Hall-Littlewood symmetric polynomials via “chip firing game”

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...work in progress...

This is a joint-work (in progress) with R. Cori and D. Senato.

Hall-Littlewood polynomials: the $R_\lambda(x;t)$-normalization

If λ is a partition with $l \leq n$ parts and $x = \{x_1, x_2, \ldots, x_n\}$ then we set

$$R_\lambda(x;t) = \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(x^\lambda \prod_{1 \leq i < j \leq n} \frac{x_i - t x_j}{x_i - x_j} \right).$$

Configurations and toppling

If $G = (V, E)$ then a **configuration** of G is a map

$$\alpha\colon v \in V \to \alpha(v) \in \mathbb{Z}.$$

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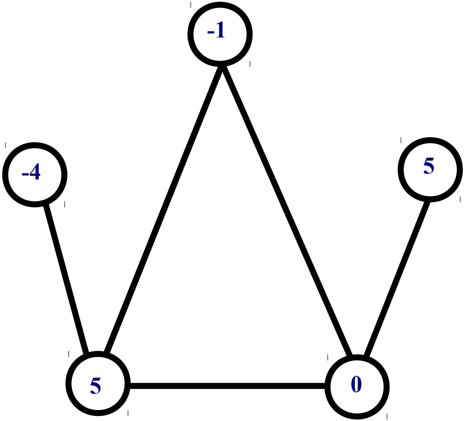
Firing a vertex v produces a **toppling** of its weight,namely

- the weight $\alpha(v)$ of v **decreases** by $deg(v)$;
- the weight $\alpha(w)$ of each neighbour w of v **increases** by 1.

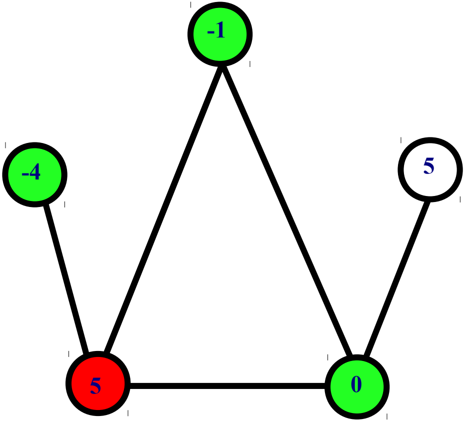
Firing a vertex: an example

Consider the following configuration



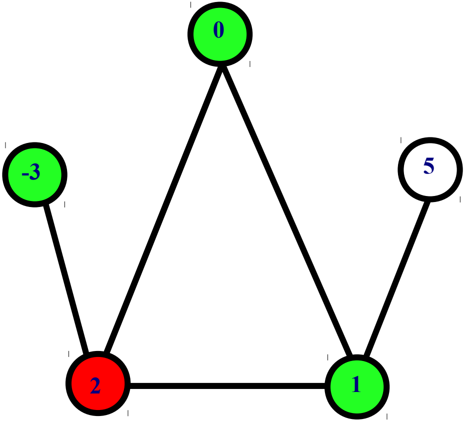
Firing a vertex: an example

Let's fire a vertex



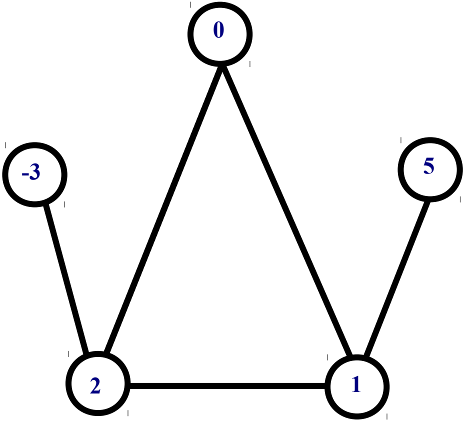
Firing a vertex: an example

new labels are



Firing a vertex: an example

and a new configuration is obtained



Toppling equivalence

If, for a fixed graph G, there exists a sequence of firings that change α in β then we say that α and β are **equivalent**, written

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Toppling equivalence has relations with parking functions and q,t-Catalan numbers [Cori and Le Borgne]

The toppling group

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Define the map $T_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ so that

$$T_i(\alpha) = \beta$$

iff

$$\beta \text{ comes from } \alpha \text{ by firing the vertex } i$$

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The **orbits** of its action on $\mathbb{Z}^n$ are the class of equivalences of $\equiv_G$:

$$\alpha \equiv_G \beta \text{ iff } \beta = T^\lambda(\alpha),$$

where $\lambda \in \mathbb{N}^n$ and $T^\lambda = T_1^{\lambda_1} T_2^{\lambda_2} \cdots T_n^{\lambda_n}$.

The partial order $\leq_G$

We set

$$\beta \leq_G \alpha \text{ iff } \beta = T^{\lambda}(\alpha)$$

for some $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n \geq 0$.

Theorem

1. *$\leq_G$ is a partial order*
2. *if $\beta \leq_G \alpha$ then there exists a* **unique** *λ such that $\lambda_n = 0$ and $\beta = T^{\lambda}(\alpha)$*

The principal order ideal $\mathcal{H}_\alpha$

Instead of studying orbits

$$\mathcal{O}_\alpha = \{\beta \,\big|\, \beta \equiv_G \alpha\}$$

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In particular we can give a description of the the series

$$\mathcal{H}_\alpha(x;t) = \sum_{\beta \leq_G \alpha} t^{dist(\lambda')} x^\beta,$$

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with λ being the unique such that $\lambda_n = 0$ and $\beta = T^\lambda(\alpha)$ and $dist(\lambda')$ is the number of distinct parts of its conjugate.

