Root Polytopes

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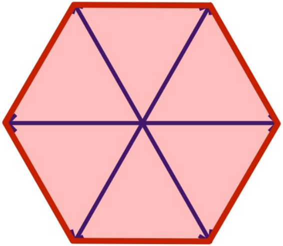
joint with Paola Cellini

Problem

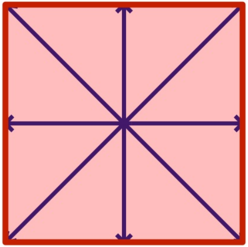
Φ = (crystallographic) irreducible root system

Root Polytope $\mathcal{P}_\Phi$:= convex hull of Φ

Type A_2



Type C_2



Problem: Find a uniform description of the Root Polytopes $\mathcal{P}_\Phi$

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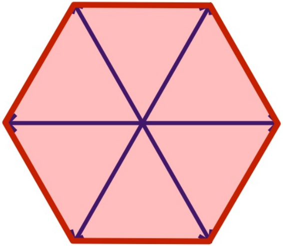

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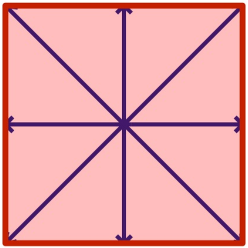

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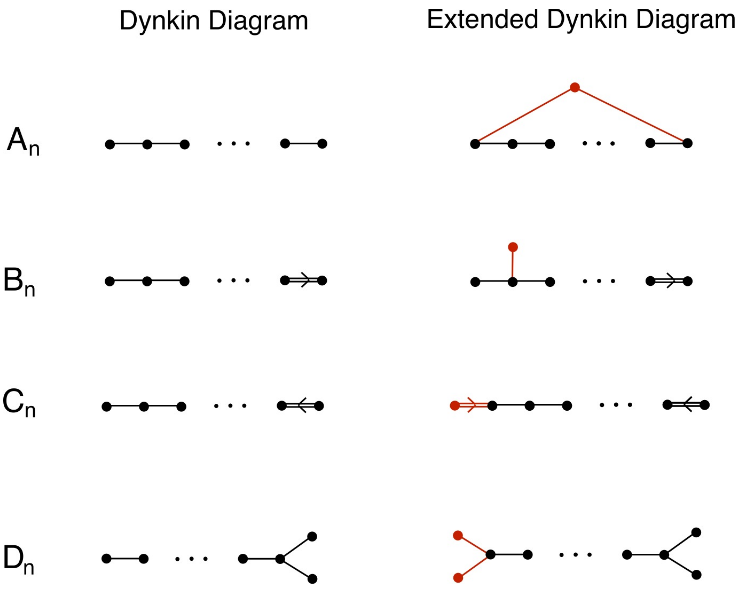
Notation

- W Weyl group of Φ
  group generated by all reflections through hyperplanes $\alpha^\perp$, $\forall \alpha \in \Phi$
  It fixes Φ: $W(\Phi) = \Phi$
- $\Pi = \{\alpha_1, \dots, \alpha_n\}$ set of simple roots
  $\Phi = \Phi^+ \cup (-\Phi^+)$ where $\Phi^+ =$ set of positive roots (roots which are positive linear combinations of Π)
- $\breve{\omega}_1, \dots, \breve{\omega}_n$ fundamental coweights (the dual basis of Π)
- $(\, , \,)$ scalar product
- $c_i(\beta)$ the coordinates of β w.r.t. Π
  $\beta = \sum c_i(\beta)\alpha_i$
- α_0 affine simple root


Dynkin Diagram
Extended Dynkin Diagram
A_n
B_n
C_n
D_n

E_6

E_7

E_8

F_4

G_2

Root poset

Partial order structure on the set Φ^+

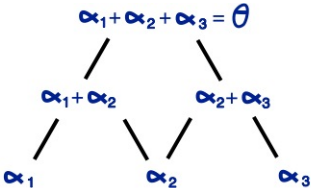
$$\alpha = \sum c_i(\alpha)\alpha_i, \quad \beta = \sum c_i(\beta)\alpha_i$$

$$\begin{aligned} \alpha \leq \beta \quad &\iff \quad \beta - \alpha \text{ is a positive combination of } \Pi \\ &\iff \quad c_i(\alpha) \leq c_i(\beta) \quad \forall i \end{aligned}$$

