Counting factorizations of Coxeter elements

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The classical situation

Cayley's formula [1889]

counting rooted trees

$$\#\{\tau_1 \cdots \tau_{N-1} = (1 \cdots N)\} = N^{N-2}$$

transpositions
reflections

long cicle
Coxeter elt

EX : $N = 3$ $\quad (123) = (12)(23)$

$= (13)(12)$

$= (23)(13)$

Thm [SSV 1997] Shapiro Shapiro Vainshtein

$$\mathrm{FACT}(t) := \sum_{g \geq 0} \frac{t^{N-1+2g}}{(N-1+2g)!} \# \{ \tau_1 \cdots \tau_{N-1+2g} = (1 \cdots N) \}$$

$$= \frac{1}{N!} \left(e^{t\frac{N}{2}} - e^{-t\frac{N}{2}} \right)^{N-1}$$

exponential gen. function of "higher genus factorizations"

Generalizing the situation

(W,S) Coxeter system

$|S| = n = \mathrm{rk}\, W$

$c = \prod_{s \in S} s$ Coxeter element

$h = \mathrm{ord}(c)$ Coxeter number

<u>Thm</u> [Deligne 1972]

$$\#\{t_1 \cdots t_n = c\} = \frac{n!}{|W|} h^n$$

<u>Ex</u>

$W = B_n$

$$\frac{n!}{2^n n!} (2n)^n = n^n$$

$n=2$ $\quad (1 2 \bar{1} \bar{2}) = (12)(2\bar{2}) = (1\bar{1})(12)$

$= (1\bar{2})(1\bar{1}) = (2\bar{2})(12)$

Thm [Chapuy-S. 2012]

$\mathrm{FACT}_W(t) :=$ $\{wsw^{-1} : w \in W, s \in S\}$

$$\sum_{\ell \geq 0} \frac{t^\ell}{\ell!} \#\{(\tau_1, \ldots, \tau_\ell) \in \overset{\shortparallel}{\mathcal{R}}{}^\ell : \tau_1 \cdots \tau_\ell = c\}$$

$$= \frac{1}{|W|} \left(e^{th/2} - e^{-th/2}\right)^n$$

Generalizing even further

Complex refl. gr.

$V = \mathbb{C}^n$, $w \in GL(V)$, $\mathrm{Fix}\, w = \ker(\mathbb{1} - w)$

w refl. if $\dim \mathrm{Fix}\, w = n-1$

$W \leq GL(V)$ **comp. refl. gr.** if

(1) W finite

(2) W gen by refl.

$\mathcal{R} = \{\text{refl}\}$ $\mathcal{R}^* = \{\text{refl hyperplanes}\}$

We consider W to be

- **irreducible** (doesn't fix proper subspace)
- **well-generated** (W gen by n reflections)

Regular & Coxeter elements

- $v \in V$ **reg** if $v \notin \bigcup_{H \in \mathcal{A}^*} H$
- $w \in W$ **reg** if it has reg. eigenvector
- w **h-reg** if the corresponding eigenvalue is a primitive h-th root of 1.
- w **Coxeter element** if w h-reg with h <u>maximal</u>

Thm [Chapuy–S. 2013]

W irred., well-gen

complex refl. gr.

Then

$$FACT_W(t) = \frac{1}{|W|}\left(e^{t|R|/n} - e^{-t|R^*|/n}\right)^n$$