The braid and the Shi arrangements and the Pak-Stanley labelling

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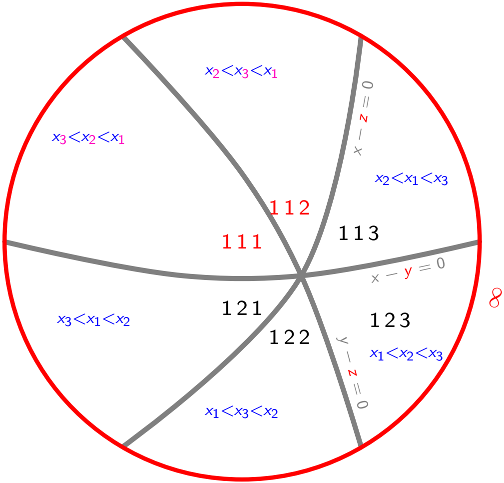
Séminaire Lotharingien de Combinatoire 73 — Strobl

Dedicated to the memory of Michel Las Vergnas

Function t



$t: \pi \mapsto f$

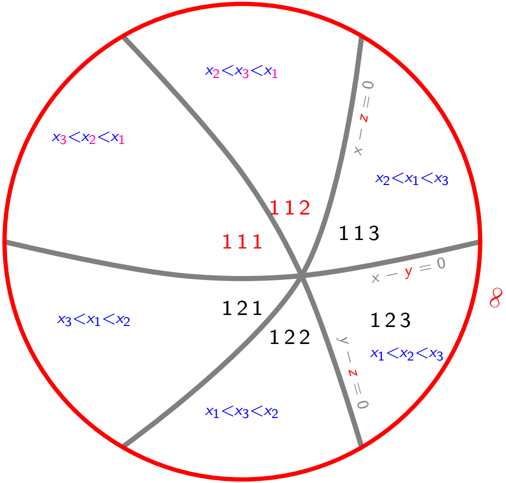
2	3	1
2		
1	3	

$\downarrow t = 112$

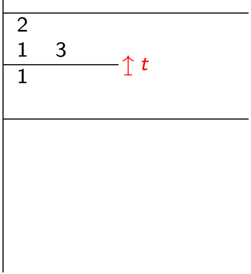
3	2	1
3		
2		
1		

$\downarrow t = 111$

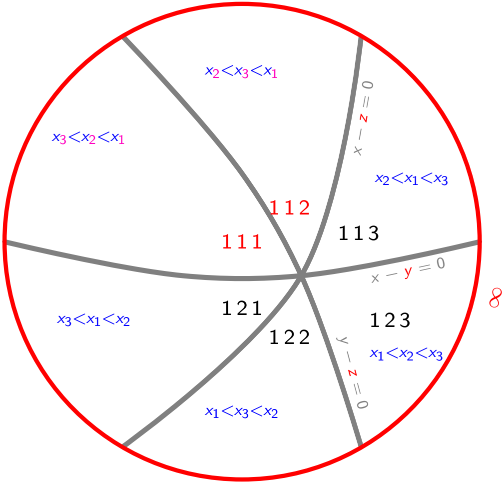
Function t^{-1}: s-parking

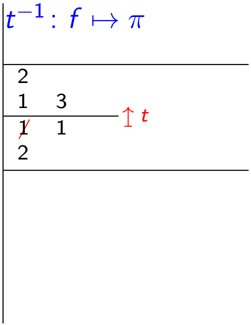


