Some Recent Works on Hankel Determinants

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Summary

Sep. 8

- 1. Determinants
- 2. Hankel determinant of the Thue-Morse sequence
- 3. Combinatorial Proof
- 4. t-Extension

Sep. 9

- 5. Jacobi continued fraction
- 6. Grafting technique and Chopping method
- 7. Miscellaneous

Sep. 10

- 8. Evolution of the proofs of the APWW theorem
- 9. Hankel continued fraction
- 10. Periodicity

1. Determinants

Basic determinant evaluation

- One square matrix of (small) fixed dimension:

$$A = \begin{pmatrix} 3 & 1 & 9 \\ -1 & -2 & 7 \\ 6 & 0 & 1 \end{pmatrix}$$

- Linear Algebra:

$$\begin{aligned} \det(A) &= 3 \times (-2) \times 1 \ + \ 1 \times 7 \times 6 \ + \ 9 \times (-1) \times 0 \\ &\quad - 9 \times (-2) \times 6 - 1 \times (-1) \times 1 - 7 \times 0 \times 3 \\ &= 145 \end{aligned}$$

- Computer Algebra System [REPLACES Human]

```
sage:  A=matrix([[3,1,9],[-1,-2,7],[6,0,1]])
sage:  A.det()
145
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Advanced determinant calculus

- One square matrix of dimension $n \times n$, where n is an unknown positive integer

$$A_n = \left(\frac{1}{i+j+1} \right)_{i,j=0..n-1}$$

- One sequence of matrices:

$$(A_0, A_1, A_2, \dots)$$

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- One sequence of matrices:

$$(A_0, A_1, A_2, \ldots)$$

• Leibniz formula. Let A be a $n \times n$ square matrix

$$A = (a_{ij})_{i,j=0,\ldots,n-1}.$$

Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, a_{0,\sigma(0)}\, a_{1,\sigma(1)} \cdots a_{n-1,\sigma(n-1)}$$

where :

$\operatorname{sgn}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}$: signature

$\operatorname{inv}(\sigma)$: the number of inversions

$$\operatorname{inv}(\sigma) = \#\{(i,j) \mid 0 \le i < j \le n-1, \sigma(i) > \sigma(j)\}.$$

- Computer Algebra System

```
> def Adet(n):
>   R=range(n)
>   F=[[1/(i+j+1) for i in R] for j in R]
>   return matrix(QQ, F).det()

> [Adet(0), Adet(1), Adet(2), Adet(3), Adet(4)]
[1, 1, 1/12, 1/2160, 1/6048000]
```

```
> Adet(n)
NameError:  name 'n' is not defined
```

- CAS does not replace, but helps Human !

Advanced determinant calculus

You need:

- Computer Algebra System
- Christian Krattenthaler

  1998: Advanced determinant calculus

  2005: Advanced determinant calculus: A complement

- Example:

$$\det_{0\le i,j\le n-1}\left(\binom{a+b}{a-i+j}\right)=\prod_{i=0}^{n-1}\frac{(a+b+i)!\,i!}{(a+i)!\,(b+i)!}$$

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Advanced determinant calculus

Christian Krattenthaler :

"Evaluating determinants is not difficult!",

if $\det(A_n) =$ NICE FORMULA(n).

Advanced determinant calculus

- Problem

  How do we proceed if $\det(A_n) \neq$ NICE FORMULA(n) ?

- Answer

  Simple. We just ignore the problem.

  Doing Enumerative Combinatorics, we avoid studying any sequence WITHOUT nice formula.

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Interlude

- Ramanujan τ-function

$$x \prod_{m\geq 1} (1 - x^m)^{24} = \sum_{n\geq 1} \tau(n) x^n$$
$$= x - 24\, x^2 + 252\, x^3 - 1472\, x^4 + 4830\, x^5 - 6048\, x^6 + \cdots$$

- Lehmer Conjecture (1947)

For each n we have $\tau(n) \neq 0$.

- I refuse to study the Lehmer Conjecture, because there is no nice formula for $\tau(n)$.

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Hankel determinant

We identify a sequence

$$\mathbf{a} = (a_0, a_1, a_2, \ldots)$$

and its generating function

$$f = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$H_n^{(k)}(\mathbf{a}) = H_n^{(k)}(f) := \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+2n-2} \end{vmatrix}$$

(constant skew-diagonals)

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Hankel determinant

Two notations. Using sequence and generating function

$$H_n^{(0)}\Big((1,1,1,1,1,\ldots)\Big) = H_n^{(0)}\Big(\frac{1}{1-x}\Big)$$

Special case $k = 0$:

$$H_n(f) = H_n^{(0)}(f) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-2} \end{vmatrix}$$

2. Hankel determinants of the Thue-Morse sequence

Thue-Morse sequence

an infinite sequence $\mathbf{t} = (e_0, e_1, e_2, \ldots)$ on $\{1, -1\}$, defined by:

- Generating function

$$\prod_{k=0}^{\infty}(1 - x^{2^k}) = \sum_{n=0}^{\infty} e_n x^n = 1 - x - x^2 + x^3 - x^4 + x^5 + \cdots$$

$$\mathbf{t} = (1, -1, -1, 1, -1, 1, 1, -1, \ldots)$$

- Recurrence relation

$$e_0 = 1$$
$$e_{2n} = e_n$$
$$e_{2n+1} = -e_n$$

Thue-Morse sequence

Negation - Concatenation

- starting with (1)
- The negation is (-1)
- Combining those two, we get $(1, -1)$
- The negation is $(-1, 1)$
- Combining those two, we get $(1, -1, -1, 1)$
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- And so on

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Thue-Morse sequence

- L-system (Lindenmayer system)

