Combinatoire rétrospective et créative

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Overview

1. Around 1980: From the early lotharingian days

2. 2013: Hermite and Laguerre are still alive!

Favorite subjects

- From the very beginning (1980)
  - Combinatorics of the symmetric group
  - Symmetric functions
  - Classical numbers and polynomials (Tangent/Secant, Genocchi, eulerian polynomials,...)
  - Hypergeometric series and special functions (classical orthogonal polynomials, q-analogues)
- Approach
  - Reveal counting properties of these objects
  - *Retrospective:*
    Understand known properties (identities) from combinatorial models
  - *Creative:*
    Use these combinatorial models to invent meaningful refinements, variations, generalizations . . .

Some of my favorite examples: bi-(multi-)linear generating functions

- Hermite polynomials (*matchings*)
  - Proof of the (bilinear) Mehler formula [Foata]
  - (Multilinear) Kibble-Slepian-Louck identity [Foata, Garsia]
- Laguerre polynomials (*injective partial functions*)
  - Proof of the (bilinear) Hille-Hardy identity [Foata, Strehl]
  - Bilinear Meixner identity [Foata]
  - Bilinear identities by Weisner and Srivastava [Strehl]
  - Multilinear extension of Hille-Hardy and Erdélyi [Foata, Strehl]
- Jacobi polynomials (*pairs of injective partial functions*)
  - Generating function [Foata, Leroux]
  - Bailey's bilinear generating function [Strehl]

The vocabulary: Hermite

- explicit

$$H_n(x) = (2x)^n {}_2F_0 \left[\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix} ; -\frac{1}{x^2} \right]$$
$$= \sum_{0 \leq k \leq n} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}$$

- generating function

$$\sum_{n \geq 0} \frac{t^n}{n!} H_n(x) = e^{2xt - t^2}$$

The vocabulary: Hermite

- combinatorial interpretation

$$\mathcal{H}[U,V] = \text{involutions } \sigma \text{ on } U \uplus V \text{ with } \mathrm{fix}(\sigma) = U$$

$$\text{valuation: } \sigma \mapsto u^{\sharp\mathrm{fix}(\sigma)} v^{2\,\sharp\mathrm{trans}(\sigma)} = u^{\sharp U} v^{\sharp V}$$

$$\begin{aligned}\mathcal{H}_n(u,v) &= \sum_{U \uplus V = [n]} \sum_{\sigma \in \mathcal{H}[U,V]} u^{\sharp\mathrm{fix}(\sigma)} v^{2\,\sharp\mathrm{trans}(\sigma)} \\ &= \sum_{K \subseteq \{1,\dots,n\}} m_{\sharp K}\, u^{n-\sharp K} v^{\sharp K}\end{aligned}$$

- special case “perfect matchings”

$$\mathcal{M}[V] = \mathcal{H}[\emptyset, V], \quad m_{2k} = \sharp\mathcal{M}[\{1,\dots,2k\}] = \frac{(2k)!}{k!\,2^k}$$

A classic: Foata's proof of the Mehler formula

JOURNAL OF COMBINATORIAL THEORY, Series A 24, 367–376 (1978)

A Combinatorial Proof of the Mehler Formula

DOMINIQUE FOATA*

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Communicated by the Managing Editors

Received October 26, 1977

DEDICATED TO JOHN RIORDAN WITH RESPECT AND ADMIRATION ON THE OCCASION OF HIS 75TH BIRTHDAY

A classic: Foata's proof of the Mehler formula

using the partitional complex techniques described below) the well-known closed formula for the exponential generating function for the Hermite polynomials

$$1 + \sum_{n \geqslant 1} H_n(a)\, u^n/n! = \exp\,(2au - u^2).$$

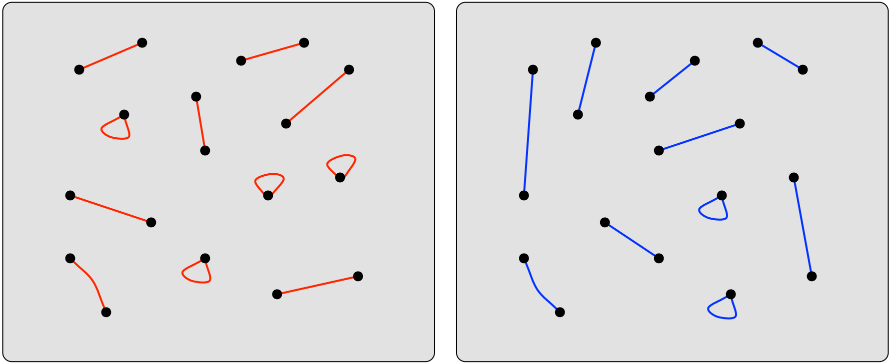
The following bilinear expansion is known as Mehler's formula

$$1 + \sum_{n \geqslant 1} H_n(a)\, H_n(b)\, u^n/n! = (1 - 4u^2)^{-1/2} \exp\left[\frac{4abu - 4(a^2 + b^2)\, u^2}{1 - 4u^2}\right]. \quad (4)$$