$t^{-1}: f \mapsto \pi$

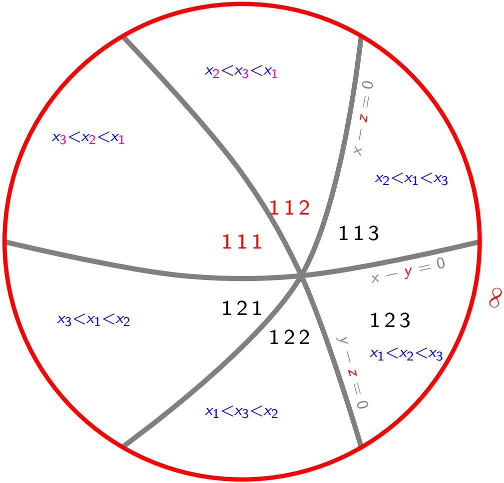


Function t^{-1}: s-parking





Function t^{-1}: s-parking

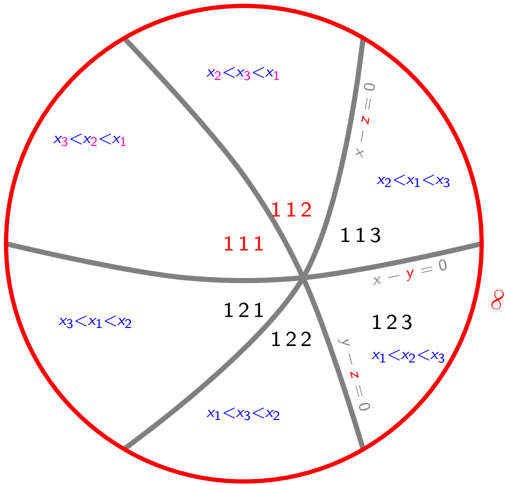


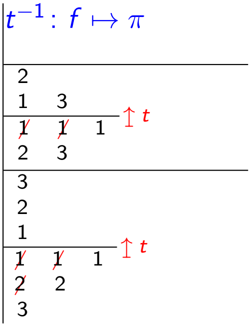
$t^{-1}: f \mapsto \pi$

2
1 3
1 1 1
2 3

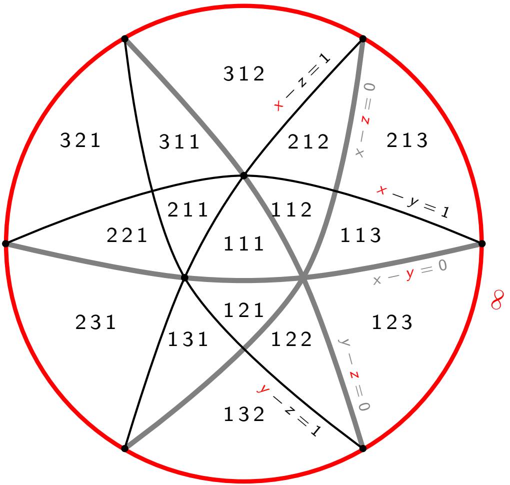
$\uparrow t$

Function t^{-1}: s-parking

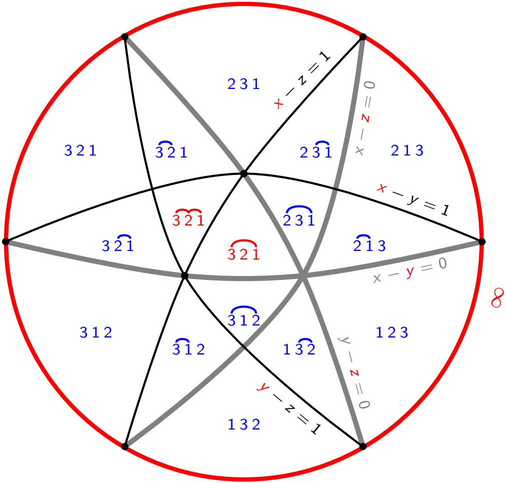




Pak-Stanley bijection λ [St96]



Pak-Stanley bijection λ [St96]



$\widehat{3\,2}\widehat{\,1}$

$z < y < x$

$y < z + 1, x < y + 1$

(but $x > z + 1$)