$\alpha_1, \ldots, \alpha_n$ are minimal elements, the highest root θ is maximum

Example

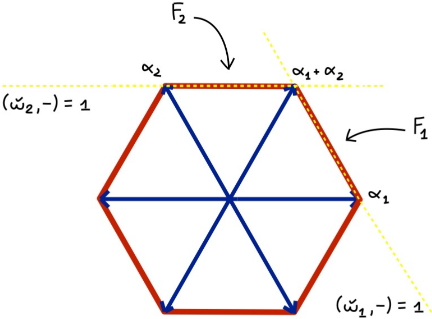
Root poset of A_3:

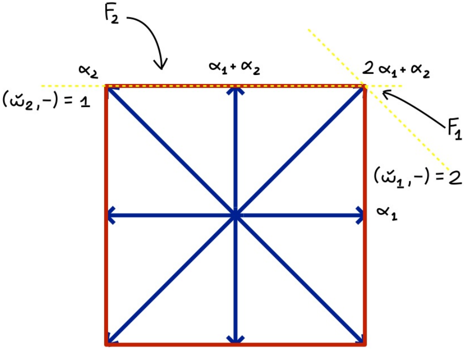


Coordinate faces

For each $i = 1, \ldots, n$

$$F_i = \mathcal{P}_\Phi \cap \{x \mid (\breve{\omega}_i, x) = c_i(\theta)\} \quad \textit{i-th coordinate face}$$





$F_1 = \text{Convex}\{\alpha_1, \alpha_1 + \alpha_2\}$
$F_2 = \text{Convex}\{\alpha_2, \alpha_1 + \alpha_2\}$

$F_1 = \{2\alpha_1 + \alpha_2\} \subset F_2$
$F_2 = \text{Convex}\{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$

Properties of coordinate faces

1. $F_i \neq F_j$ if $i \neq j$
2. the sum of two roots in F_i is never a root
3. if $\alpha, \beta \in \Phi$, $\alpha \in F_i$, $\beta \geq \alpha$, then $\beta \in F_i$
   (dual order ideal in the root poset)
4. $F_i \cap \Phi$ is an interval in the root poset, i.e. $\exists \eta_i \in \Phi^+$ such that
   $$F_i \cap \Phi = [\eta_i, \theta]$$
5. $\dim F_i = \#\{k \mid c_k(\eta_i) \neq c_k(\theta)\}$
6. The barycenter of F_i is parallel to $\breve{\omega}_i$
7. Two coordinate faces are never in the same W-orbit

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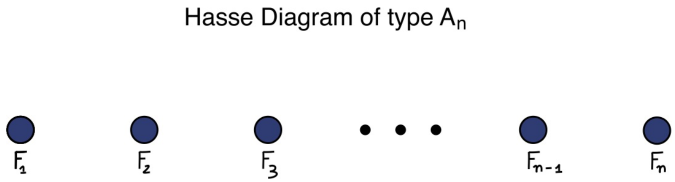
Proposition

- F_i *is a facet* $\iff$ F_i *is maximal among the coordinate faces*
- $F_i \subseteq F_j \iff$ *every path from* α_j *to* α_0 *in the extended Dynkin diagram contains* α_i

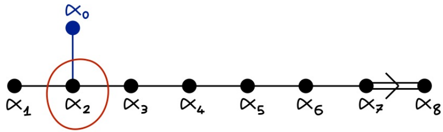
Remark $F_i \subseteq F_j$ means that every root having maximal i-th coordinate (w.r.t. Π) has also maximal j-th coordinate