```
variables 1, -1
constants none
start 1
rules (1 -> 1 -1), (-1 -> -1 1)
```

1
1 -1
1 -1 -1 1
1 -1 -1 1 -1 1 1 -1
$\vdots$

1998

Allouche, Peyrière, Wen, Wen proved:

Theorem [APWW]. Let

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}).$$

Then $H_n(P_2(x)) \neq 0$ for every positive integer n.

Table

n	H_n	n	H_n	n	H_n	n	H_n
0	1	7	-64	14	8192	21	28311552
1	1	8	128	15	-16384	22	-94371840
2	-2	9	-256	16	32768	23	62914560
3	4	10	-1536	17	-65536	24	8388608
4	8	11	-3072	18	-393216	25	16777216
5	-16	12	2048	19	-2359296		
6	-32	13	4096	20	14155776		

- I resigned, because there is no nice formula for H_n.

Table

n	$H_n/2^{n-1}$	n	$H_n/2^{n-1}$	n	$H_n/2^{n-1}$	n	$H_n/2^{n-1}$
0	2	7	−1	14	1	21	27
1	1	8	1	15	−1	22	−45
2	−1	9	−1	16	1	23	15
3	1	10	−3	17	−1	24	1
4	1	11	−3	18	−3	25	1
5	−1	12	1	19	−9		
6	−1	13	1	20	27		

Theorem APWW(i)

$$\frac{H_n(P_2)}{2^{n-1}} \quad \text{is odd.}$$

Period-doubling sequence

Recall: The Thue-Morse sequence $\mathbf{t}$

$$\begin{array}{l} \mathbf{t} = \; 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad \dots \\ \mathbf{d} = \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad \dots \end{array}$$

The *period-doubling sequence* $\mathbf{d} = (1, 0, 1, 1, 1, 0, \dots)$ is derived from the Thue–Morse sequence

$$d_k := \frac{1}{2}|e_k - e_{k+1}| \qquad (k \geq 0).$$

Theorem APWW(ii).

$$H_n(\mathbf{d}) \quad \text{is odd.}$$

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Recall: The Thue-Morse sequence $\mathbf{t}$

$$\begin{array}{lccccccccccccccc} \mathbf{t} = & 1 & & -1 & & -1 & & 1 & & -1 & & 1 & & 1 & & \dots \\ \mathbf{d} = & & 1 & & 0 & & 1 & & 1 & & 1 & & 0 & & \dots & \end{array}$$

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\begin{array}{lccccccccccccccc}
\mathbf{t} = & 1 & & -1 & & -1 & & 1 & & -1 & & 1 & & 1 & & \ldots \\
\mathbf{d} = & & 1 & & 0 & & 1 & & 1 & & 1 & & 0 & \ldots &
\end{array}
$$

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Theorem APWW(ii).

$$H_n(\mathbf{d}) \quad \text{is odd.}$$

Theorem APWW(iii)

Let M_n be the $n \times n$ matrix derived from the Hankel matrix of the period-doubling sequence by replace the last column by $(1, 1, 1, 1, \ldots, 1)^t$. Then

$$\det(M_n) \quad \text{is odd.}$$

- Theorems APWW(i) and APWW(iii) are equivalent.

$$\mathbf{t} = (1,-1,-1,1,-1,1,1,-1,-1,\ldots)$$

$$\mathbf{d} = (1,0,1,1,1,0,1,0,\cdots)$$

$$H_4(\mathbf{t}) = \begin{vmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -2 & 1 \\ 0 & -2 & 2 & -1 \\ -2 & 2 & -2 & 1 \\ 2 & -2 & 0 & 1 \end{vmatrix}$$

[Col(i):=Col(i)-Col(i+1)   i=0,1,2]

$$H_4(\mathbf{t})/2^3 = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} \equiv \det(M_4) \equiv 1 \pmod 2$$

$$H_4(\mathbf{d}) = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \equiv 1 \pmod 2$$

APWW's Proof

- Sixteen relations

- Sudoku method

Irrationality exponent

Let ξ be an irrational, real number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q.

Irrationality exponent

Well known:

- $\mu(\xi) \geq 2$
- $\mu(\xi) = 2$ if ξ is algebraic irrational number
- $\mu(\xi) = 2$ for almost all real numbers ξ (Lebesgue measure)
- $P_2(1/m)$ is transcendental for every integer $m \geq 2$

Theorem [Bugeaud, 2011]

$$\mu(P_2(1/m)) = 2.$$

Proof. Using Theorem APWW, ...

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Coons (2011)

The Gros sequence [Louis Gros, 1872]

$$S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}$$

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, \ldots$$

Theorem

$$H_n(S_2) \quad \text{is odd.}$$

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$$\mu(S_2(1/m)) = 2.$$

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Theorem

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Coons's Proof

APWW's method

3. Combinatorial Proof

Combinatorial Proof / Bugeaud-Han

Infinite sets of integers:

$$N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\}$$

$$J = \{(2n+1)2^{2k} - 1 \mid n, k \in N\}$$
$$= \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \ldots\}$$

$$K = \{(2n+1)2^{2k+1} - 1 \mid n, k \in N\}$$
$$= \{1, 5, 7, 9, 13, 17, \ldots\}$$

Property: $N = J + K$ (disjoint union)

$m \geq 1$: integer

$\mathfrak{S}_m = \mathfrak{S}_{\{0,1,\ldots,m-1\}}$: all permutations on $\{0, 1, \ldots, m-1\}$

Theorem J.

(J1):

$$\#\{\sigma \in \mathfrak{S}_m \mid i + \sigma(i) \in J \text{ for } i = 0, 1, \ldots, m-1\} \text{ is odd.}$$

(J2):

$$\#\{\sigma \in \mathfrak{S}_m \mid i + \sigma(i) \in J \text{ for } i = 0, 1, \ldots, m-2\} \text{ is odd.}$$

Enumerative Combinatorics

permutations, involutions, inversion number, major index, the number of descents, the number of excedances, the number of fixed points, Denert statistics, Young tableau, charge statistics, pattern-avoiding permutations, signed permutations, André permutations, derangements, desarrangements, wreath product, RSK algorithm, dez, maz, pix, two-pix, ides, imaj, rmail, rmal, cyc, surfix, maf, mak, ...

Today: $i + \sigma(i)$

Proof of Theorem J / Notations

Three representations for permutations

- one-line

$$\sigma \in \mathfrak{S}_9 = 516280374$$

- two-lines

$$\sigma \in \mathfrak{S}_9 = \begin{pmatrix} 012345678 \\ 516280374 \end{pmatrix}$$

- product of disjoint cycles

$$\sigma \in \mathfrak{S}_9 = (0,5)(1)(2,6,3)(4,8)(7)$$

Involution

- An *involution* is a permutation σ such that $\sigma = \sigma^{-1}$.

- In the cycle representation of an involution every cycle is either a *fixed point* (b) or a *transposition* (c, d).

- Definition.

For every set A, a transposition (c, d) is said to be an A-*transposition* if:

$$c + d \in A \text{ and } c + d \text{ is odd.}$$

- Remark.

There is an even number and an odd number in every A-transposition. We write the *even number before the odd number*.

(Usually the order of the two numbers in a transposition does not matter)

A: finite set
B: infinite set
f: non-negative integer

Definition.

$\nu(A, f, B)$: the number of involutions in $\mathfrak{S}_A$ such that

- all transpositions are B-transpositions
- have exactly f fixed points.

Example.

$A = \{0, 1, 2, 3, 4, 5, 6\}$

$K = \{(2n+1)2^{2k+1} - 1 \mid n, k \in N\} = \{1, 5, 7, 9, 13, 17, \ldots\}$

$f = 1$

$$\nu(A, f, K) = 11$$

The possible transpositions:

$$(01), (05), (14), (16), (23), (25), (34), (36), (45)$$

List of 11 involutions:

$$(0)(14)(52)(36),\ (0)(16)(23)(45),\ (0)(16)(25)(34)$$

$$(01)(2)(63)(45),\ (01)(23)(6)(45),\ (01)(25)(6)(34),\ (01)(25)(63)(4)$$

$$(05)(14)(2)(36),\ (05)(14)(23)(6),\ (05)(16)(2)(34),\ (05)(16)(23)(4)$$

Proof of Theorem J / Basic Lemmas

- Two infinite sets of integers:

$$
\begin{aligned}
P &= \{k \mid k \equiv 0, 3 \pmod 4\} \\
&= \{0, 3, 4, 7, 8, 11, 12, 15, 16, \ldots\} \\
Q &= \{k \mid k \equiv 1, 2 \pmod 4\} \\
&= \{1, 2, 5, 6, 9, 10, 13, 14, 17, \ldots\}
\end{aligned}
$$

- Property : $N = P + Q$ (disjoint union)

- Let $A|_m$ be the set composed of the smallest m integers in A.

Example: $P|_5 = \{0, 3, 4, 7, 8\}$

Transformation:

$$\beta : N \to N; \qquad k \mapsto \begin{cases} k/2 & \text{if } k \text{ is even} \\ (k-1)/2 & \text{if } k \text{ is odd} \end{cases}$$

β is extended to the involutions σ on $N|_m$ whose transpositions are K-transpositions, by applying β on every number in the cycle representation of σ.

$$\beta((7)(0,5)(6,3)(2)(8,1)(4)) = (3)(0,2)(3,1)(1)(4,0)(2).$$

