Nonnesting partitions and the cluster complex

Marko Thiel

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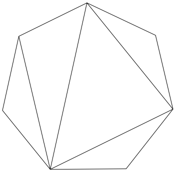

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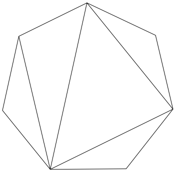

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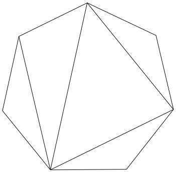

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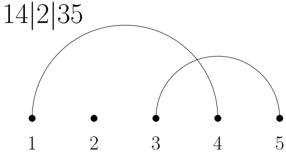

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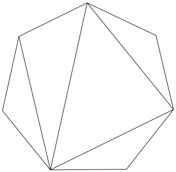

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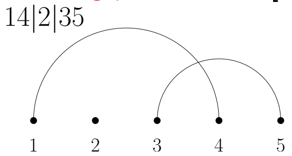

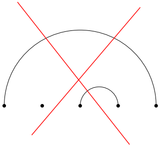

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We will prove a relationship between them, the $\mathrm{H} = \mathrm{F}$ conjecture.

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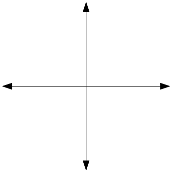
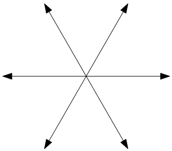


A_2



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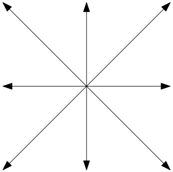

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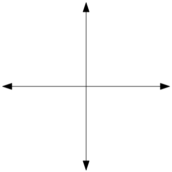

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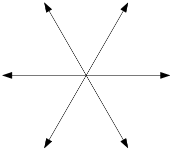

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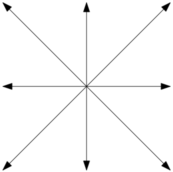

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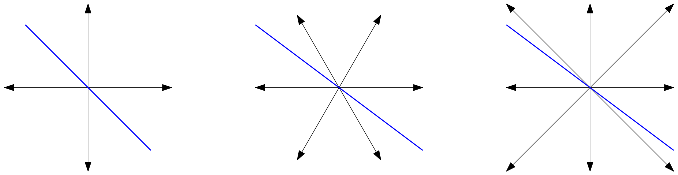

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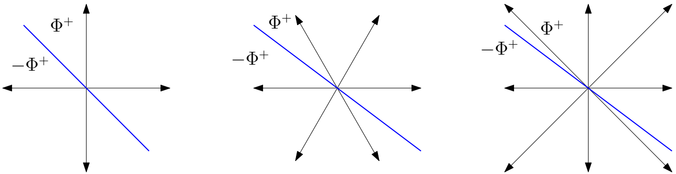

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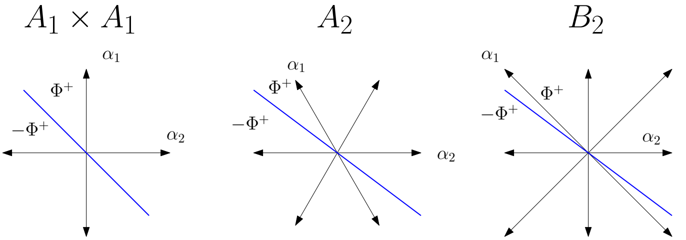

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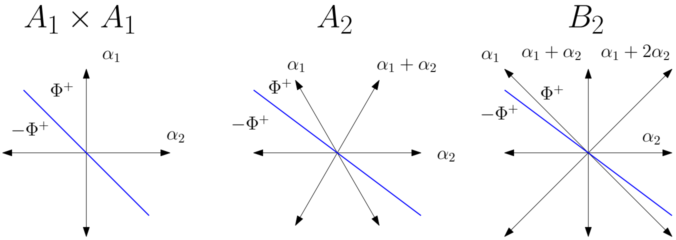