A first result

$\mathcal{T}$ acts on monomials x^α by following

$$T_i \cdot x^\alpha = x^{T_i(\alpha)}.$$

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We have

$$\mathcal{H}_\alpha(x;t) = \underbrace{\prod_{1 \le i \le n-1} \frac{1-(1-t)[i]}{1-[i]}}_{\mathcal{T}} \cdot x^\alpha,$$

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A first result: a sketch proof

Since

$$T^\lambda = \begin{array}{lllll} R_1 & R_1 & \dots & R_1 & R_1 \\ R_2 & R_2 & \dots & R_2 & \\ \vdots & \vdots & & & \\ R_n & \dots & R_n & & \end{array} \qquad \begin{array}{l} \lambda_1 \text{ times} \\ \lambda_2 \text{ times} \\ \\ \lambda_n \text{ times} \end{array},$$

by multiplying along columns

$$T^\lambda = [\lambda_1'][\lambda_2'] \cdots$$

A first result: a sketch proof

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$$T^\lambda = \begin{matrix} R_1 & R_1 & \dots & R_1 & R_1 & & \lambda_1 \text{ times} \\ R_2 & R_2 & \dots & R_2 & & & \lambda_2 \text{ times} \\ \vdots & \vdots & & & & & \\ R_n & \dots & R_n & & & & \lambda_n \text{ times} \end{matrix},$$

by multiplying along columns

$$T^\lambda = [\lambda'_1][\lambda'_2]\cdots$$

Finally, by expanding τ as a series

$$\tau \cdot x^\alpha = \sum_{\lambda'_i \le n-1} t^{dist(\lambda')}[\lambda'_1][\lambda'_2]\cdots = \sum_{\ell(\lambda)\le n-1} t^{dist(\lambda')} T^\lambda \cdot x^\alpha.$$

Acting on polynomials: basis attached on each graph

An action of $\mathcal{T}$ on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is obtained by means of

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By **symmetrizing** a basis (attached on each G) of symmetric polynomials is obtained

$$\mathcal{H}^*_{\alpha}(x;t) = \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(\mathcal{H}_{\alpha}(x;t)\right).$$

Some manipulation

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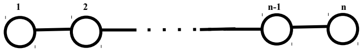
and in particular

$$\tau \cdot x^\alpha = \tau(x;t)\, x^\alpha,$$

for some $\tau(x,t) \in \mathbb{Z}[t]\left[\left[x_1^{\pm 1}, x_2^{\pm 2}, \ldots, x_n^{\pm 1}\right]\right]$ **uniquely determined** by G.

A special case

If $G = L_n$ is the graph



then $[i] = T_1 T_2 \cdots T_i$ "produces" $\alpha_i - 1$ and $\alpha_{i+1} + 1$ and then

$$\prod_{1 \le i \le n-1} \frac{1-(1-t)[i]}{1-[i]} \cdot x^{\alpha} = \prod_{1 \le i \le n} \frac{x_i - (1-t)x_{i+1}}{x_i - x_{i+1}} x^{\alpha}.$$

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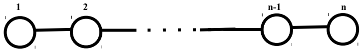

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N.B. $\tau(x;t)$ is obtained by expanding in powers of x_{i+1}/x_i

From toppling to $R_\lambda(x;t)$: looking for elements $[i,j]$'s

The idea is to find elements $[i,j]$ in the toppling group that for L_n give

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so that if

$$\hat{\mathcal{H}}_\alpha(x;t) = \prod_{1\leq i<j\leq n} \frac{1-(1-t)[i,j]}{1-[i,j]}\cdot x^\alpha,$$

then

$$\hat{\mathcal{H}}^*_\alpha(x;t) = \sum_{\sigma\in\mathfrak{S}_n} \sigma\left(\hat{\mathcal{H}}_\alpha(x;t)\right) = R_\alpha(x,1-t)$$

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This is obtained by setting

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Also we have the following combinatorial interpretation:

$$\hat{\mathcal{H}}_\alpha(x;t)=\prod_{1\le i<j\le n}\frac{1-(1-t)[i,j]}{1-[i,j]}\cdot x^\alpha=\sum_{\beta\le_G\alpha}K_{\alpha,\beta}(t)x^\beta,$$

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where

$$K_{\alpha,\beta}(t) = \sum_{\beta=[i_1,j_1]^{a_1}[i_2,j_2]^{a_2}\cdots[i_k,j_k]^{a_k}(\alpha)} t^k,$$

with the $[i_1,j_1],[i_2,j_2],\ldots,[i_k,j_k]$ pairwise distinct.

Reassuming

We have

1. via topplings, for each graph we define a partial order $\leq_G$
2. (for each graph!) we construct a basis of symmetric polynomials $(\hat{\mathcal{H}}^*_\alpha(x;t))_\alpha$
3. such basis encodes principal order ideals of $\leq_G$ and reduces to Hall-Littlewood symmetric polynomials when $G = L$

What's more?

1. more parameters t, q, z related to further statistics of the unique λ and more general polynomials $(\hat{\mathcal{H}}^*_\alpha(x; t, q, z))_\alpha$ may be constructed (do they share some property with Macdonald polynomials?)

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3. do special choices of G (trees, cyclic graphs,...) give rise to interesting basis?

Thanks

Many thanks for your attention!