• β for involutions is reversible, even though β on N is not reversible.

$$\beta((?)(0,5)(6,3)(2)(8,1)(4)) = (3)(0,2)(3,1)(1)(4,0)(2).$$

We do not know a priori whether the fixed point 3 is obtained from 6 or from 7.

• We must look at the transposition $(3,1)$ first. It is obtained from the permutation $(6,3)$ since we know that an even number is always *before* an odd number. Thus, we can recover the K-transpositions $(6,3)(0,5)(8,1)$.

• All the other numbers are fixed points $(7)(2)(4)$.

$$P = \{k \mid k \equiv 0, 3 \pmod 4\}$$

$$\beta : N \to N; \qquad k \mapsto \begin{cases} k/2 & \text{if } k \text{ is even} \\ (k-1)/2 & \text{if } k \text{ is odd} \end{cases}$$

$$\gamma : P \to N; \qquad k \mapsto \begin{cases} k/4 & \text{if } k \text{ is even} \\ (k-3)/4 & \text{if } k \text{ is odd} \end{cases}$$

$$\begin{array}{rcl} \beta : (7)(0,5)(6,3)(2)(8,1)(4) & \to & (3)(0,2)(3,1)(1)(4,0)(2) \\ & & \| \\ \gamma^{-1} : (15)(0,11)(12,7)(4)(16,3)(8) & \leftarrow & (3)(0,2)(3,1)(1)(4,0)(2) \end{array}$$

By the bijection $\gamma^{-1}\beta$,

Lemma. For $m \geq 1$ we have

$$\nu(N|_m, 0/1, K) \equiv \nu(P|_m, 0/1, J)$$
$$\nu(N|_{2m-1}, 1, J) \equiv \nu(P|_{2m-1}, 1, K)$$
$$\nu(N|_{2m}, 0/2, J) \equiv \nu(P|_{2m}, 0/2, K)$$

where $\nu(A, f/g, B) := \nu(A, f, B) + \nu(A, g, B)$

Again, by

$$\delta : N \to N; \qquad k \mapsto \begin{cases} k+1 & \text{if } k \text{ is even} \\ k-1 & \text{if } k \text{ is odd} \end{cases}$$

$$\delta : P \to Q$$

Lemma. For $m \geq 1$ we have

$$\nu(N|_m, 0/1, K) \equiv \nu(P|_m, 0/1, J) \equiv \nu(Q|_m, 0/1, J)$$
$$\nu(N|_{2m-1}, 1, J) \equiv \nu(P|_{2m-1}, 1, K) \equiv \nu(Q|_{2m-1}, 1, K)$$
$$\nu(N|_{2m}, 0/2, J) \equiv \nu(P|_{2m}, 0/2, K) \equiv \nu(Q|_{2m}, 0/2, K)$$

Proof of Theorem J / Proof of (J1)

(J1): $\#\{\sigma \in \mathfrak{S}_m \mid i + \sigma(i) \in J \text{ for all } i\}$ is odd.

We count the permutations in (J1) modulo 2.

Recall

$$\begin{aligned} J &= \{(2n+1)2^{2k} - 1 \mid n, k \in N\} \\ &= \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \ldots\} \end{aligned}$$

- Fact1: J contains all even numbers;
- Fact2: If an odd number $x \in J$, then $x \equiv 3 \pmod 4$.

Proof

- Take a permutation σ in (J1)

• If σ contains more than two columns $\binom{odd}{odd}$, select the first two such columns $\binom{i_1}{j_1}$ and $\binom{i_2}{j_2}$.

• Define permutation τ obtained from σ by exchanging j_1 and j_2 in the bottom line.

• This procedure is reversible.

• By Fact1, τ is also a valid permutation in (J1).

• So that we can delete the pair σ and τ, and there only remain the permutations containing 0 or 1 column $\binom{odd}{odd}$, in particular, having 0 or 1 odd fixed point.

• Similarly with an even number, there only remain permutations containing 0 or 1 column $\binom{even}{even}$. Consequently, the only remaining permutations have $0, 1$ or 2 fixed points.

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Proof

- Thanks to the bijection $\sigma \mapsto \sigma^{-1}$, we only need consider the involutions .
- We can check that all transpositions are J-transpositions.
- If m is odd, then the involution contains one fixed point, and the number of such involutions is $\nu(N|_m, 1, J)$
- By Fact2 the two numbers in every J-transposition are either both in P or both in Q. This means that no J-transposition takes one number in P and another in Q.
- The involutions are of type

$$\begin{array}{c|c} P & Q \\ (\bullet\bullet)(\bullet\bullet)(\bullet\bullet) & (\bullet)(\bullet\bullet)(\bullet\bullet)(\bullet\bullet)(\bullet\bullet) \end{array}$$

- The two parts composed by numbers from P and from Q are INDEPENDENT

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Proof

- IF $m = 2k+1$, $\mathrm{fix}(\sigma) = 1$

$$\nu(N|_{2k+1}, 1, J) \equiv \nu(P|_k, 0/1, J) \times \nu(Q|_{k+1}, 0/1, J)$$

- We need evaluate $\nu(P|_k, 0/1, J)$
- IF $k = 2\ell$, $\mathrm{fix} = 0$. Apply γ to an involution in $\nu(P|_k, 0/1, J)$

$$\gamma((0,3)(8,7)(4,11)) = (0,0)(2,1)(1,2)$$

which can be identified with the permutation on $\{0,1,2\}$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\nu(P|_{2\ell}, 0, J) = \#\{\sigma \in \mathfrak{S}_\ell \mid i + \sigma(i) \in J \text{ for all } i\}$$

- By induction, $\nu(P|_{2\ell}, 0, J) \equiv 1 \pmod 2$.
- ELSE, ELSE ... ... [Still 3 pages]

For the complete proof, see Bugeaud-Han, Elec J Comb, 2014.

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Theorem.

$$\text{(J1)} = \text{APWW(ii)}$$

$$\text{(J2)} = \text{APWW(iii)} = \text{APWW(i)}$$

Proof. Recall:

- the period-doubling sequence $\mathbf{d} = (1, 0, 1, 1, 1, 0, \ldots)$.
- $J = \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \ldots\}$

We have

$$d_j = 1 \quad \text{if and only if} \quad j \in J$$

4. t-Extensions

t-Hankel determinants

t-Hankel determinant / Fu-Han

t : an indeterminate

$$H_n(\mathbf{a}, t) = \begin{vmatrix} ta_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & ta_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & ta_4 & a_5 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & ta_6 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & ta_{2n-2} \end{vmatrix}$$

- $H_n(\mathbf{a}, t)$ is a polynomial in t of degree $\leq n$
- When $t = 1$, $\quad H_n(\mathbf{a}, 1) = H_n(\mathbf{a})$

Recall : period-doubling sequence $\mathbf{d} = (1, 0, 1, 1, 1, 0, \ldots)$

Theorem APWW(ii).

$$H_n(\mathbf{d}) \equiv 1 \pmod 2$$

Theorem.

The t-Hankel determinant $H_n(\mathbf{d}, t)$ is a polynomial in t of degree n, whose leading coefficient is the only one to be an odd integer.

$$H_n(\mathbf{d}, t) \equiv t^n \pmod 2$$

n	$H_n(\mathbf{d},t)$	$H_n(\mathbf{d},t) \pmod 2$	$H_n(\mathbf{d},1)$
0	1	1	1
1	t	t	1
2	t^2	t^2	1
3	$t^3 - 2t$	t^3	-1
4	$t^4 - 4t^2$	t^4	-3
5	$t^5 - 6t^3 + 2t^2 + 4t$	t^5	1
6	$t^6 - 8t^4 + 4t^3 + 12t^2 - 8t$	t^6	1
7	$t^7 - 12t^5 + 10t^4 + 24t^3 - 24t^2$	t^7	-1
8	$t^8 - 16t^6 + 16t^5 + 48t^4 - 64t^3$	t^8	-15

$|\cdots$ coefficients are even $\cdots|$

Idea of the Proof.

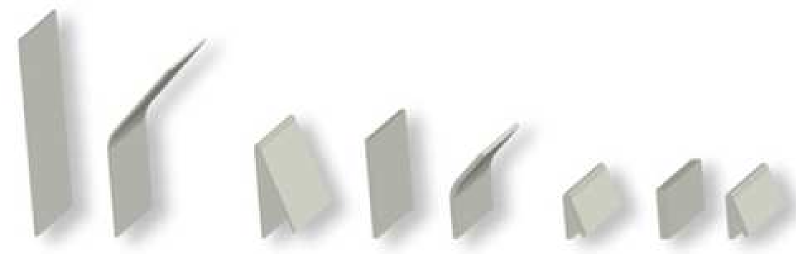
Using the same combinatorial set-up. The parameter t counts the number of fixed points of a permutaion.

By Leibniz formula, the t-Hankel determinant

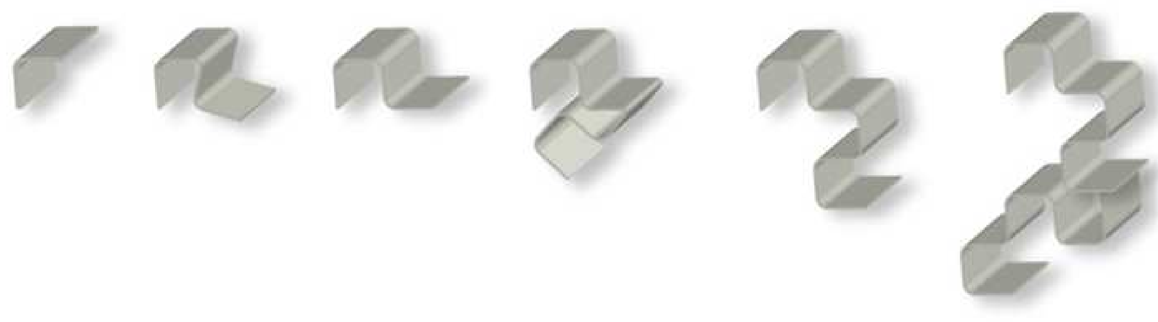
$$H_k(\mathbf{d}, t) = \sum_{\sigma \in \mathfrak{S}_k} t^{\mathrm{fix}(\sigma)} (-1)^{\mathrm{inv}(\sigma)} d_{0+\sigma(0)} d_{1+\sigma(1)} \cdots d_{k-1+\sigma(k-1)}$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points of σ.

Regular paperfolding sequence



(Source: Wikipedia)



1= Left turn, 0=Right turn

$$\mathbf{r} = (1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, \ldots)$$

- Generating function

$$G_{0,2}(x) = \sum_{n \geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n - 1}}{1 - x^{2^{n+2}}}.$$

- Recurrence relations:

$$u_{4n} = 1, \quad u_{4n+2} = 0, \quad u_{2n+1} = u_n$$

- String substitution rules

11	→	1101
01	→	1001
10	→	1100
00	→	1000

11
1101
11011001
1101100111001001
11011001110010011101100011001001

Coons and Vrbik conjectured (2012) and Guo, Wu and Wen (2013) proved

Theorem GWW.

The parities of the Hankel determinants of the regular paper-folding sequence $\mathbf{r}$ are periodic of period 10

$$(H_k(\mathbf{r}))_{k=0,1,\dots} \equiv (1,1,1,0,0,1,0,0,1,1)^* \pmod 2.$$

Proof.

APWW's method.

Trying to find a combinatorial proof and t-extension, we obtain:

Theorem FH.

The t-Hankel determinant $H_k(\mathbf{r}, t)$ is a polynomial in t of degree less than or equal to 3.

k	$H_k(\mathbf{r}, t)$	k	$H_k(\mathbf{r}, t)$
0	1	5	$-t^3 + 2t^2 + 2t - 2$
1	t	6	$2t^2 - 2t - 4$
2	-1	7	$3t^3 - 6t^2 - 7t + 6$
3	$-2t$	8	$-9t^2 + 12t + 16$
4	$-t^2 + 2t + 1$	9	$-15t^3 + 20t^2 + 46t - 40$

Proof of Theorem FH

$R = \{(4k+1)2^n - 1 \mid n, k \in N\} = \{0, 1, 3, 4, 7, 8, 9, 12, 15, 16, \ldots\}$

Lemma.
For each $k \geq 0$ the integer $r_k = 1$ if and only if $k \in R$

$H_k(\mathbf{r}, t) =$

$$\sum_{\sigma \in \mathfrak{S}_k} t^{\mathrm{fix}(\sigma)} (-1)^{\mathrm{inv}(\sigma)} r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}.$$

Proof

The product

$$(*) \qquad r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}$$

is equal to 1 if $i+\sigma(i) \in R$ for $i = 0, 1, \cdots, k-1$, and is equal to 0 otherwise.

- Consider a permutation σ such that the product $(*)$ is non-zero and $\mathrm{fix}(\sigma) \geq 4$.
- Thinking the two-line representation of σ

$$\sigma \in \mathfrak{S}_9 = 516280374 = \begin{pmatrix} 012345678 \\ 516280374 \end{pmatrix}$$

- An even number $m \in R$ if and only if $m \equiv 0 \pmod 4$, so that all fixed points are even.

Proof

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$$(*) \qquad r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}$$

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$$\sigma \in \mathfrak{S}_9 = 516280374 = \begin{pmatrix} 012345678 \\ 516280374 \end{pmatrix}$$

- An even number $m \in R$ if and only if $m \equiv 0 \pmod 4$, so that all fixed points are even.

Proof

The product

$$(*) \qquad r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}$$

is equal to 1 if $i+\sigma(i) \in R$ for $i = 0, 1, \cdots, k-1$, and is equal to 0 otherwise.

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- Since the number of odd integers in $\{0, 1, \ldots, k-1\}$ is equal to $\lfloor k/2 \rfloor$, there are at least 3 columns of type $\binom{odd}{odd}$
- Let $\binom{i_1}{j_1}$, $\binom{i_2}{j_2}$ and $\binom{i_3}{j_3}$ be the first three such columns. By the Pigeonhole Principle, there are at least two numbers among j_1, j_2, j_3 which are congruent modulo 4.
- Without loss of generality, we assume that j_1 and j_2 are congruent modulo 4.
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Statistics of σ and τ:

- $\mathrm{inv}(\sigma) = \mathrm{inv}(\tau) \pm 1$, so that $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\tau)$.
- $i_1 + j_2 \in R$ and $i_2 + j_1 \in R$. Since $i_1 + j_1$ and $i_2 + j_2$ are in R and are even, hence must be congruent to 0 modulo 4. Consequently, $i_1 + j_2$ and $i_2 + j_1$ are congruent to 0 modulo 4.
- $\mathrm{fix}(\sigma) = \mathrm{fix}(\tau)$, i.e., no fixed point has been created.

Thus, the contributions by σ and τ compensate each other. We can delete the pair $\{\sigma, \tau\}$ from $\mathfrak{S}_k$.

After deleting all the permutations such that $\mathrm{fix}(\sigma) \geq 4$, all remaining permutations have at most 3 fixed points.

Hence, $H_k(\mathbf{r}, t)$ is a polynomial in t of degree ≤ 3.

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5. Jacobi continued fraction

Jacobi Continued Fraction

$$\mathbf{u} = (u_1, u_2, \ldots)$$

$$\mathbf{v} = (v_0, v_1, v_2, \ldots)$$

$$\cfrac{v_0}{1 + u_1 x - \cfrac{v_1 x^2}{1 + u_2 x - \cfrac{v_2 x^2}{1 + u_3 x - \cfrac{v_3 x^2}{\ddots}}}}$$

Notation:

$$\mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{J}\begin{bmatrix} u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}$$

How to find and prove the J-Fraction

Example. Let

$$f = \frac{(1-x)(1+2x) - \sqrt{(1-x)(1-2x)(1+3x)(1+2x-4x^2)}}{4x^2(1-x)}$$

Find: by computer

The J-fraction of f is

$$f = \mathbf{J}\left[\begin{matrix} (-\frac{1}{2}, -\frac{1}{2}, 2)^* \\ 1, (\frac{1}{4}, 2, 2)^* \end{matrix}\right].$$

Proof. $\mathbf{u}$ and $\mathbf{v}$ are periodic. It suffices to check:

$$f = \cfrac{1}{1 - \frac{1}{2}x - \cfrac{\frac{1}{4}x^2}{1 - \frac{1}{2}x - \cfrac{2x^2}{1 + 2x - 2x^2 f}}}$$

QED.

Fundamental relation

between J-fractions and Hankel determinants

$$H_n\left(\mathbf{J}\begin{bmatrix} u_1, u_2, \cdots \\ v_0, v_1, v_2, \dots \end{bmatrix}\right) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}$$

Well known. See, for example:

- Wall: 1948
- Flajolet: 1980
- Viennot : 1983
- Kratthenthaler : 1998

Jacobi Continued Fraction

- Hankel determinants
- Orthogonal polynomials
- Stieltjes algorithm
- Combinatorial aspects (Motzkin paths, Permutations, ...)

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J-Fraction of P2

Thue–Morse sequence

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

where

$$\mathbf{u} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$\mathbf{v} = 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3, 1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15, 1/15, -1/15, -15, 13, -3/13, 3/13, 13/3, -19/3, 3/19, -3/19, \ldots$$

Too bad

No nice formula for v_n. Even worse, there are rational numbers.

We cannot prove anything about the Hankel determinants.

J-Fraction of S2

The Gros Sequence

$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^n}} = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

$\mathbf{u} = -2, 7/3, 23/3, -167/21, -169/21, 7, 7, -629/105, -631/105,$
$7, 7, -57/7, -55/7, 65/9, 391/63, -17663/3255, -17677/3255,$
$391/63, 65/9, -55/7, -57/7, 7, 7, -211/35, -209/35, 7, 7,$
$-73/9, -71/9, \ldots$

$\mathbf{v} = 1, -3, -1/9, -63, -1/441, -63, -1, -35, -1/11025, -35, -1,$
$-63, -1/49, -63, -49/81, -1395/49, -1/216225, -1395/49,$
$-49/81, -63, -1/49, -63, -1, -35, -1/1225, -35, -1,$
$-63, -1/81, -63, \ldots$

Again, no nice formula for u_n, v_n

Main idea

let p be a prime number and f a sequence. We want to prove that $H_n(f) \not\equiv 0 \pmod{p}$.

- No nice formula for the coefficients in the J-fraction of f;
- We try to find a "nice" sequence g such that
- (1) $f \equiv g \pmod{p}$
- (2) g has simple J-fraction

By (2) we know $H(g)$.

By (1) we know $H_n(f) \equiv H_n(g) \pmod{p}$

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Question

How to find a $nice$ sequence g such that

- $g \equiv f$

- The coefficients in the J-fraction of g have nice formula ?

Fractional congruence

p: prime number

Let a, b, c be four integers such that $(p, b) = 1$.

The fraction a/b and the integer c are said to be

$$\textit{congruent modulo } p \text{ if } a \equiv cb \pmod{p}.$$

We write $a/b \equiv c \pmod{p}$.

$$8/9 \equiv 3/4 \equiv -3 \equiv 2 \pmod{5}$$

$\equiv$: means $\equiv \pmod p$

Lemma

Let $\mathbf{J}[\mathbf{u},\mathbf{v}] = f$ and $\mathbf{J}[\hat{\mathbf{u}},\hat{\mathbf{v}}] = \hat{f}$.

(1) If $f \equiv \hat{f}$, then $H(f) \equiv H(\hat{f})$.

(2) If $\mathbf{u} \equiv \hat{\mathbf{u}}$ and $\mathbf{v} \equiv \hat{\mathbf{v}}$, then $f \equiv \hat{f}$.

(3) If $\mathbf{v} \equiv \hat{\mathbf{v}}$, then $H(f) \equiv H(\hat{f})$.

J-Fraction of S2

$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^n}} = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

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Only one even number in $\mathbf{u}$ and $\mathbf{v}$. Let

$$g = \mathbf{J}\begin{bmatrix} 0, 1, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}$$

J-Fraction of S2

Let

$$g = \mathbf{J}\begin{bmatrix} 0, 1, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}.$$