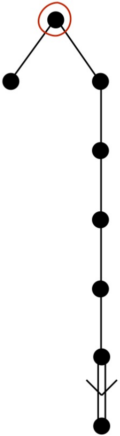
Example



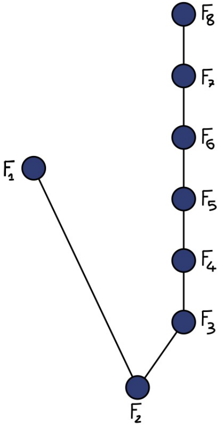
Example

B_8





flip for the Hasse Diagram



Standard parabolic faces

For each $I \subseteq \{1, \ldots, n\}$

$$F_I := \cap_{i \in I} F_i \quad \text{standard parabolic face}$$

Properties

1. $F_I \cap \Phi$ is an interval in the root poset, i.e. $\exists \eta_I \in \Phi^+$ such that

$$F_I \cap \Phi = [\eta_I, \theta]$$

2. $\dim F_I = \#\{k \mid c_k(\eta_I) \neq c_k(\theta)\}$
3. The barycenter of F_I is in the cone generated by $\breve{\omega}_i$, $i \in I$
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$$F_I = F_J \not\Rightarrow I = J$$

Question: For which J we have that F_J equals a prescribed F_I?

Answer: Again the extended Dynkin diagram comes out
$\widehat{\Pi} := \Pi \cup \{\alpha_0\}$
$\Pi_I := \{\alpha_i \mid i \in I\}$
$(\widehat{\Pi} \setminus \Pi_I)_{\alpha_0} :=$ connected component of α_0 in $\widehat{\Pi} \setminus \Pi_I$

Definitions
closure of I

$$\bar{I} \quad := \quad \{k \mid \alpha_k \notin (\widehat{\Pi} \setminus \Pi_I)_{\alpha_0}\}$$

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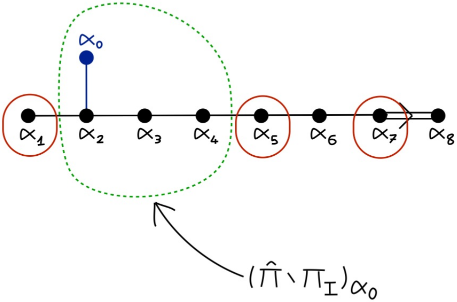
Example

B_8

$I = \{1,5,7\}$



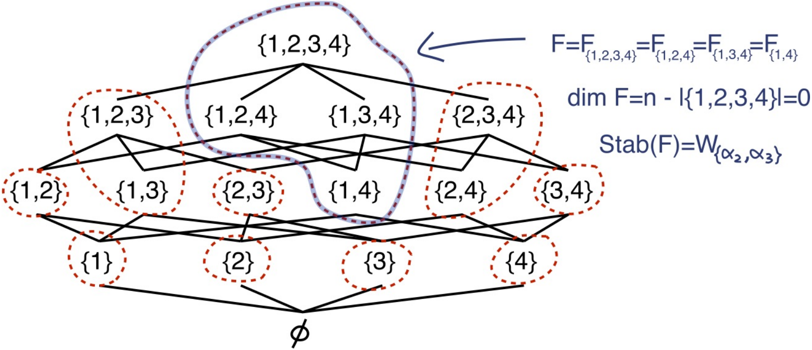
$\bar{I} = \{1,5,6,7,8\}$ $\partial I = \{1,5\}$

Theorem

Fix $I \subseteq \{1,2,\ldots,n\}$. *Then*

1. $F_J = F_I \iff \partial I \subseteq J \subseteq \bar{I}$
2. $\dim F_I = n - |\bar{I}|$
3. $\mathrm{Stab}(F_I) = W_{\Pi \setminus \Pi_{\partial I}}$ $\left(\begin{array}{c} \textit{parabolic subgroup of } W \\ \textit{generated by the refls} \notin \Pi_{\partial I} \end{array} \right)$

Example: type A_4



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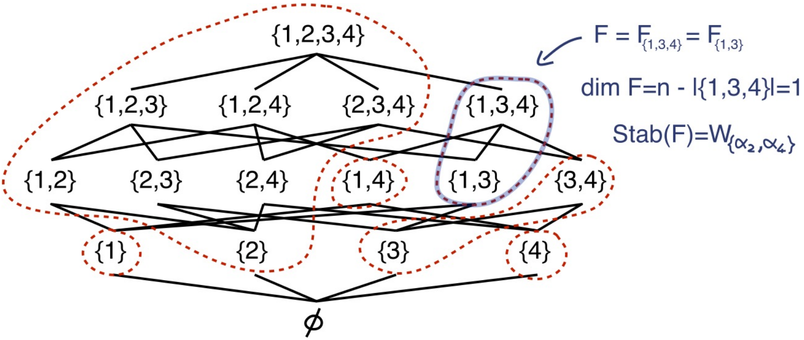