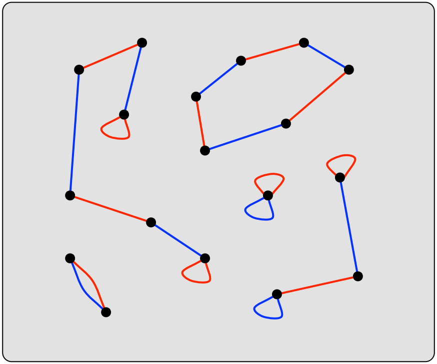
There exist several classical analytic proofs of (4) (see, e.g., Watson [12]). The identity plays an essential role in the study of the positivity of the Poisson kernels for series of orthogonal polynomials (see Askey [1]). It has also been extended to the multilinear case by Carlitz [3] and Slepian [11].

The purpose of this paper is to show how the above interpretation of the Hermite polynomials in terms of statistical distributions over involution sets also provides a combinatorial proof of Mehler's formula.

A classic: Foata's proof of the Mehler formula

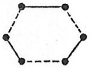
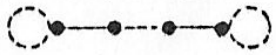
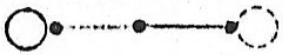


A classic: Foata's proof of the Mehler formula



A classic: Foata's proof of the Mehler formula

TABLE I

Type	n	Set	$\mu(y)$	card set	$\mu\{\text{set}\}$
	Even	A_n	$(-2)^n$	$(n-1)!$	$\mu\{A_n\} = 2^n(n-1)!$
	Odd	ϕ	—	0	$= 0$
	Even	B_n	$(-2)^{n-1}(2a)^2$	$n!/2$	$\mu\{B_n\} = -a^2\, 2^n\, n!$
	Odd	ϕ	—	0	$= 0$
	Even	C_n	$(-2)^{n-1}(2b)^2$	$n!/2$	$\mu\{C_n\} = -b^2\, 2^n\, n!$
	Odd	ϕ	—	0	$= 0$
	Even	ϕ	—	0	$\mu\{D_n\} = 0$
	Odd	D_n	$(-2)^{n-1} 2a\, 2b$	$n!$	$= 2ab\, a^n n!$

A classic: Foata's proof of the Mehler formula

The combinatorial proof of Mehler's formula is then completed.

ACKNOWLEDGMENTS

The author wishes to thank professor Schützenberger for conveying him his belief that Mehler's formula was to be proved combinatorially, and professor Askey for encouraging him to write up the proof.

The vocabulary: Laguerre

- explicit

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n\\1+\alpha\end{matrix};x\right] = \sum_{0\le k\le n}(-1)^k\,\frac{(1+\alpha+k)_{n-k}}{k!\,(n-k)!}\,x^k$$

- generating function

$$\sum_{n\ge 0} t^n L_n^{(\alpha)}(x) = \frac{1}{(1-t)^{1+\alpha}}\,e^{-xt/(1-t)}$$

The vocabulary: Laguerre

- combinatorial interpretation

$$\mathcal{L}[X,Y] = \text{injective mappings } \lambda : X \to X \uplus Y$$

$$\text{valuation: } \lambda \mapsto \alpha^{\mathsf{cyc}(\lambda)} x^{\sharp X} y^{\sharp Y}$$

$$\mathcal{L}_n^{(\alpha)}(x,y) = \sum_{X \uplus Y = [n]} \sum_{\lambda \in \mathcal{L}[X,Y]} \alpha^{\mathsf{cyc}(\lambda)} x^{\sharp X} y^{\sharp Y}$$

- Simple fact:

$$\sum_{\lambda \in \mathcal{L}[X,Y]} \alpha^{\mathsf{cyc}(\lambda)} = (\alpha + \sharp Y)_{\sharp X}$$

Bilinear Laguerre: Hille-Hardy

$$\sum_{n\geq 0} \frac{t^n}{n!} \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{(\alpha+1)_n} = \frac{1}{(1-t)^{\alpha+1}} e^{-(x+y)t/(1-t)} \sum_{n\geq 0} \frac{1}{n!(\alpha+1)_n} \left(\frac{xyt}{(1-t)^2}\right)^n$$

CRAS 1981

C. R. Acad. Sc. Paris, t. 293 (16 novembre 1981) Série I – 517

FONCTIONS SPÉCIALES. – *Une extension multilinéaire de la formule d'Erdélyi pour les produits de fonctions hypergéométriques confluentes.* Note (*) de **Dominique Foata** et **Volker Strehl,** transmise par Marcel-Paul Schützenberger.