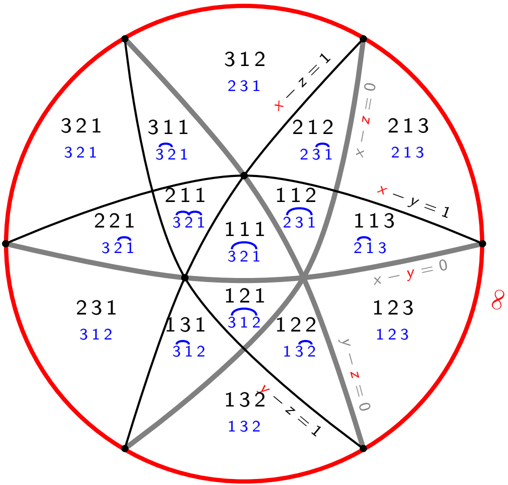
$\widehat{3\,2\,1}$

$z < y < x$

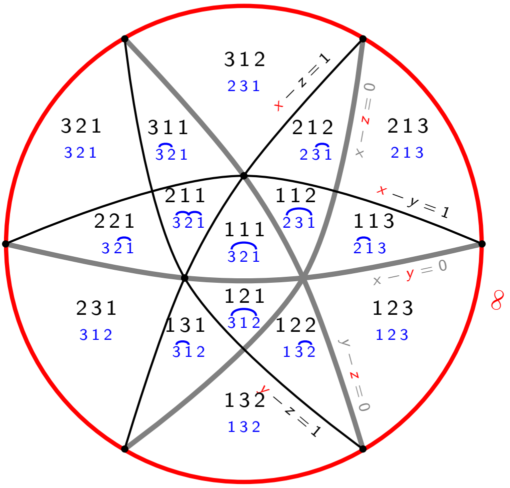
$x < z + 1$

($\Rightarrow$ $x < y + 1, y < z + 1$)

Pak-Stanley bijection λ [St96]



Pak-Stanley bijection λ [St96]



Images:

t: Central p. functions

$\{f : [n] \to [n] \mid (f_1, \ldots, f_n) \preceq (1, \ldots, n)\}$

λ: Parking functions

$\{f : [n] \to [n] \mid (f_1, \ldots, f_n) \preceq \pi \in \mathfrak{S}_n\}$

Problem 1: invert λ

Problem 1

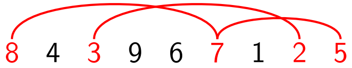
An example from [R. P. Stanley, An introduction to hyperplane arrangements, in *Geometric Combinatorics* (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Mathematics Series, vol.**13**, A.M.S. (2007)]

8 4 3 9 6 7 1 2 5

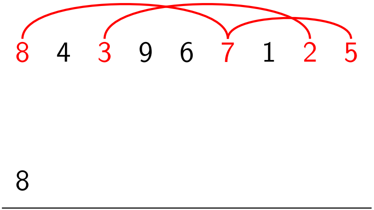
$$\Big\{x \in \mathbb{R}^9 \ \Big|\ x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5,$$

$$x_7 < x_8 + 1 \ (\Rightarrow x_4, x_3, x_6 < x_8 + 1),$$
$$x_1 > x_8 + 1 \ (\Rightarrow x_2, x_5 > x_8 + 1),$$
$$x_2 < x_3 + 1 \ (\Rightarrow x_1 < x_3 + 1),$$
$$x_5 < x_7 + 1 \ (\Rightarrow x_1, x_2 < x_7 + 1)\Big\} \in \mathcal{R}(\mathrm{Shi}_9)$$