Then

$$g = \frac{1}{1 - x^2 f},$$

where

$$f = \mathbf{J}\begin{bmatrix} 1, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}.$$

We have

$$f = \frac{1}{1 - x - x^2 f}$$

$$f = -\frac{1 - x - \sqrt{1 - 2\,x - 3\,x^2}}{2x^2}, \qquad g = \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x}$$

Proof of Coons's Theorem

Let

$$S_2 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}$$

$$g = \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x}$$

Since

$$H(g) = (1)^*$$

$$g \equiv S_2 \pmod 2$$

We have

$$H(S_2) \equiv (1)^* \pmod 2.$$

QED.

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Crucial Fact

$$(a + x)^p \equiv a^p + x^p \pmod{p}$$

So that

$$f(x^p) \equiv f(x)^p \pmod{p}$$

Proof of Coons's Theorem

$$x^2 S_2(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^n}}{1 + x^{2^n}} = x S_2(x) - \frac{x}{1+x} \pmod{2}$$

$$x S_2(x)^2 \equiv S_2(x) - \frac{1}{1+x} \pmod{2}$$

We get

$$S_2(x) \equiv \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x} \pmod{2}.$$

QED.

New results

Theorem

Let

$$P_3 = P_3(x) = \prod_{k \geq 0} (1 - x^{3^k}).$$

Then $H_n(P_3) \equiv (-1)^{n-1} \pmod 3$

Remark

$$P_4 = \prod_{k \geq 0} (1 - x^{4^k}), \quad P_5 = \prod_{k \geq 0} (1 - x^{5^k}), \quad \cdots$$

$H_n(P_m) \neq 0$ for all n NOT TRUE when $m \geq 4$

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Proof. We successively have

$$P_3(x) = (1-x)P_3(x^3) \equiv (1-x)P_3(x)^3 \pmod 3,$$

$$P_3(x)^2 \equiv \frac{1}{1-x} \pmod 3,$$

$$P_3(x) \equiv \sqrt{\frac{1}{1-x}} \pmod 3.$$

$$\sqrt{\frac{1}{1-x}} = \mathbf{J}\left[\begin{matrix} (-1/2)^* \\ 1, 1/8, (1/16)^* \end{matrix}\right] \equiv \mathbf{J}\left[\begin{matrix} (1)^* \\ 1, -1, (1)^* \end{matrix}\right] \pmod 3$$

QED

6. Grafting technique and Chopping method

J-Fraction of P2

Thue–Morse sequece

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

$$\mathbf{u} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$\mathbf{v} = 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3,$$
$$1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15, \ldots$$

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$$\mathbf{u} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$\mathbf{v} = 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3, \\ 1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15, \ldots$$

Let

$$g = \mathbf{J}\begin{bmatrix}1, 1, 1, 1, 1, 1, \ldots\\ 1, 0, 1, 1, 1, 1, \ldots\end{bmatrix}$$

J-Fraction of P2

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

Let

$$g = \mathbf{J}\begin{bmatrix}1, 1, 1, 1, 1, 1, \ldots\\ 1, {\color{red}0}, 1, 1, 1, 1, \ldots\end{bmatrix}$$

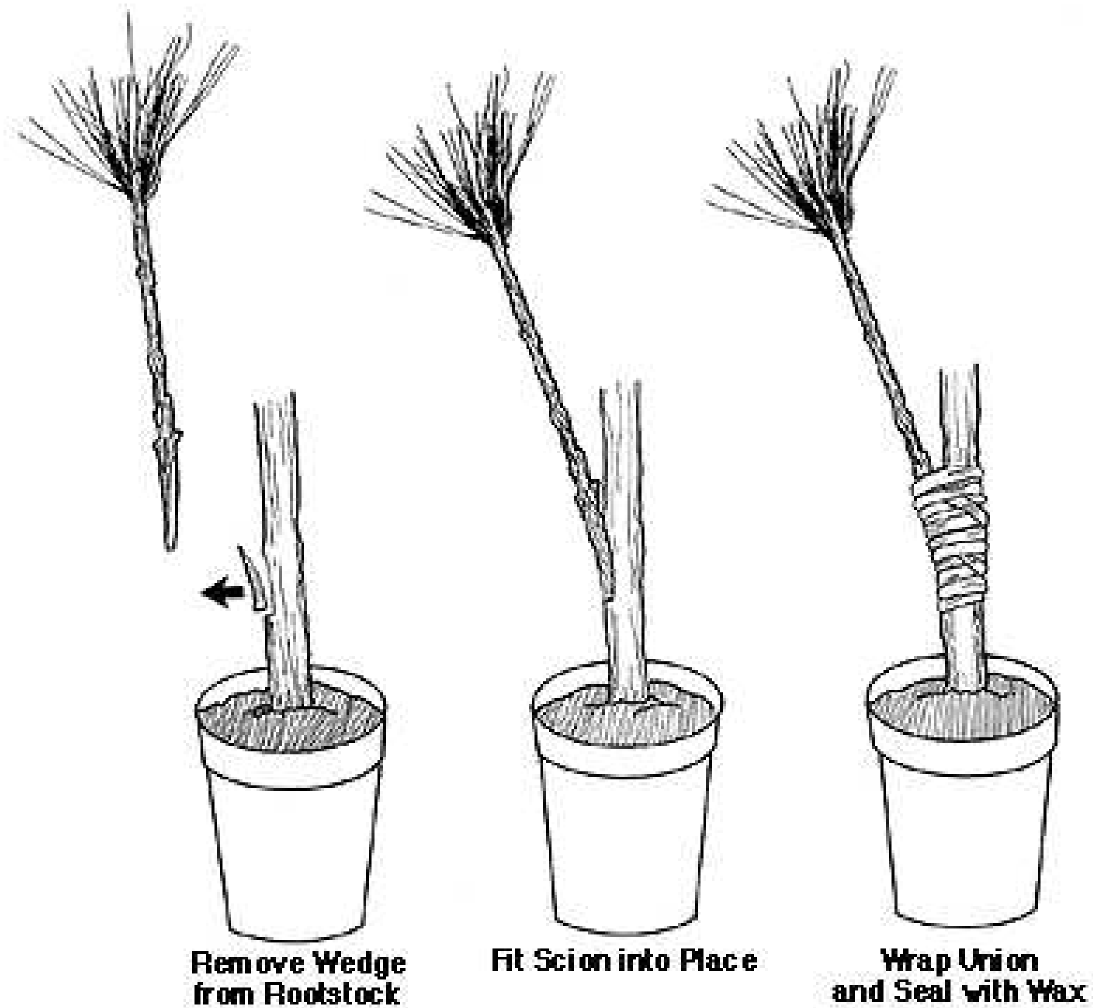
We have

$$g = \frac{1}{1-x}$$

- $P_2 \equiv g \pmod 2$
- $H_n(g) = 0$ for $n \geq 1$
- so that $H_n(P_2) \equiv 0 \pmod 2$.

- Yes! But we want to prove $H_n(P_2)/2^{n-1} \equiv 1 \pmod 2$

Grafting technique



Source: http://www.ces.ncsu.edu/depts/hort/hil/grafting.html

Grafting technique

$$F(x) = \mathbf{J}\begin{bmatrix} u_1, u_2, u_3, \cdots \\ v_0, v_1, v_2, v_3, \cdots \end{bmatrix}, \quad G(x) = \mathbf{J}\begin{bmatrix} a_1, a_2, a_3, \cdots \\ 1, b_1, b_2, b_3, \cdots \end{bmatrix}$$

The *grafting* of $G(x)$ into $F(x)$ of order k:

$$F|^k G = \mathbf{J}\begin{bmatrix} u_1, u_2, \cdots, u_k, a_1, a_2, a_3, \cdots \\ v_0, v_1, v_2, \cdots, v_k, b_1, b_2, b_3, \cdots \end{bmatrix}.$$

Grafting technique

$$F(x) = \mathbf{J}\left[\begin{matrix} & u_1, u_2, u_3, \cdots \\ v_0, & v_1, v_2, v_3, \cdots \end{matrix}\right], \quad G(x) = \mathbf{J}\left[\begin{matrix} & a_1, a_2, a_3, \cdots \\ 1, & b_1, b_2, b_3, \cdots \end{matrix}\right]$$

The *grafting* of $G(x)$ into $F(x)$ of order k:

$$F|^k G = \mathbf{J}\left[\begin{matrix} & u_1, u_2, \cdots, u_k, a_1, a_2, a_3, \cdots \\ v_0, & v_1, v_2, \cdots, v_k, b_1, b_2, b_3, \cdots \end{matrix}\right].$$

Advantage:

- Keep u_i, v_i
- Take $a_i, b_i \pmod{p}$.

Grafting technique

$$F(x) = \mathbf{J}\left[\begin{matrix} & u_1, u_2, u_3, \cdots \\ v_0, & v_1, v_2, v_3, \cdots \end{matrix}\right], \quad G(x) = \mathbf{J}\left[\begin{matrix} & a_1, a_2, a_3, \cdots \\ 1, & b_1, b_2, b_3, \cdots \end{matrix}\right]$$

The *grafting* of $G(x)$ into $F(x)$ of order k:

$$F|^k G = \mathbf{J}\left[\begin{matrix} & u_1, u_2, \cdots, u_k, a_1, a_2, a_3, \cdots \\ v_0, & v_1, v_2, \cdots, v_k, b_1, b_2, b_3, \cdots \end{matrix}\right].$$

Advantage:

- Keep u_i, v_i
- Take $a_i, b_i \pmod{p}$.

Let $F|G := F|^1 G$ and $F||G := F|^2 G$, for short.

J-Fraction of P2

- The J-fraction of the Thue-Morse sequence:

$$P_2 = \mathbf{J}\begin{bmatrix} & 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, \cdots & \end{bmatrix}$$

- Let $P_2 = P_2 | g$ where

$$g = \mathbf{J}\begin{bmatrix} & -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, \cdots & \end{bmatrix}$$

J-Fraction of P2

- The J-fraction of the Thue-Morse sequence:

$$P_2 = \mathbf{J}\begin{bmatrix} 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, \cdots \end{bmatrix}$$

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$$g = \mathbf{J}\begin{bmatrix} -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, \cdots \end{bmatrix}$$

- Only odd numbers in g. Let

$$\bar{g} = \mathbf{J}\begin{bmatrix} (1)^* \\ (1)^* \end{bmatrix}$$

- Finally we define

$$\bar{P}_2 = P_2|\bar{g} = \mathbf{J}\begin{bmatrix} 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots \\ 1, -2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots \end{bmatrix}$$

J-Fraction of P2

- The J-fraction of the Thue-Morse sequence:

$$P_2 = \mathbf{J}\begin{bmatrix} 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, \cdots \end{bmatrix}$$

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- Only odd numbers in g. Let

$$\bar{g} = \mathbf{J}\begin{bmatrix} (1)^* \\ (1)^* \end{bmatrix}$$

- Finally we define

$$\bar{P}_2 = P_2|\bar{g} = \mathbf{J}\begin{bmatrix} 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots \\ 1, -2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots \end{bmatrix}$$

- $H_n(P_2) \neq 0 \Leftrightarrow H_n(\bar{P}_2) \neq 0$.

Proof of APWW's Theorem

Define g by

$$P_2 = \frac{1}{1 + x + 2x^2 g}, \qquad g = \frac{1}{2x^2}(\frac{1}{P_2} - 1 - x).$$

We have

$$H_n(P_2) = (-2)^{n-1} H_{n-1}(g)$$

$$(?) \qquad \frac{1}{P_2} \equiv \sqrt{(1-x)(1+3x)} \pmod 4,$$

$$g \equiv \frac{1}{2x^2}\left(1 + x - \sqrt{(1-x)(1+3x)}\right) \pmod 2.$$

$$g = \mathbf{J}\left[\begin{matrix} (1)^* \\ (-1)^* \end{matrix}\right],$$

so that $H_n(g) \equiv 1 \pmod 2$. Hence, $H_n(P_2) \neq 0$.

Crucial Lemma

Lemma:

$$\sqrt{1-4x} \equiv 1 + 2\sum_{k=0}^{\infty} x^{2^k} \pmod 4.$$

Proof of APWW's Theorem

Let

$$f = \sqrt{\frac{1}{(1-x)(1+3x)}}$$

Then

$$(1-x)f(x) = \sqrt{1 - \frac{4x}{1+3x}} \equiv 1+2\sum_{k=0}^{\infty}\left(\frac{x}{1+3x}\right)^{2^k} \pmod 4$$

$$(1-x)f(x) \equiv 1+2\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{2^k} \pmod 4$$

and

$$(1-x^2)f(x^2) \equiv (1-x)f(x) - \frac{2x}{1+x} \pmod 4.$$

Proof of APWW's Theorem

On the other hand,

$$(1-x)P_2(x^2) = P_2(x),$$
$$P_2(x) = \frac{1}{1+x} \pmod 2,$$
$$(1-x^2)P_2(x^2) = (1+x)(1-x)P_2(x^2) = (1+x)P_2(x).$$

$$(1-x^2)P_2(x^2) \equiv (1-x)P_2(x) + \frac{2x}{1+x} \pmod 4.$$

Hence,

$$f \equiv P_2 \pmod 4.$$

QED

Chopping method

Proposition

Let