La formule bilinéaire de Hille-Hardy pour les produits de polynômes de Laguerre a été étendue au cas des fonctions hypergéométriques confluentes par Erdélyi. Nous proposons ici une version multilinéaire de l'identité d'Erdélyi, qui est l'analogue pour les polynômes de Laguerre de la formule de Kibble-Slepian pour les polynômes d'Hermite.

The Hille-Hardy bilinear formula for products of Laguerre polynomials has been extended to the confluent hypergeometric functions by Erdélyi. Here a multilinear version of the latter identity is proposed that is the analog for the Laguerre polynomials of the Kibble-Slepian formula for the Hermite polynomials.

ICM 1983

Proceedings of the International Congress of Mathematicians
August 16–24, 1983, Warszawa

DOMINIQUE FOATA

Combinatoire des identités sur les polynômes orthogonaux

L'étude combinatoire récente des identités sur les polynômes orthogonaux est passée en revue. A titre d'illustration, on établit, par des méthodes combinatoires, une extension de la formule du noyau de Poisson pour les polynômes de Meixner.

ICM 1983

L'étude combinatoire des identités sur les fonctions spéciales a été entreprise dans les dernières années par différentes écoles, bostonienne ([27], [28], [29], [35], [36], [38]), californienne ([19], [24], [25], [26], [44]), lotharingienne ([13], [14], [16], [17], [18], [22], [39], [40], [41]), québécoise ([20], [21], [31], [33]) et viennoise ([10], [11], [30]). Comme le dit fort justement notre ami Adriano Garsia [26], "les fonctions spéciales et les identités des mathématiques classiques recèlent une information abondante. Cette information s'exprime sous forme de correspondances entre structures finies qu'il s'agit de dégager. Les identités classiques apparaissent alors comme de simples relations entre ces structures comptées suivant des statistiques appropriées. Une étude systématique est en cours et a pour but de déterrer cette information de la littérature classique. Ce riche inventaire de correspondances a permis d'établir de nouvelles identités et d'obtenir aussi des démonstrations très explicites des formules classiques."

ICM 1983

Pour établir (1.1) il suffit donc d'établir l'identité *polynômiale*

$$\sum w_\beta(\sigma) w(\gamma, -x, -a; \varphi) w(\delta, -y; -b; \psi)$$
$$= \sum \binom{n}{q,r,s,i,j} (\beta)_q (\beta)_r (\beta+r)_i (\beta+r)_j (2r+i+j)_s \times$$
$$\times (\gamma+r+i)_{n-r-i} (\delta+r+j)_{n-r-j} (-x)_{r+i} (-y)_{r+j} \times$$
$$\times (-a)^{r+i} (-b)^{r+j} \quad (q+r+s+i+j=n). \qquad (2.7)$$

3. Lemmes combinatoires

L'identité (2.7) est beaucoup moins effrayante qu'il n'y paraît, car tous les termes ont une signification combinatoire qu'on va maintenant donner. Les trois lemmes ci-après sont extraits de l'article sur la formule d'Erdélyi–Hille–Hardy pour les polynômes de Laguerre [22]. Comme dans (2.3), si h est une injection d'un ensemble fini, on pose $w_\beta(h) = \beta^{\mathrm{cyc}(h)}$.

- In October 2012, D. Babusci, G. Dattoli (both from Roma), K Górska (Kraków) and K.A. Penson (Paris) published a list of 12 lacunary generating series involving Laguerre (and Hermite) polynomials:

  Generating Functions for Laguerre Polynomials: New identities for Generating Series, arXiv1210.3710 [math-ph]
- Karol Penson asked me, what a combinatorialist might see in these identities
- My résumé today:
  All identieties are *very combinatorial*, and combinatorics can help to systematize and extend them considerably

Lacunary Laguerre series – what do they really count?