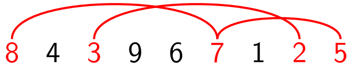
Problem 1



Problem 1



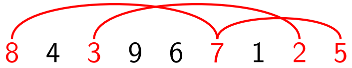
Problem 1



8

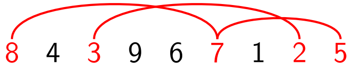
4

Problem 1



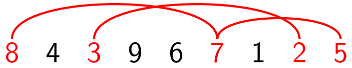
8
4
3

Problem 1



8	
4	
3	9

Problem 1



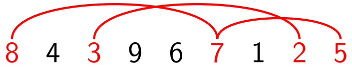
8		
4		
3	6	9

Problem 1

8 4 3 9 6 7 1 2 5

8			9					
4		6	7					
3		1	2				5	

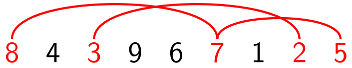
Problem 1



8			9					
4		6	7					
3		1	2				5	

3

Problem 1



8			9					
4		6	7					
3		1	2				5	

3 4

Decomposition

Problem 1

8	4	3	9	6	7	1	2	5

8			9					
4		6	7					
3		1	2				5	

3 4 1

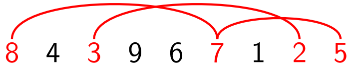
Problem 1

8	4	3	9	6	7	1	2	5

8			9					
4		6	7					
3		1	2				5	

3	4	1	1

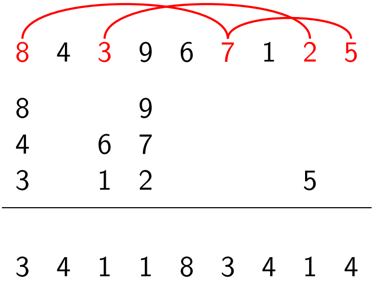
Problem 1



8			9					
4		6	7					
3		1	2				5	

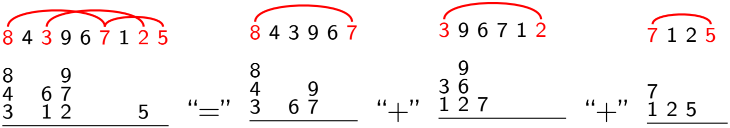
3	4	1	1	8	3	4	1	4

Problem 1

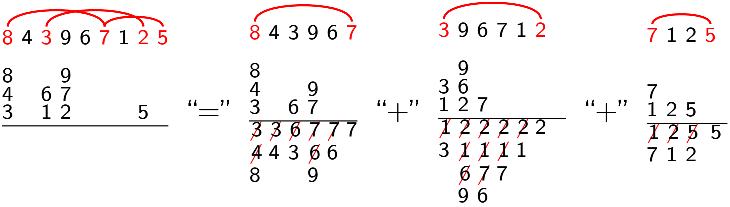


Problem 1: invert λ —
Go from 3 4 1 1 8 3 4 1 4 to 8 4 3 9 6 7 1 2 5

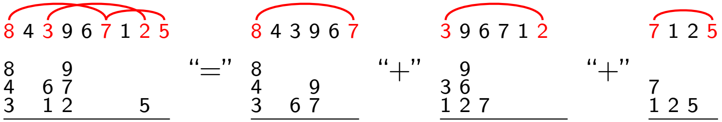
λ in terms of t



λ in terms of t



λ in terms of t



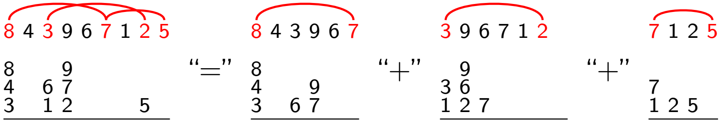
Definition

Let $f \in \mathrm{PF}_n$ and $X = \{x_1, \ldots, x_m\} \subseteq [n]$. We say that X *is f-central* if

$$f(x_i) \leq i\,, \quad i = 1, \ldots, m\,.$$

The centre of f is the (unique) maximal f-central subset $X(f)$ of $[n]$.

λ in terms of t



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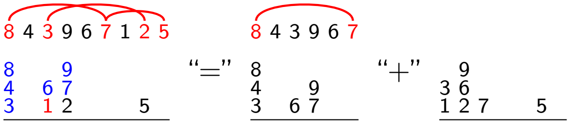
Lemma

Let $f = \lambda(w, \mathfrak{I}) \in \mathrm{PF}_n$, $X = X(f)$ and $m = |X| < n$. Then

$$\tilde{w} = w_1 \cdots w_m \ \text{and} \ \tilde{\mathfrak{I}} = \{[i,j] \in \mathfrak{I} \mid j \leq m\}.$$

λ in terms of t



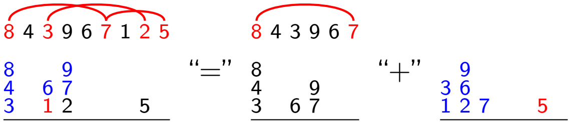
Definition

- $b := \min f\left([n] \setminus X\right), a := \max\left(f^{-1}(\{b\}) \setminus X\right)$;
- if $b > m$, $c := b$;
  if $b \leq m$, let $c > 1$ be the greatest element $j \in [m]$ such that

$$j + |w([j,m]) \cap [a-1]| = b$$

- $$\begin{array}{rccl} g: & Z := [n] \setminus w([1,c-1]) & \to & [n-c+1] \\ & i & \mapsto & \begin{cases} f_i - |Y \cap [i-1]|, & \text{if } i \in X \cap Z; \\ f_i - c + 1, & \text{if } i \in Z \setminus X. \end{cases} \end{array}$$

λ in terms of t



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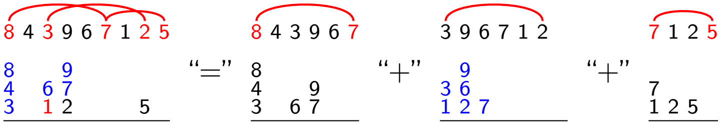
λ in terms of t