Corollary

There is an inclusion preserving bijection σ

$$\{\text{standard parabolic faces}\} \overset{\sigma}{\longleftrightarrow} \left\{\begin{array}{c}\text{connected subdiagrams of the}\\ \text{ext. Dynkin diag. containing } \alpha_0\end{array}\right\}$$

$$F_I \mapsto (\widehat{\Pi} \setminus \Pi_I)_{\alpha_0}$$

$$F_{\{1,\dots,n\}\setminus\Gamma} \leftarrow\!\shortmid \Pi_\Gamma$$

Remark: $\dim F_I = \#$ *of vertices of* $\sigma(F_I) - 1$

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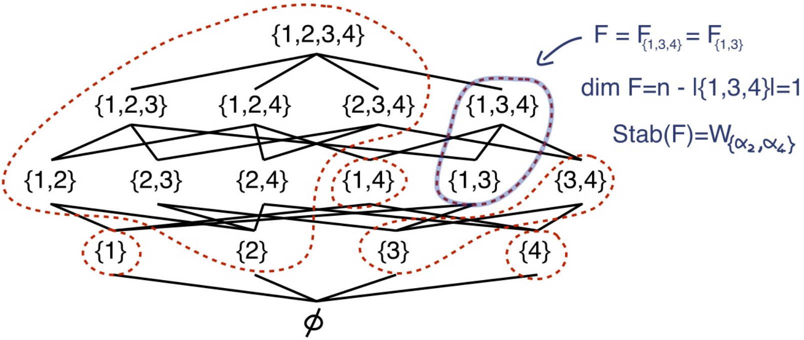

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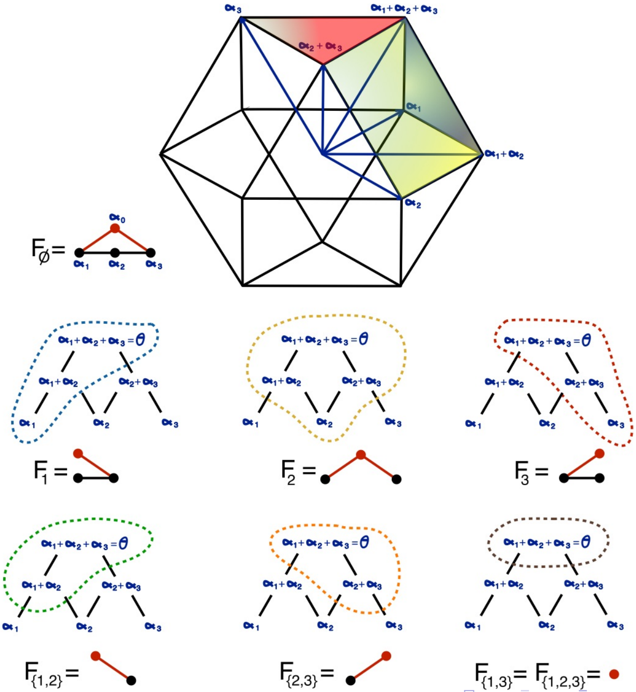

α3
α1+α2+α3
α2 + α3
α1
α1+α2
α2
F∅ =
α0
α1 α2 α3
α1+α2+α3 = θ
α1+α2
α2+α3
α1 α2 α3
F1 =
F2 =
F3 =
F{1,2} =
F{2,3} =
F{1,3} = F{1,2,3} =

Theorem

The standard parabolic faces form a complete set of representatives of the W-orbits of faces (of all dimensions).