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}(x) = e^t \sum_{r\geq 0} \frac{(i\,x\,\sqrt{t})^r}{(r!)^2}\, H_r(i\,\sqrt{t})$$

$$\sum_{n\geq 0} \frac{t^{2n}}{(2n)!} \frac{(2n)!}{n!2^n} L_{2n}(x) = \ldots$$

$$\sum_{n\geq 0} \frac{t^{2n}}{(2n)!}\, m_{2n}\, L_{2n}(x) = \ldots$$

$$\sum_{n\geq 0} \frac{\mathcal{H}_n(u,v)}{n!} \, \frac{\mathcal{L}_n(x,y)}{n!} = \quad ?$$

$$\sum_{n\geq 0} \frac{\mathcal{H}_n(u,v)}{n!} \, \frac{\mathcal{L}_n^{(\alpha)}(x,y)}{(\alpha)_n} = \quad ?$$

Lacunary Laguerre series – what do they really count?

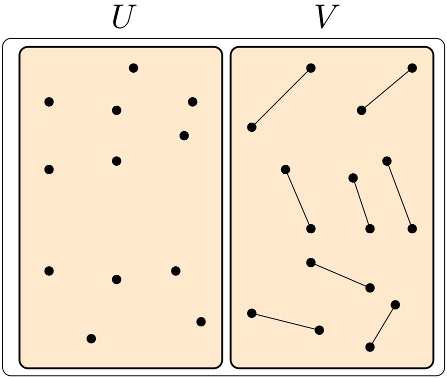
- The lacunary generating series

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}(x) = e^t \sum_{r\geq 0} \frac{(i\,x\,\sqrt{t})^r}{(r!)^2}\, H_r(i\,\sqrt{t})$$

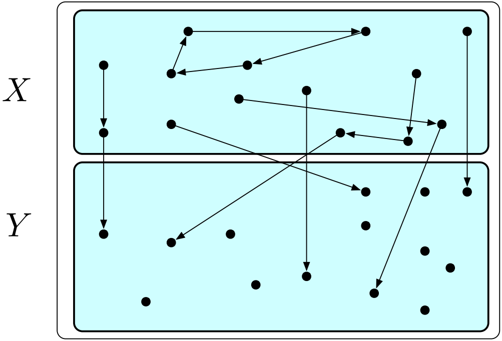
  is essentially a (very) special case ($u = 0$) of a joint bilinear generating function for Hermite and Laguerre
- Combinatorics tells us, what the full bilinear gf should be:

$$\sum_{n\geq 0} \frac{\mathcal{H}_n(u,v)}{n!}\,\frac{\mathcal{L}_n^{(\alpha)}(x,y)}{(\alpha)_n} = \mathcal{M}(xv) \sum_{s,p\geq 0} \frac{\mathcal{H}_s(xyv^2,yv)}{s!}\,\frac{\mathcal{L}_p^{(\alpha+s)}(xu,yu)}{p!\,(\alpha)_{s+p}}$$

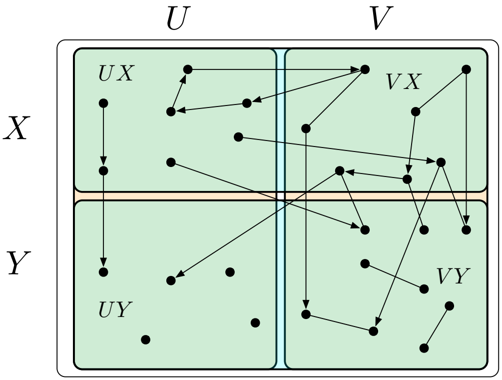
- The fully combinatorial proof is based on a superposition of Hermite configurations and Laguerre configurations



