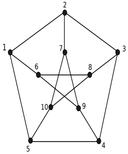
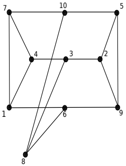
In the case of graphs with an adjacency matrix A, an isomorphism applies *the same* permutation to the rows and the columns, therefore $P = Q^{-1}$.

A different point of view: Incidence structures / hypergraphs (4)

But consider the adjacency matrix A of a graph as an incidence matrix of a hypergraph with equal number of vertices and blocks. Then, TF-automorphisms (TF-isomorphisms) become automorphisms on (isomorphisms between) hypergraphs.

Petersen & cousin again





Neighbourhoods of Petersen:
1: {2,5,6}
2: {1,3,7}
3: {2,4,8}
4: {3,5,9}
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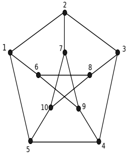

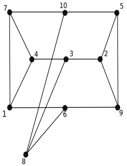

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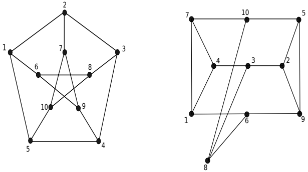

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In the adjacency matrix of the Petersen graph:
Keep the rows fixed (giving $\alpha =$ id).
Interchange columns 1 and 9, columns 2 and 4, and columns 5 and 7, giving

$$\beta = (1 \quad 9)(2 \quad 4)(5 \quad 7).$$

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  MSZ specialised this problem starting with a matrix A which is already symmetric, therefore posing the following question: can a given graph G be TF-isomorphic to another graph (which may be isomorphic to G itself) via a non-trivial TF-isomorphism? This question leads to the notion of graph stability!

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The realisability problem restricted to bipartite graphs therefore becomes: given a bipartite graph G, is there a graph K such that G is the canonical double cover of K?

Finally: the right context to study TF-orbitals?

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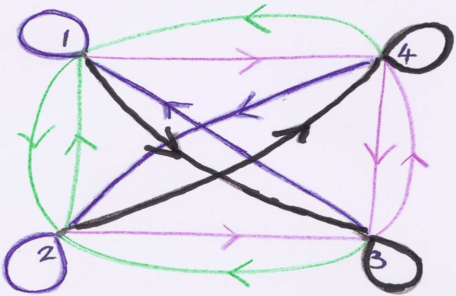
$$(a, b) = ((1234), (12)(34))$$

$$G = \langle (a,b) \rangle = \{(a,b), (a^2, b^2), (a^3, b^3), (a^4, b^4) = (id, id)\}$$

$$X = \{1, 2, 3, 4\}$$

Diagram shows Two-Fold orbitals of G acting on $X \times X$.



THANK YOU!