$$f = f(x) = \frac{1}{x^4} \sum_{k=1}^{\infty} \frac{x^{2^{k+1}}}{1 - x^{2^k}}.$$

Then, $H(f) \equiv (1,1,1,1,1,1,0,0)^* \pmod 2$.

Proof. We successively have

$$x^8 f(x^2) = \sum_{k=2}^{\infty} \frac{x^{2^{k+1}}}{1 - x^{2^k}} = \sum_{k=1}^{\infty} \frac{x^{2^{k+1}}}{1 - x^{2^k}} - \frac{x^4}{1 - x^2} = x^4 f(x) - \frac{x^4}{1 - x^2}$$

$$x^4 f(x^2) = f(x) - \frac{1}{1 - x^2}$$

$$x^4 f(x)^2 \equiv f(x) - \frac{1}{1 - x^2} \pmod 2$$

$$f(x) \equiv \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4} \pmod 2$$

$$f \equiv g \pmod 2$$

where

$$g = \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4}$$

By the next Lemma,

$$H(g) = (1, 1, 1, 1, -1-1, -2, -2, 1, 1, 3, 3, -1, -1, -4, -4, \ldots)$$

$$H(f) \equiv H_n(g) \equiv (1, 1, 1, 1, 1, 1, 0, 0)^* \pmod 2$$

QED.

Lemma A. Let

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4}.$$

Then

$$f = \mathbf{J}\left[\begin{matrix} & & & & & & & (0)^* & & & & & & \\ 1, & 1, & 1, & -1, & -1, & 2, & \frac{1}{2}, & -\frac{1}{2}, & -2, & 3, & \frac{1}{3}, & -\frac{1}{3}, & -3, & \ldots \end{matrix}\right].$$

Proof. $z = 1$ in the next Lemma.

Lemma B. Let

$$f(x;z) = \frac{1-(2z-1)x^2 - \sqrt{(1-x^2)(1-x^2-4x^4)}}{2x^2((1-z)+(1-z+z^2)x^2-x^4)}.$$

Then

$$f = \mathbf{J}\Big[1, z, \tfrac{1}{z}, -\tfrac{1}{z}, -z, z+1, \overset{(0)^*}{\tfrac{1}{z+1}}, -\tfrac{1}{z+1}, -(z+1), \ldots\Big].$$

Proof of Lemma B. Easy to check that $f(x;z)$ verifies the following functional equation:

$$f(x;z) = \cfrac{1}{1 - \cfrac{zx^2}{1 - \cfrac{\frac{1}{z}x^2}{1 + \cfrac{\frac{1}{z}x^2}{1 + zx^2 f(x;z+1)}}}}.$$

QED.

The most difficult part for proving Lemma A is to find an appropriate generalization, namely, Lemma B.

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4}$$

$$\Downarrow ?$$

$$f(x;z) = \frac{1 - (2z-1)x^2 - \sqrt{(1-x^2)(1-x^2-4x^4)}}{2x^2((1-z) + (1-z+z^2)x^2 - x^4)}$$

Chopping method

- We need prove that $f = f_1$ where

$$f_1 = \mathbf{J}\left[1, 1, 1, -1, -1, 2, \tfrac{1}{2}, \overset{(0)^*}{-\tfrac{1}{2}}, -2, 3, \tfrac{1}{3}, -\tfrac{1}{3}, -3, \ldots \right].$$

Chopping method

- We need prove that $f = f_1$ where

$$f_1 = \mathbf{J}\left[\begin{matrix} & & & & & & & (0)^* & & & & & \\ 1, & 1, & 1, & -1, & -1, & 2, & \frac{1}{2}, & -\frac{1}{2}, & -2, & 3, & \frac{1}{3}, & -\frac{1}{3}, & -3, \ldots \end{matrix}\right].$$

- Define f_2 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_1:

$$f_2 = \mathbf{J}\left[\begin{matrix} & & & (0)^* & & & & & \\ 1, & 2, & \frac{1}{2}, & -\frac{1}{2}, & -2, & 3, & \frac{1}{3}, & -\frac{1}{3}, & -3, \ldots \end{matrix}\right].$$

Chopping method

- We need prove that $f = f_1$ where

$$f_1 = \mathbf{J}\left[\begin{matrix} & & & & & & & (0)^* & & & & & & \\ 1, & 1, & 1, & -1, & -1, & 2, & \frac{1}{2}, & -\frac{1}{2}, & -2, & 3, & \frac{1}{3}, & -\frac{1}{3}, & -3, & \dots \end{matrix}\right].$$

- Define f_2 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_1:

$$f_2 = \mathbf{J}\left[\begin{matrix} & & & (0)^* & & & & & \\ 1, & 2, & \frac{1}{2}, & -\frac{1}{2}, & -2, & 3, & \frac{1}{3}, & -\frac{1}{3}, & -3, & \dots \end{matrix}\right].$$

- We get the first coefficients of f_2

$$f_2 = (1, 0, 2, 0, 5, 0, 12, 0, 30, 0, 75, 0, 190, 0, 483, 0, 1235, \dots).$$

Chopping method

- We need prove that $f = f_1$ where

$$f_1 = \mathbf{J}\left[\substack{(0)^* \\ 1,1,1,-1,-1,2,\frac{1}{2},-\frac{1}{2},-2,3,\frac{1}{3},-\frac{1}{3},-3,\ldots}\right].$$

- Define f_2 by deleting the first four pairs u_i, v_i $(i = 1,2,3,4)$ from the J-fraction of f_1:

$$f_2 = \mathbf{J}\left[\substack{(0)^* \\ 1,2,\frac{1}{2},-\frac{1}{2},-2,3,\frac{1}{3},-\frac{1}{3},-3,\ldots}\right].$$

- We get the first coefficients of f_2

$$f_2 = (1,0,2,0,5,0,12,0,30,0,75,0,190,0,483,0,1235,\ldots).$$

- With the help of a CAS, we guess that f_2 satisfies the equation

$$(x^6 - 3x^4 + x^2)f_2^2 + (-3x^2+1)f_2 - 1 = 0.$$

- Define f_3 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_2 and guess

$$(x^6 - 7x^4 + 2x^2)f_3^2 + (-5x^2 + 1)f_3 - 1 = 0,$$

- Repeat these steps,

$$(x^6 - 13x^4 + 3x^2)f_4^2 + (-7x^2 + 1)f_4 - 1 = 0,$$
$$\cdots$$

- Define f_3 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_2 and guess

$$(x^6 - 7x^4 + 2x^2)f_3^2 + (-5x^2 + 1)f_3 - 1 = 0,$$

- Repeat these steps,

$$(x^6 - 13x^4 + 3x^2)f_4^2 + (-7x^2 + 1)f_4 - 1 = 0,$$
$$\ldots$$

- We guess the general equation valid for every z (vertical guess)

$$(x^6-(z^2-z+1)x^4+(z-1)x^2)f_z^2+(-(2z-1)x^2+1)f_z-1 = 0.$$

- Solving the above equation yields the series $f(x; z)$

Example. Using the chopping method, we prove that

$$\frac{-2zx^2-(sx-x^2y-1)-\sqrt{(sx-x^2y-1)^2-(2x^2)^2}}{2x^2(x^2+x^2z^2+z(sx-x^2y-1))}$$

$$=\cfrac{1}{1-sx-\cfrac{\frac{\alpha_1}{\alpha_0}x^2}{1-\cfrac{\frac{\alpha_0}{\alpha_1}x^2}{1-sx-\cfrac{\frac{\alpha_2}{\alpha_1}x^2}{1-\cfrac{\frac{\alpha_1}{\alpha_2}x^2}{\ddots}}}}}$$

where α_n is defined by

$$\sum_n \alpha_n x^n = \frac{1+zx}{1+yx+x^2}.$$

7. Miscellaneous

Misc 1. Nice formula for v

Thue–Morse sequence

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

where

$$\mathbf{u} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$\mathbf{v} = 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3,$$
$$1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15,$$
$$1/15, -1/15, -15, 13, -3/13, 3/13, 13/3, -19/3, 3/19, -3/19, \ldots$$

Misc 1. Nice formula for v

Theorem

Let

$$\prod_{k=0}^{\infty}(1-x^{2^k}) = \cfrac{v_0}{1+u_1x-\cfrac{v_1x^2}{1+u_2x-\cfrac{v_2x^2}{1+u_3x-\cfrac{v_3x^2}{\ddots}}}}$$

Then, $u_n = (-1)^{n-1}$ for each $n = 1, 2, 3, \ldots$

Misc 2. A magical sequence

$$\sqrt{\frac{1}{(1-x)(1+3x)}} = 1 - x + 3\,x^2 - 7\,x^3 + 19\,x^4 - 51\,x^5 + \cdots$$

integral coefficients

$$s = 1 \quad -1 \quad 3 \quad -7 \quad 19 \quad -51 \quad 141 \quad -393 \quad 1107 \quad -3139$$

$$s \pmod 4 \equiv 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1$$

$= P_2$ (Thue-Morse)

$$s \pmod 3 \equiv 1 \quad -1 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1$$

$= P_3$

Misc 2. A magical sequence

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Misc 2. A magical sequence

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Misc 2. A magical sequence

$$\sqrt{\frac{1}{(1-x)(1+3x)}} = 1 - x + 3\,x^2 - 7\,x^3 + 19\,x^4 - 51\,x^5 + \cdots$$

$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1 - x^{2^k}) \pmod 4$$

$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1 - x^{3^k}) \pmod 3$$

Provide proofs of Theorems based on integer congruence instead of fractional congruence

Misc 3. Another magical sequence

$$\phi(x) = \frac{1}{1+x}\prod_{k=0}^{\infty}\Big(1 - (\frac{x}{1+x})^{2^k}\Big)$$

$$= 1 - 2\,x + 2\,x^2 - 6\,x^4 + 20\,x^5 - 48\,x^6 + 96\,x^7 - 166\,x^8 + \cdots$$

Misc 3. Another magical sequence

$$\phi(x) = \frac{1}{1+x} \prod_{k=0}^{\infty} \Big(1 - (\frac{x}{1+x})^{2^k}\Big)$$

$$= 1 - 2\,x + 2\,x^2 - 6\,x^4 + 20\,x^5 - 48\,x^6 + 96\,x^7 - 166\,x^8 + \cdots$$

Then,

$$H_n(\phi(x)) = H_n(\phi(x^2)) \qquad \text{for each } n$$

Misc 3. Another magical sequence

$$\phi(x) = 1 - 2\,x + 2\,x^2 + 0x^3 - 6\,x^4 + \cdots$$
$$\phi(x^2) = 1 - 2\,x^2 + 2\,x^4 + 0x^6 - 6\,x^8 + \cdots$$

$$H_3(\phi(x)) = \begin{vmatrix} 1 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & -6 \end{vmatrix} = 4$$

$$H_3(\phi(x^2)) = \begin{vmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{vmatrix} = 4$$