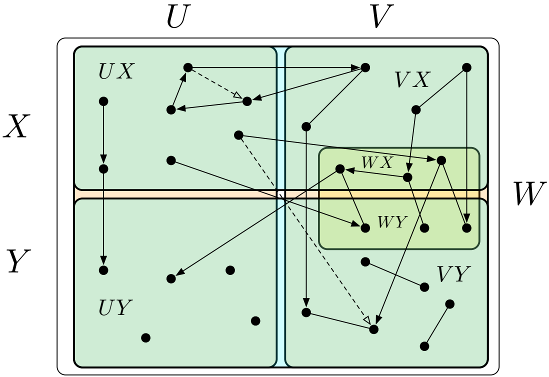
$$\sigma \in \mathcal{H}[U, V]$$



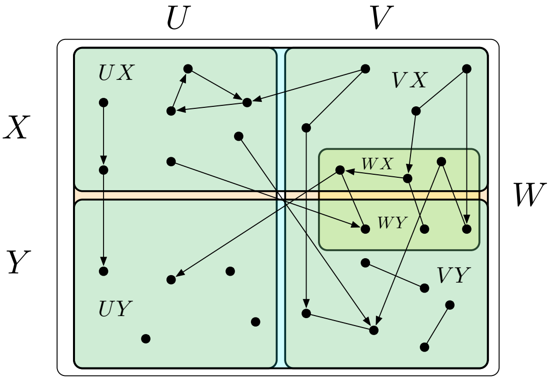
$$\lambda \in \mathcal{L}[X, Y]$$



$$(\sigma, \lambda) \in \mathcal{H}[U, V] \times \mathcal{L}[X, Y]$$



β : transpositions of σ between X and Y



$$\sigma_{VX} \in \mathcal{H}[\emptyset, VX \setminus WX], \quad \sigma_{VY} \in \mathcal{H}[WY, VY \setminus WY]$$

$$\beta \in \mathcal{B}[WY, WX], \quad \lambda_{VX} \in \mathcal{L}[VX, UX \uplus Y], \quad \lambda_{UX} \in \mathcal{L}[UX, Y]$$

A new kind of lacunary series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(1)}(x) = e^t \sum_{r=0}^{\infty} \frac{p_2(r;x,t)}{r!(r+3)!} (ix\sqrt{t})^r H_r(i\sqrt{t}),$$

$$p_2(r;x,t) = (1+2t)r^2 + (5-4xt+10t)r + (6+12t-12xt+2tx^2);$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(2)}(x) = e^t \sum_{r=0}^{\infty} \frac{p_4(r;x,t)}{r!(r+6)!} (ix\sqrt{t})^r H_r(i\sqrt{t}), \tag{4}$$

$$\begin{aligned} p_4(r;x,t) = {} & (2+10t+4t^2)r^4 + [36+(180-20x)t+(72-16x)t^2]r^3 \\ & + [238+(1190-300x+10x^2)t+(476-240x+24x^2)t^2]r^2 \\ & + [684+(3420-1480x+110x^2)t+(1368-1184x+264x^2-16x^3)t^2]r \\ & + 720+(3600-2400x+300x^2)t \\ & + (1440-1920x+720x^2-96x^3+4x^4)t^2; \end{aligned} \tag{5}$$

A new kind of lacunary series

- Some obvious questions
  - Why are these identities interesting? Ask the authors
  - Are these identities true? Yes!
  - Is it true that for any $k \geq 0$ there is always a polynomial $p_{2k}(r;x,t)$ of degree $2k$ in r with integer coefficients s.th.

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}^{(k)}(x) = e^t \sum_{r\geq 0} \frac{p_{2k}(r;x,t)}{r!(r+3k)!} (i\,x\,\sqrt{t})^r H_r(i\,\sqrt{t})$$

  - How does this generalize to

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}^{(\alpha+k)}(x) = \ldots \quad ?$$

  - What about

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{3n}^{(\alpha+k)}(x) = \ldots \quad ?$$

  - What is the counting behind these identities? You will see ...

A new kind of lacunary series

Assuming the existence of a polynomial $p_{2k}(r;x,t)$ of degree $2k$ in

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}^{(k)}(x) = e^t \sum_{r\geq 0} \frac{p_{2k}(r;x,t)}{r!(r+3k)!} (i\,x\sqrt{t})^r H_r(i\sqrt{t})$$

you can ask your CA software to compute it for any specific k ...

- We know the cases $k = 0, 1, 2$
- Here is what you get for $k = 3$ ...