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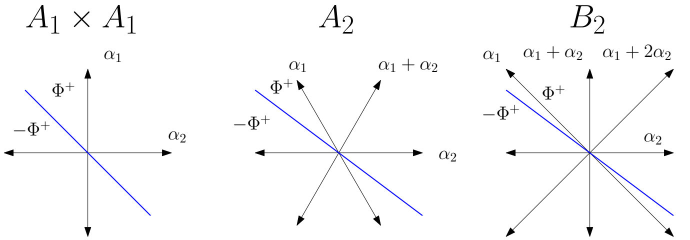

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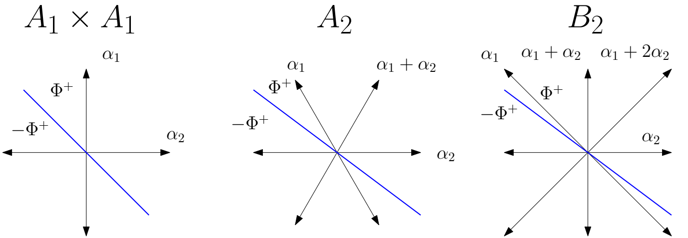

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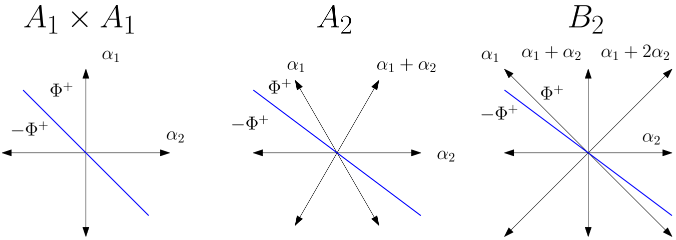

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A root system is determined up to isomorphism by its Dynkin diagram, a (multi)graph with vertex set S, and $\frac{4\langle \alpha,\beta\rangle^2}{\langle \alpha,\alpha\rangle\langle \beta,\beta\rangle}$ edges between α and β, for $\alpha \neq \beta$.

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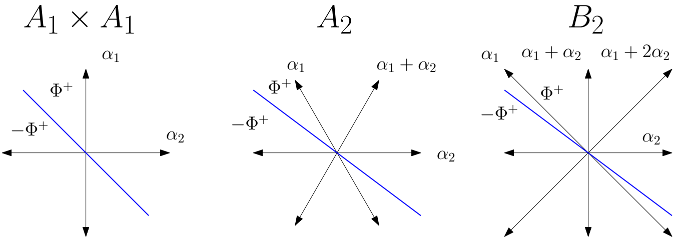

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Define the rank of Φ as $dim(V) = |S|$.

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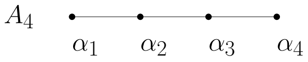

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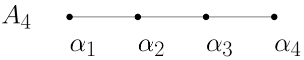

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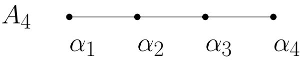
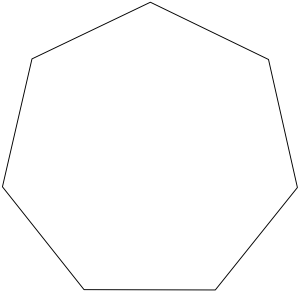


The simple roots are $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, and the positive roots are all sums $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq n-1$.

The cluster complex

Take a convex $(n+2)$-gon.



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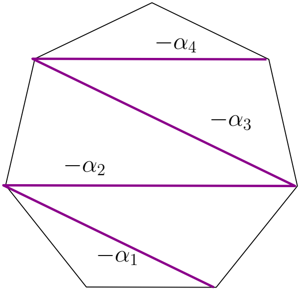

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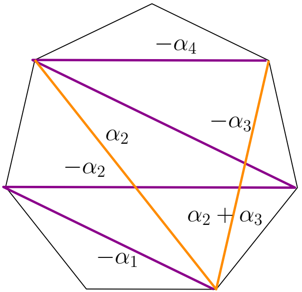

Label the diagonals in a snake by the negative simple roots of Φ, a root system of type A_{n-1}.
Label the other diagonals by minus the sum of the labels of the snake diagonials they cross.

