On the hypergeometric nature of certain Legendre-type polynomials

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SLC71 & XIX-IICA, September 15th-18th, 2013

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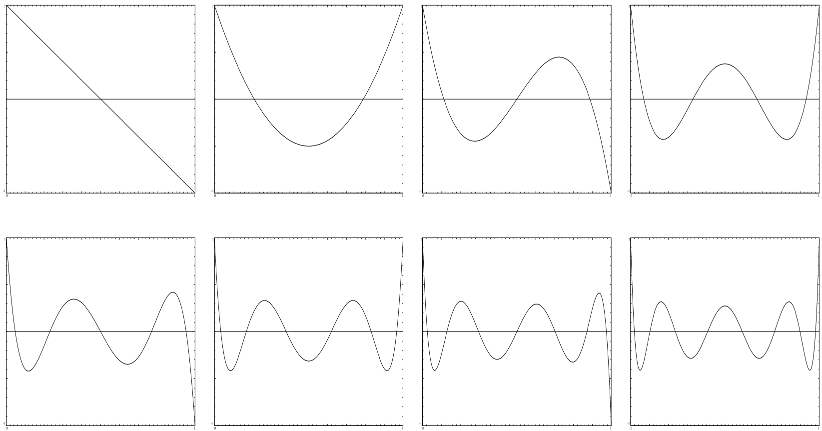
  $P_6(x) = 924x^6-2772x^5+3150x^4-1680x^3+420x^2-42x+1$,

$y = P_n(x)$, $x \in [0,1]$, $n = 1, 2, 3, 4, 5, 6, 7, 8$



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- Then $(1-x)F(x,y) = G(\frac{x}{x-1}, \frac{y}{1-x})$.
- Since $H(w,z) := \sum_{m,n\geq 0} \binom{n+m}{m} w^m z^n$, then $G(w^2,z^2) = H(w^2,z^2) * H(w^2,z^2)$ by definition, where $*$ denotes the Hadamard product; and by the formula $\left(G(w^2,z^2)\right)^2 = H(w,z)H(w,-z)H(-w,z)H(-w,-z)$.

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If $\alpha \notin \mathbb{Q}$, then $\mu(\alpha) \geq 2$ is the irrationality measure of α.

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  - linear recurrence relation.

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- $\mu(\log 2) < 4.662\ldots$ with $P_n(x) = D_n\big(x^n(1-x)^n\big)$.

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  More generally, let $p_1, \ldots, p_m, q_1, \ldots, q_m \geq 0$ be integers.

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- $D_{6n}\left(\frac{w^{7n}}{(1-w)^{8n+1}}\right) = \binom{7n}{n} w^n \, {}_2F_1\left(\begin{matrix} 8n+1 & 7n+1 \\ & n+1 \end{matrix}; w\right).$

  More generally, let $p_1, \ldots, p_m, q_1, \ldots, q_m \geq 0$ be integers.

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Another representation

(M, 201?)

- Change of variable: $x = \frac{w}{w-1}$, i.e. $1 - x = (1-w)^{-1}$.

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- $(1-w)\mathcal{S}_m(p_1, q_1; \ldots; p_m, q_m; w) = \mathcal{L}_m(p_1, q_1; \ldots; p_m, q_m; x)$.

General definition

- $\mathcal{L}_0(x) := 1$,
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  One would expect a linear recurrence in the form $\sum_{i=0}^{3} c_i(n,x)P_{n+i}(x) = 0$ for some $c_0, \dots, c_3 \in \mathbb{Z}[n,x]$ such that $(c_0(n,\zeta), \dots, c_3(n,\zeta)) \neq (0, \dots, 0)$ for any $\zeta \in \mathbb{C}$.

Theorem

For any $p_1, \ldots, p_m > 0$ and $q_1, \ldots, q_m \geq 0$ integers, there exist effectively computable polynomials $\alpha_{l,r}(n)$, $l = 0, \ldots, m$, $r = 0, \ldots, s$ not all zero such that for all $k = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$

$$\sum_{l=0}^{m} \sum_{r=0}^{s} \alpha_{l,r}(n) \binom{k - r + p_1(n+l)}{(p_1 + q_1)(n+l)} \cdots \binom{k - r + p_m(n+l)}{(p_m + q_m)(n+l)} = 0.$$

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- By taking $k=yn$, and letting $n\to\infty$ in

$$\sum_{l=0}^{m}\sum_{r=0}^{s}\alpha_{l,r}(n)\binom{k-r+p_1(n+l)}{(p_1+q_1)(n+l)}\cdots\binom{k-r+p_m(n+l)}{(p_m+q_m)(n+l)}=0,$$

one finds the roots of the characteristic polynomial of the recurrence for $\mathcal{S}_m(p_1n,q_1n;\dots;p_mn,q_mn;w)$, i.e. for $\mathcal{L}_m(p_1n,q_1n;\dots;p_mn,q_mn;x)$.

Results

- With $m = 3$ and $(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; x) = (4, 5, 3; 1, 2, 0; -1)$ one obtains $\mu(\log 2) < 3.57455\ldots$;

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- With $m = 4$ and $(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; x) = (5, 6, 7, 4; 1, 2, 3, 0; -1)$ one obtains $\mu_2(\log 2) < 12.84161\ldots$;

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$$\mu_2(\log 2) < 15.65142\ldots \quad \text{(M, 2009)}.$$

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- Generalize the classical case:
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Supposing that the characteristic of the ground field is zero:

- is $T_3(w,z) = \sum_{k,n\geq 0} \binom{k+n}{n}^3 w^k z^n$ algebraic?
- is $T_m(w,z) = \sum_{k,n\geq 0} \binom{k+np_1}{n(p_1+q_1)} \cdots \binom{k+np_m}{n(p_m+q_m)} w^k z^n$ algebraic, for general $p_1,\ldots,p_m; q_1,\ldots,q_m$ and $m > 2$?