$$\begin{aligned}
&\left(8\,t^3+48\,t^2+54\,t+6\right)r^6\\
&+\left(48\,xt^3+312\,t^3+192\,xt^2+1872\,t^2+108\,xt+2106\,t+234\right)r^5\\
&+\left(120\,x^2t^3+1680\,xt^3+288\,x^2t^2+5000\,t^3+6720\,xt^2+54\,x^2t\right.\\
&\left.+30000\,t^2+3780\,xt+33750\,t+3750\right)r^4\\
&+\left(160\,x^3t^3+3600\,x^2t^3+192\,x^3t^2+23280\,xt^3+8640\,x^2t^2\right.\\
&\left.+42120\,t^3+93120\,xt^2+1620\,x^2t+252720\,t^2+52380\,xt+284310\,t+31590\right)r^3\\
&+\left(120\,x^4t^3+3840\,x^3t^3+48\,x^4t^2+40200\,x^2t^3+4608\,x^3t^2+159600\,xt^3\right.\\
&+96480\,x^2t^2+196592\,t^3+638400\,xt^2+18090\,x^2t\\
&\left.+1179552\,t^2+359100\,xt+1326996\,t+147444\right)r^2\\
&+\left(48\,x^5t^3+2040\,x^4t^3+30560\,x^3t^3+816\,x^4t^2+198000\,x^2t^3\right.\\
&+36672\,x^3t^2+541152\,xt^3+475200\,x^2t^2+481728\,t^3+2164608\,xt^2+89100\,x^2t\\
&\left.+2890368\,t^2+1217592\,xt+3251664\,t+361296\right)r\\
&+8\,x^6t^3+432\,x^5t^3+8640\,x^4t^3+80640\,x^3t^3+3456\,x^4t^2+362880\,x^2t^3\\
&+96768\,x^3t^2+725760\,xt^3+870912\,x^2t^2+483840\,t^3+2903040\,xt^2\\
&+163296\,x^2t+2903040\,t^2+1632960\,xt+3265920\,t+362880
\end{aligned}$$

... where the coefficient polynomials don't simplify in any reasonable way

- ... but, if you change the priority of the variables, you can get

$$
\begin{aligned}
&8\,x^6t^3 + 48\,x^5t^3\,(r+9) + 24\,t^2\,(r+9)\,(r+8)\,(5\,t+2)\,x^4 \\
&+ 32\,t^2\,(r+9)\,(r+8)\,(r+7)\,(5\,t+6)\,x^3 \\
&+ 6\,t\left(20\,t^2+48\,t+9\right)(r+9)\,(r+8)\,(r+7)\,(r+6)\,x^2 \\
&+ 12\,t\left(4\,t^2+16\,t+9\right)(r+5)\,(r+9)\,(r+8)\,(r+7)\,(r+6)\,x \\
&+ 8\,(r+9)\,(r+8)\,(r+7)\,(r+6)\,(r+5)\,(r+4)\,t^3 \\
&+ 31590\,r^3 + 6\,r^6 + 234\,r^5 + 3750\,r^4 + 147444\,r^2 \\
&+ 48\,(r+9)\,(r+8)\,(r+7)\,(r+6)\,(r+5)\,(r+4)\,t^2 \\
&+362880+54\,(r+9)\,(r+8)\,(r+7)\,(r+6)\,(r+5)\,(r+4)\,t+361296\,r
\end{aligned}
$$

- This suggest to look at cases $k = 1$ and $k = 2$ again:
  - $k = 1$

$$2\,x^2 t + 4\,xt\,(r+3) + 2\,(r+3)\,(r+2)\,t + 6 + r^2 + 5\,r$$

  - $k = 2$

$$\begin{aligned}4\,x^4 t^2 + 16\,x^3 t^2\,(r+6) + 2\,t\,(r+6)\,(r+5)\,(12\,t+5)\,x^2 \\ + 4\,t\,(r+5)\,(r+4)\,(r+6)\,(4\,t+5)\,x \\ + 4\,(r+5)\,(r+4)\,(r+3)\,(r+6)\,t^2 \\ +10\,(r+5)\,(r+4)\,(r+3)\,(r+6)\,t + 238\,r^2 + 684\,r + 2\,r^4 + 36\,r^3 + 720\end{aligned}$$

- A pattern emerges!

- This suggests that in general one should have

$$\sum_{n\geq 0} L_{2n}^{(k)}(x)\frac{t^n}{n!} = e^t \sum_{r\geq 0} \frac{\tilde{p}_{2k}(r;-x,t)}{r!}\,(i\,x\,\sqrt{t})^r\,H_r(i\sqrt{t})$$

where

$$\tilde{p}_{2k}(r;x,t) = \sum_{i=0}^{2k}\sum_{j=0}^{k} x^i\,t^j\binom{2j}{i}\frac{1}{(r+k+i)!}\,a_{k,j}$$

with (positive integer) coefficients $(a_{k,j})_{0\leq j\leq k}$.