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Misc 4. Thue-Morse and Gros

Thue-Morse sequence

$$P_2 = P_2(x) = \prod_{k=0}^{\infty} (1 - x^{2^k})$$

Gros sequence

$$S_2 = S_2(x) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}$$

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$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1 - x^{2^n}}$$

Trivial sequence

$$\frac{1}{1-x} = \prod_{k=0}^{\infty}(1 + x^{2^k})$$

Misc 4. Thue-Morse and Gros

$$f(b) := \prod_{k=0}^{\infty} (1 + b x^{2^k})$$

$$= \mathbf{J} \left[\begin{matrix} -b, b, -b, b, -b, b, -b, b, -b, b, -b, b, \ldots \\ 1, b - b^2, 1, -(1 + b + b^2), \frac{-1}{1+b+b^2}, \frac{1+b-b^4}{1+b+b^2}, -\frac{(1+b+b^2)^2}{1+b-b^4}, \ldots \end{matrix} \right]$$

Misc 4. Thue-Morse and Gros

$$f(b) := \prod_{k=0}^{\infty} (1 + bx^{2^k})$$

$$= \mathbf{J}\left[\begin{matrix} -b, b, -b, b, -b, b, -b, b, -b, b, \ldots \\ 1, b - b^2, 1, -(1+b+b^2), \frac{-1}{1+b+b^2}, \frac{1+b-b^4}{1+b+b^2}, -\frac{(1+b+b^2)^2}{1+b-b^4}, \ldots \end{matrix}\right]$$

$b = 1$:

$$\frac{1}{1-x} = \mathbf{J}\left[\begin{matrix} -1, 1, -1, 1, -1, 1, -1, \ldots \\ 1, 0, 1, -3, \frac{-1}{3}, \frac{1}{3}, -9, \ldots \end{matrix}\right]$$

$$(1-x)S_2(x^2) = \mathbf{J}\left[\begin{matrix} 1, -1, 1, -1, 1, -1, \ldots \\ 1, 1, -3, \frac{-1}{3}, \frac{1}{3}, -9, \ldots \end{matrix}\right]$$

Misc 4. Thue-Morse and Gros

Thue-Morse sequence

$$P_2 = P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k})$$

Gros sequence

$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1 - x^{2^n}}$$

Roughly speaking (!)

$$S_2 \simeq \prod_{k=0}^{\infty}(1 + x^{2^k})$$

7. Evolution of the proofs of the APWW theorem

Hankel determinant

$$\mathbf{a} = (a_0, a_1, a_2, \ldots)$$

$$f = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$H_n(f) := \begin{vmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \ldots & a_{2n-2} \end{vmatrix}$$

Allouche, Peyrière, Wen, Wen (1998)

Theorem

Let

$$P_2 = P_2(x) = \prod_{k=0}^{\infty} (1 - x^{2^k})$$

be the Thue-Morse sequence.

Then, the Hankel determinant $H_n(P_2) \neq 0$ for every positive integer n.

Coons (2011)

Theorem

Let
$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^n}}.$$

Then $H_n(S_2) \equiv 1 \pmod 2$.

Remark: In fact, it is equivalent to the APWW Theorem

First proof [Allouche, Peyrière, Wen, Wen]

- “Sudoku method”
- Sixteen recurrence relations between determinants
- 12 pages

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- Combinatorial proof
- Count the number of permutations modulo 2
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Third proof [H.]

- Using Jacobi continued fraction
- 1 page

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- 1 page

Fourth proof [H.]

Encore . . . ?

- Automatic computer proof

Fourth proof [H.]

Encore . . . ?

- Automatic computer proof

Regular paperfolding sequence

- Regular paperfolding sequence:

$$G_{0,2}(x) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+2}}}.$$

- Coons and Vrbik (2012) wrote a C++ program for computing the Hankel determinants $H_n(G_{0,2}) \pmod 2$ upto $n = 8196$

1,1,1,0,0,1,0,0,1,1,1,1,1,0,0,1,0,0,1,1,1,1,1,0,0,1,0,0,1,1,1,...

and conjectured the sequence is periodic with period 10:

$$H(G_{0,2}) \pmod 2 = (1,1,1,0,0,1,0,0,1,1)^*.$$

- Guo, Wu and Wen (2013) proved.

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$$H(G_{0,2}) \pmod 2 = (1,1,1,0,0,1,0,0,1,1)^*.$$

- Guo, Wu and Wen (2013) proved.

Automatic computer proof

My program

- computes the first values of the Hankel determinants $H_n(G_{0,2}) \pmod 2$

1,1,1,0,0,1,0,0,1,1,1,1,1,0,0,1,0,0,1,1,1,1,1,0,0,1,0,0,1,1,1,...

- and proves the periodicity

$$H(G_{0,2}) \pmod 2 = (1,1,1,0,0,1,0,0,1,1)^*.$$

New results

Theorem [H, 2014].

For each pair of positive intergers a, b, let

$$G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}}.$$

Then $H(G_{a,b}) \pmod 2$ is periodic.

The following relations are calculated and proved by a computer program automatically.

$$H(G_{0,0}) \equiv (1)^*;$$

Michael Coons, 2013; APWW, 1998

$$H(G_{0,1}) \equiv 1, 1, (0)^*;$$
$$H(G_{1,0}) \equiv (1)^*;$$

$$H(G_{0,2}) \equiv (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*;$$

Guo, Wu, Wen, 2013

" $\equiv$ " means " $\equiv \pmod 2$"

$$H(G_{1,1}) \equiv (1,1,0,0,1,1)^*;$$

$$H(G_{2,0}) \equiv (1,1,0,0)^*;$$

$$H(G_{0,3}) \equiv \big(1^5 0^2 1^1 0^6 1^3 0^2 1^2 0^2 1^2 0^4 1^1 0^4 1^1 0^2 1^1 0^2 1^1 \\ 0^4 1^1 0^4 1^2 0^2 1^2 0^2 1^3 0^6 1^1 0^2 1^4\big)^*;$$

[period is 74]

$$H(G_{1,2}) \equiv 1,1,1,(0)^*;$$

$$H(G_{2,1}) \equiv (1,1,1,1,1,1,0,0)^*;$$

Chopping method

$$H(G_{3,0}) \equiv (1,1,0,0,0,0,0,0)^*;$$

$$H(G_{0,4}) \equiv (1^9 0^2 1^1 0^2 \cdots 1^1 0^2 1^8)^* \quad \text{[period is } \mathbf{1078}\text{];}$$

Oh là là

$$H(G_{1,3}) \equiv (1,1,1,1,1,0,0,0,0,1,1,0,0,0,0,1,1,1,1,1)^*;$$
$$H(G_{2,2}) \equiv (1,1,0,0,0,0,0,0,1,1,0,0)^*;$$
$$H(G_{3,1}) \equiv (1,1,1,0,0,0,0,1,1,1,0,0,0,0,0,0)^*;$$
$$H(G_{4,0}) \equiv (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^*.$$

8. Hankel continued fraction

Jacobi Continued Fraction (J-Fraction)

$\mathbf{u} = (u_1, u_2, \ldots)$

$\mathbf{v} = (v_0, v_1, v_2, \ldots)$

$$\cfrac{v_0}{1 + u_1 x - \cfrac{v_1 x^2}{1 + u_2 x - \cfrac{v_2 x^2}{1 + u_3 x - \cfrac{v_3 x^2}{\ddots}}}}$$

Fundamental relation

$$H_n\left(\mathbf{J}\begin{bmatrix} u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}\right) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}$$

Stieltjes Continued Fraction (S-Fraction)

$\mathbf{a} = (a_0, a_1, a_2, \ldots)$

$$\cfrac{a_0}{1 + \cfrac{a_1 x}{1 + \cfrac{a_2 x}{1 + \cfrac{a_3 x}{\ddots}}}}$$

Contraction Theorem (Relation between S-fraction and J-fraction):

$$
\begin{aligned}
u_1 &= a_1; \\
u_k &= a_{2k-2} + a_{2k-1}; \qquad (k \geq 2) \\
v_0 &= a_0; \\
v_k &= a_{2k-1} a_{2k}; \qquad (k \geq 1)
\end{aligned}
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v_0 &= a_0; \\
v_k &= a_{2k-1} a_{2k}; \qquad (k \geq 1)
\end{aligned}
$$

Fundamental relation

$$H_n(S(x)) = a_0^n (a_1 a_2)^{n-1} (a_3 a_4)^{n-2} \cdots (a_{2n-3} a_{2n-2})$$

Real number and continued fraction

Real numbers $\longleftrightarrow$ Continued fractions

$$\sqrt{2} \qquad 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

Real number and continued fraction

Real numbers	$\longleftrightarrow$	Continued fractions
$\sqrt{2}$		$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$
Quadratic numbers	$\longleftrightarrow$	Periodic continued fractions

Similar result for J-fraction ?

Real numbers	$\longleftrightarrow$	Continued fractions
$\sqrt{2}$		$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$
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Formal power series	$\longleftarrow$	Jacobi continued fractions

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Real numbers	$\longleftrightarrow$	Continued fractions
Quadratic numbers	$\longleftrightarrow$	Periodic continued fractions
Formal power series	$\longleftarrow$	Jacobi continued fractions

Remark: The $\longrightarrow$ *in the third relation is missing.*

Condition: The Jacobi continued fraction of a power series $F(x)$ *exists if and only if all the Hankel determinants of* $F(x)$ *are nonzero.*

Similar result for S-fraction ?

Real numbers	$\longleftrightarrow$	Continued fractions
Quadratic numbers	$\longleftrightarrow$	Periodic continued fractions