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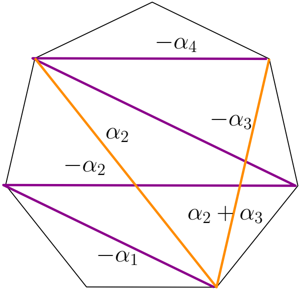

This gives a bijection between the set of diagonals of the $(n+2)$-gon and $\Phi_{\geq -1} = \Phi^+ \sqcup -S$, the set of almost positive roots of Φ.

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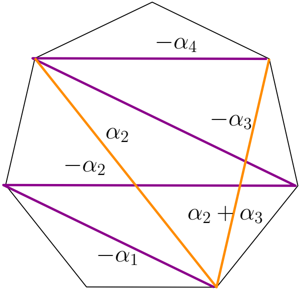

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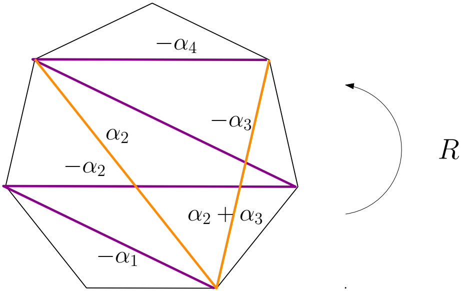

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There is a rotation R on $\Phi_{\geq -1}$ such that α and β are compatible if and only if $R(\alpha)$ and $R(\beta)$ are compatible.

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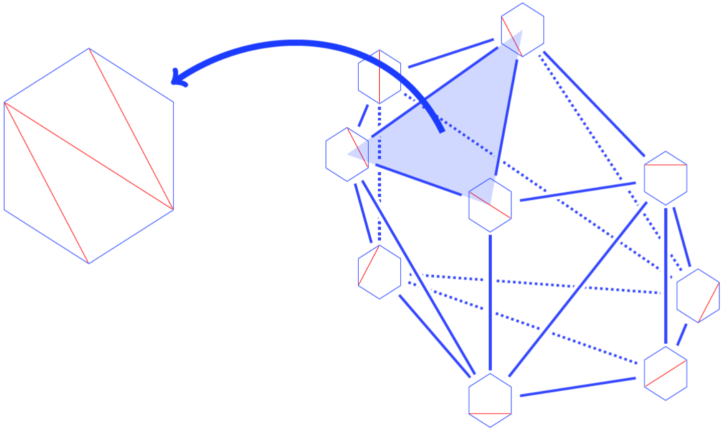
The simplicial associahedron



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Theorem (F. Chapoton '04)

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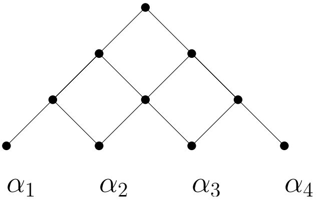
A_4



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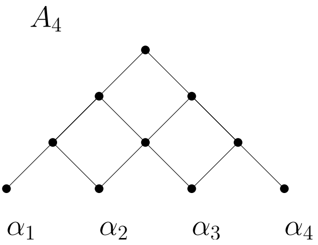

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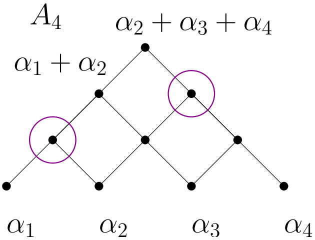

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Proof.

Case by case check, using the classification of irreducible crystallographic root systems. □

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Thanks for your attention!

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