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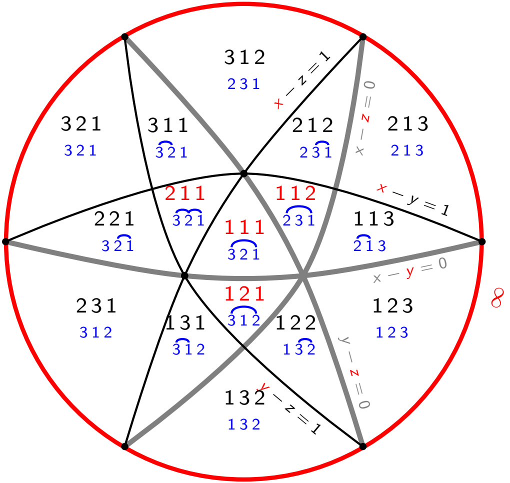
Problem 2



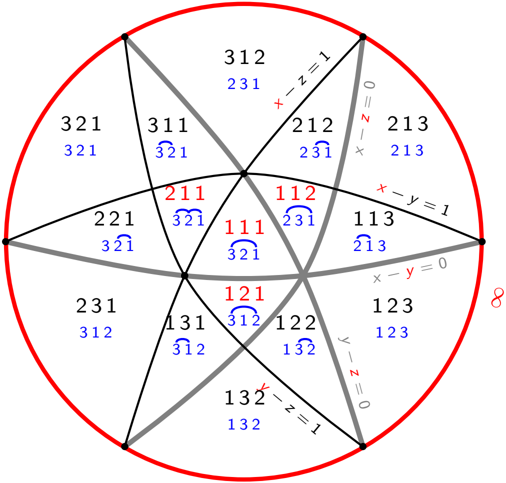
Prime p. functions:

$\left|f^{-1}([i])\right| \gneq i, \forall_{i\in[n-1]}$

$\leftrightsquigarrow$

bounded regions

Problem 2



Prime p. functions:

$\left|f^{-1}([i])\right| \gneq i, \forall_{i\in[n-1]}$

$\leftrightsquigarrow$

bounded regions

(Central p. f.)

$\left|f^{-1}([i])\right| = i \implies$

f : $\boxed{\leq i}$ $\boxed{> i}$

[AL99](A simple bij. ...)

AL99 AL99 C. Athanasiadis, S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes. *Discrete Math.* **204** (1999) 27–39.

GH96 GH96 A. M. Garsia and M. Haiman, A Remarkable q, t-Catalan Sequence and q-Lagrange Inversion, *J. Algebr. Comb.* **5** (1996) 191–244.

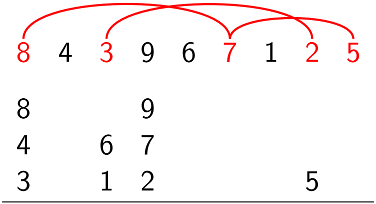
KW66 KW66 A.G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Appl. Math.* **14** (1966), 1266–1274.

Shi86 Shi86 J. Y. Shi, *The Kazhdan-Lusztig Cells in certain Affine Weyl Groups*, Lecture Notes in Mathematics **1179** (1986), Springer-Verlag .

St07 St07 R. P. Stanley, An introduction to hyperplane arrangements, in *Geometric Combinatorics* (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Mathematics Series, vol.**13**, A.M.S. (2007), 389–496.

St96 St96 R. P. Stanley, Hyperplane arrangements, interval orders and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

S-parking directly



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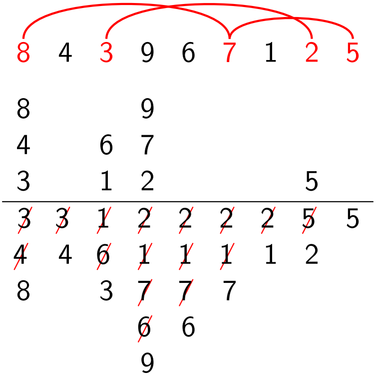

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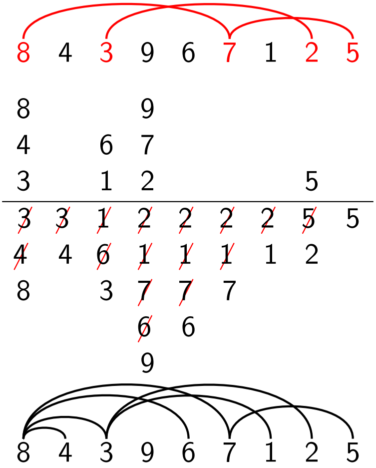

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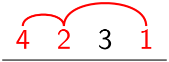

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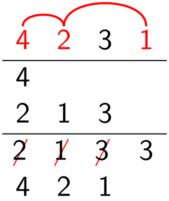

S-parking directly