- The triangular scheme for the coefficients $(a_{k,\ell})_{0\leq\ell\leq 6}$

$k\backslash\ell$	0	1	2	3	4	5	6
0	1						
1	1	2					
2	2	10	4				
3	6	54	48	8			
4	24	336	492	176	16		
5	120	2400	5100	2920	560	32	
6	720	19440	55800	45240	13680	1632	64

which agrees with the recurrence

$$a_{k,\ell} = 2a_{k-1,\ell-1} + (k+2\ell)\, a_{k-1,\ell}$$

- It is more convenient to consider the numbers

$$\overline{a}_{k,\ell} = a_{k,\ell}\,\frac{\ell!}{k!}$$

for which

$$k\,\overline{a}_{k,\ell} = 2\,\ell\,\overline{a}_{k-1,\ell-1} + (k+2\ell)\,\overline{a}_{k-1,\ell}$$

and

$$\sum_{k\geq 0}\sum_{0\leq\ell\leq k} \overline{a}_{k,\ell}\,t^{\ell}\,z^{k} = \frac{1-z}{1-2(1+t)\,z+(1+t)\,z^2}$$

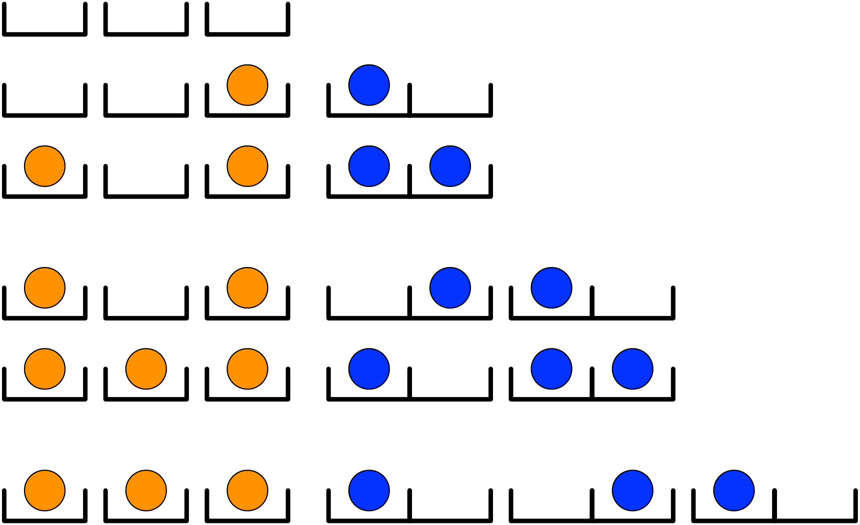
- with first values

$k\backslash\ell$	0	1	2	3	4	5	6
0	1						
1	1	2					
2	1	5	4				
3	1	9	16	8			
4	1	14	41	44	16		
5	1	20	85	146	112	32	
6	1	27	155	377	456	272	64

Balls into boxes

- These numbers are known to the OEIS, counting *order-consecutive partitions* (Hwang-Mallows, JCT-A 70, 1995)
- Another combinatorial interpretation is more convenient here:
  - k (ordred) singleton boxes which can contain at most one ball
  - ℓ (ordered) double boxes, which can contain at most one ball in each of two compartments
  - $\overline{a}_{k,\ell}$ is the number of distributions of balls into the boxes such that
    - each double box contains at least one ball
    - the total number of balls in singleton boxes equals the total number of balls in double boxes

Illustration for $k = 3, 0 \leq \ell \leq 3$



The proof of

$$\sum_{n\geq 0} L_{2n}^{(k)}(x)\frac{t^n}{n!} = e^t \sum_{r\geq 0} \frac{\tilde{p}_{2k}(r;-x,t)}{r!}\,(i\,x\,\sqrt{t})^r\, H_r(i\sqrt{t})$$

follows from

$$m_{2n}\binom{2n}{s}\binom{2n+k}{k}$$
$$= \sum_{2a+2b+i+j=2n} \binom{2n}{2a,2b,i,j} m_{2a}\, m_{2b}\, m_{i+j}\, 2^{2b-s+i}\binom{b}{s-i-b}\overline{a}_{k,(i+j)/2}$$

Remember Foata's principle

Pour établir (1.1) il suffit donc d'établir l'identité *polynômiale*

$$\sum w_\beta(\sigma)\, w(\gamma,\, -x,\, -a;\varphi)\, w(\delta,\, -y;\, -b;\, \psi)$$
$$= \sum \binom{n}{q,r,s,i,j} (\beta)_q (\beta)_r (\beta+r)_i (\beta+r)_j (2r+i+j)_s \times$$
$$\times (\gamma+r+i)_{n-r-i} (\delta+r+j)_{n-r-j} (-x)_{r+i} (-y)_{r+j} \times$$
$$\times (-a)^{r+i} (-b)^{r+j} \qquad (q+r+s+i+j=n). \tag{2.7}$$

3. Lemmes combinatoires

L'identité (2.7) est beaucoup moins effrayante qu'il n'y paraît, car tous les termes ont une signification combinatoire qu'on va maintenant donner. Les trois lemmes ci-après sont extraits de l'article sur la formule d'Erdélyi–Hille–Hardy pour les polynômes de Laguerre [22]. Comme dans (2.3), si h est une injection d'un ensemble fini, on pose $w_\beta(h) = \beta^{\mathrm{cyc}(h)}$.