Corollary

1. *The W-orbits are parametrized by the connected subdiagrams of the extended Dynkin diagram which contain the affine simple root α_0 (also follows from Vinberg's results)*
2. *Half-space representation:*

$$\mathcal{P}_\Phi = \{x \mid (w\check{\omega}_i, x) \leq c_i(\theta)\}$$

   for all i s. t. $\widehat{\Pi} \setminus \{\alpha_i\}$ is connected and $w \in W^{\alpha_i}$ (coset repr.)
3. *f-polynomial:*

$$\sum_\Gamma [W : W_{\Pi \setminus \Pi_{\partial(\Pi \setminus \Gamma)}}] t^{|\Gamma|}$$

   where the sum is over $\Gamma \subseteq \Pi$ such that $\Gamma \cup \{\alpha_0\}$ is connected

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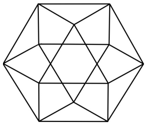
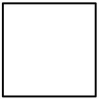
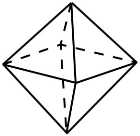
$\mathcal{P}_{A_3} = \mathcal{P}_{B_3} =$ cuboctahedron



$\mathcal{P}_{B_n} = \mathcal{P}_{D_n}$
$\mathcal{P}_{A_n}$
} distinct generalizations of the cuboctahedron

$\mathcal{P}_{C_n} =$ n-dimensional cross-polytope





$\cdots$

$\mathcal{P}_{F_4} = \mathcal{P}_{B_4} = \mathcal{P}_{D_4}$

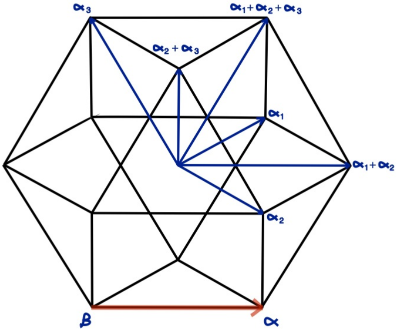
$\mathcal{P}_{G_2} = \mathcal{P}_{A_2}$

The 1-skeleton

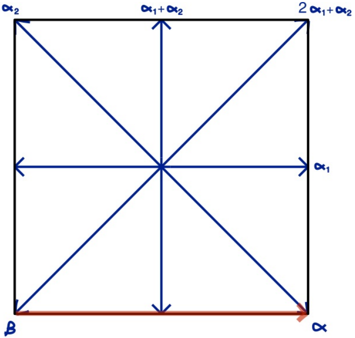
The 1-skeleton is "made out of roots"

Proposition

Let F be a 1-dimensional face with vertices α and β. Then $\alpha - \beta$ is either a root or twice a short root.



$\alpha - \beta = \alpha_1 + \alpha_2$



$\alpha - \beta = 2\alpha_1$

Faces and Abelian ideals of the Borel subalgebras

$\mathfrak{g}$ = complex simple Lie algebra
$\mathfrak{h}$ = Cartan subalgebra
Φ = root system
Π = set of simple roots of Φ
$\mathfrak{b}$ = Borel subalgebra associated with Π

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha \qquad \mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

$\mathfrak{g}_\alpha =$ root space
An Abelian ideal of $\mathfrak{b}$ is a subspace $\mathfrak{i} \subseteq \mathfrak{b}$ such that

1. $[\mathfrak{b}, \mathfrak{i}] \subseteq \mathfrak{i}$
2. $[\mathfrak{i}, \mathfrak{i}] = \{0\}$

These are of the form $\mathfrak{i} = \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ for $\Gamma \subseteq \Phi^+$ such that

1. $(\Phi^+ + \Gamma) \cap \Phi^+ \subseteq \Gamma$
2. $(\Gamma + \Gamma) \cap \Phi^+ = \emptyset$

Recall that, for every standard parabolic face F_I of the root polytope $\mathcal{P}_\Phi$:

1. the sum of two roots in F_I is never a root
2. the roots in F_I form a dual order ideal of the root poset
3. the roots in F_I form an interval $[\eta_I, \theta]$ in the root poset

1, 2, 3 imply that

$$\sum_{\alpha \in F_I} \mathfrak{g}_\alpha$$

is a principal Abelian ideal of $\mathfrak{b}$ (generated by any non-zero vector in $\mathfrak{g}_{\eta_I}$)