There are the two special cases

- $k = 0$

$$m_{2n}\binom{2n}{s} = \sum_{2a+2b=2n} \binom{2n}{2a, 2b} m_{2a}\, m_{2b}\, 2^{2b-s} \binom{b}{s-b}$$

  We know this already!

- $s = 0$

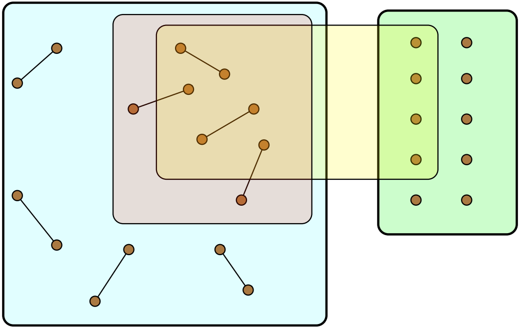
$$m_{2n}\binom{2n+k}{k} = \sum_{2a+2c=2n} \binom{2n}{2a, 2c} m_{2a}\, m_{2c}\, \overline{a}_{k,c}$$

  This is immediate from a combinatorial picture

The general case is by amalgamating these combinatorial views

Illustration fo the second special case

- $n = 8, k = 10, \ell = 4$



The α-extension

$$\sum_{n\geq 0} \frac{t^{2n}}{n!} \frac{(2n)!}{(\alpha)_{2n}} L_{2n}^{(\alpha+k-1)}(x)$$

$$= e^t \sum_{r\geq 0} \frac{1}{r!} \, p_k^{(\alpha)}(r,-x,t) \, (i\,x\sqrt{t})^r \, H_r(i\sqrt{t})$$

holds with an appropriate extension of the polynomials $p_k^{(\alpha)}(r,x,t)$

three step Lacunary Laguerre, $k = 1$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{3n}^{(1)}(x) = \mathrm{e}^t \sum_{r=0}^{\infty} \frac{q_3(r;x,t)}{(r+4)!} \left[\sum_{s=0}^{[r/3]} \frac{(-tx^3)^s (ix\sqrt{3t})^{r-3s}}{s!(r-3s)!} H_{r-3s}\left(i\frac{\sqrt{3t}}{2} \right) \right];$$

$$\begin{aligned} q_3(r;x,t) &= (1+3t)r^3 + (9+27t-9tx)r^2 + (26+78t-63tx+9tx^2)r \\ &+ (24+72t-108tx+36tx^2-3tx^3); \end{aligned}$$

three step Lacunary Laguerre, general case

$$\sum_{n\geq 0} L_{3n}^{(k)} \frac{z^n}{n!} = e^t \cdot \sum_{r\geq 0} q_k(r;-x,t) \sum_{s\geq 0}^{r/3} \frac{(-tx^3)^2\,(ix\sqrt{3t})^{r-3s}}{(r-3s)!\,s!} H_{r-3s}(i\sqrt{3t}/2)$$

where

$$q_k(r;x,t) = \sum_{i=0}^{3k}\sum_{j=0}^{k} x^i t^j \binom{3j}{i} \frac{1}{(r+k+i)!} b_{k,j}$$

and where the polynomials

$$b_k(t) = \sum_{\ell=0}^{k} b_{k,\ell}\, t^\ell$$

are defined via the “balls into boxes” linear recurrence

$$b_{k,\ell} = 3\, b_{k-1,\ell-1} + (k+3\ell)\, b_{k-1,\ell},$$

now using triple boxes (not yet in OEIS)

A combinatorial exercise for you

Find out what kind of counting is behind

$$\sum_{n=0}^{\infty} t^n L_{2n}^{(\alpha-2n)}(x) = (1-t)^{\frac{\alpha}{2}} \cosh\left[\sqrt{t}x - i\alpha \arcsin\left(\frac{\sqrt{t}}{\sqrt{t-1}}\right)\right].$$