Formal power series	$\longleftarrow$	S-fractions

Remark: The $\longrightarrow$ *in the third relation is missing.*

Condition: The S-fraction of a power series $F(x)$ *exists if and only if all the Hankel determinants of* $F(x)$ *are nonzero.*

C-Fraction

Johann Cigler (2013) : A special class of Hankel determinants

C-Fraction

$\mathbf{a} = (a_0, a_1, a_2, \ldots),\ a_i \neq 0$
$\mathbf{q} = (q_0, q_1, q_2, \ldots),\ q_0 \geq 0, q_i \geq 1$

$$\cfrac{a_0 x^{q_0}}{1 + \cfrac{a_1 x^{q_1}}{1 + \cfrac{a_2 x^{q_2}}{1 + \cfrac{a_3 x^{q_3}}{\ddots}}}}$$

Frank 1946 :

power series $\longleftrightarrow$ C-fraction

C-Fraction

Conjecture (Paul Barry, 2012.05)

$$f = \cfrac{a_0 x^{q_0}}{1 + \cfrac{a_1 x^{q_1}}{1 + \cfrac{a_2 x^{q_2}}{1 + \cfrac{a_3 x^{q_3}}{\ddots}}}}$$

If all $a_i = \pm 1$, then $H_n(f) = \pm 1$.

Several interesting examples are given !

C-Fraction

Paul Barry, 2012.12 (a second paper)

"*We study the Hankel transforms of sequences whose generating function can be expressed as a C-fraction. In particular, we relate the index sequence of the non-zero terms of the Hankel transform to the powers appearing in the monomials defining the C- fraction. A closed formula for the Hankel transforms studied is given. As every power- series can be represented by a C-fraction, this gives in theory a closed form formula for the Hankel transform of any sequence. The notion of multiplicity is introduced to differentiate between Hankel transforms.*"

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This result is exactly we need !

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This result is exactly we need ! But ...

C-Fraction

Johann Cigler (2013)

- found a counter example for “Barry’s Theorem”.

C-Fraction

Johann Cigler (2013)

- found a counter example for “Barry’s Theorem”.
- correctly stated the theorem.

C-Fraction

Theorem (Johann Cigler, 2013)

$$\mathbf{a} = (a_0, a_1, a_2, \ldots), \quad a_i \neq 0$$
$$\mathbf{b} = (b_0, b_1, b_2, \ldots), \quad b_{-1} = -1, \quad b_0 = 0, \quad b_{k+2} - b_k \geq 1$$

$$f = \cfrac{1}{1 - \cfrac{a_0 x^{b_1 - b_{-1}}}{1 - \cfrac{a_1 x^{b_2 - b_0}}{1 - \cfrac{a_2 x^{b_3 - b_1}}{\ddots}}}}$$

Then alll non-vanishing Hankel determinants are given by

$$H_{b_k}(f) = (-1) a_0^{b_k - b_0} a_1^{b_k - b_1} a_2^{b_k - b_2} \cdots a_{k-1}^{b_k - b_{k-1}}$$

C-Fraction

Theorem (Buslaev 2010, Cigler, 2013)

$$\mathbf{a} = (a_0, a_1, a_2, \ldots), \quad a_i \neq 0$$
$$\mathbf{b} = (b_0, b_1, b_2, \ldots), \quad b_{-1} = -1, \quad b_0 = 0, \quad b_{k+2} - b_k \geq 1$$

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After Cigler's correction, every power series has a unique C-fraction expansion, but not all C-fractions have Hankel determinant formula.

Fraction type	Fraction existence	Hankel det. formula
S, J-fraction	No	Yes
C-fraction	Yes	No

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Fraction type	Fraction existence	Hankel det. formula
S, J-fraction	No	Yes
C-fraction	Yes	No
(Today)	Yes	Yes

Main Definition (H., 2014)

A Hankel continued fraction (H-fraction) is a continued fraction of the following form

$$F(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+2}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+2}}{1 + u_3(x)x - \ddots}}}$$

where

- $v_j \neq 0$ are contants,
- k_j are nonnegative integers
- $u_j(x)$ are polynomials of degree less than or equal to k_{j-1}.

By convention, 0 is of degree -1.

Fundamental Theorem (H., 2014)

(i) Each H-fraction defines a power series, and conversely, for each power series $F(x)$, the H-fraction expansion of $F(x)$ exists and is unique.

$$\text{power series} \longleftrightarrow \ H\text{-fraction}$$

Fundamental Theorem (H., 2014)

(i) Each H-fraction defines a power series, and conversely, for each power series $F(x)$, the H-fraction expansion of $F(x)$ exists and is unique.

$$\text{power series} \longleftrightarrow \quad H\text{-fraction}$$

(ii) All non-vanishing Hankel determinants of $F(x)$ are given by

$$H_{s_j}(F(x)) = (-1)^{\epsilon} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}},$$

where $\epsilon = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 0$.

Proof

(ii) Well known method: Cigler, Andrews, Wimp, Buslaev, ...

Lemma

Let k be a nonnegative integer and let $F(x), G(x)$ be two power series satisfying

$$F(x) = \frac{x^k}{1 + u(x)x - x^{k+2}G(x)},$$

where $u(x)$ is a polynomial of degree less than or equal to k. Then,

$$H_n(F) = (-1)^{k(k+1)/2} H_{n-k-1}(G).$$

Proof of Lemma

Let $F(x) = \sum_j f_j x^j$.
Let $x^k/F(x) = \sum_j b_j x^j$
Let $G(x) = \sum_j g_j x^j$.

Define four matrices by

$$\begin{aligned} \mathbf{F}_1 &= (f_{i-j+k})_{0\le i,j\le n-1}, \\ \mathbf{G} &= \mathrm{Diag}\big((b_{i+j-k})_{0\le i,j\le k},\ (g_{i+j})_{0\le i,j\le n-k-1}\big), \\ \mathbf{F} &= (f_{i+j})_{0\le i,j\le n-1}, \\ \mathbf{B} &= (b_{j-i})_{0\le i,j\le n-1}, \end{aligned}$$

We can prove that

$$\mathbf{F}_1 \times \mathbf{G} = \mathbf{F} \times \mathbf{B}.$$

Take determinants.

Example 1

Let

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1+x}}}{2x^4} \in \mathbb{Q}[[x]].$$

Then

$$f(x) = \cfrac{1}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{1 + x - \cfrac{x^4}{\ddots}}}}}}.$$

Hence

$$H(f) = (1, 1, 0, 0, -1, -1, 0, 0)^*.$$

Example 2

The number of distinct partitions

$$g(x) = \prod_{n\geq 1}(1+x^k) \in \mathbb{Q}[[x]]$$

$$=1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+\cdots$$

Then $g(x) =$

$$\cfrac{1}{1-x-\cfrac{x^3}{1+x+\cfrac{x^5}{1-x+x^2-x^3-\cfrac{x^5}{1+x+x^2+\cfrac{x^3}{1-x+\cfrac{x^3}{\ddots}}}}}}$$

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Then $g(x) =$

$$\cfrac{1}{1-x-\cfrac{x^3}{1+x+\cfrac{x^5}{1-x+x^2-x^3-\cfrac{x^5}{1+x+x^2+\cfrac{x^3}{1-x+\cfrac{x^3}{\ddots}}}}}}$$

- integral coefficients ? No

9. Periodicity

Main Theorem (H., 2014)

(Roughly speaking)

The H-fraction of a quadratic power series

- exists, is unique,
- is ultimately periodic,
- can be *entirely* calculated by CAS.

Main Theorem (H., 2014)

Let p be a prime number and $F(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[x]$ are three polynomials with one of the following conditions

(i) $B(0) = 1,\ C(0) = 0,\ C(x) \neq 0;$

(ii) $B(0) = 1,\ C(x) = 0;$

(iii) $B(0) = 1,\ C(0) \neq 0,\ A(0) = 0;$

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

Then, the Hankel continued fraction expansion of $F(x)$ exists and is ultimately periodic. Also, the Hankel determinant sequence $H(F)$ is ultimately periodic.

Algorithm NextABC

Prototype: $(A^*, B^*, C^*; k, A_k, D) = \texttt{NextABC}(A, B, C)$

Input: $A(x), B(x), C(x) \in \mathbb{F}[x]$ three polynomials such that $B(0) = 1$, $C(0) = 0, C(x) \neq 0, A(x) \neq 0$;

Output: $A^*(x), B^*(x), C^*(x) \in \mathbb{F}[x]$, $k \in \mathbb{N}^+$, $A_k \neq 0 \in \mathbb{F}$, $D(x) \in \mathbb{F}[x]$ a polynomial of degree less than or equal to $k+1$ such that $D(0) = 1$.

Step 1 [Define k, A_k]. Since $A(x) \neq 0$, let $A(x) = A_k x^k + O(x^{k+1})$ with $A_k \neq 0$.

Step 2. Let

$$F(x) = \frac{-B + \sqrt{B^2 - 4AC}}{2C};$$

or

$$F(x) = \frac{-A(x)}{B(x) + C(x)F(x)}.$$

Get the first terms of

$$F(x) = -A_k x^k + \cdots + O(x^{2k+2});$$

$$\frac{-A_k x^k}{F(x)} = 1 + \cdots + O(x^{k+2}).$$

Step 3 [Define D]. Define $D(x), G(x)$ by

$$\frac{-A_k x^k}{F(x)} = D(x) - x^{k+2} G(x)$$

where $D(x)$ is a polynomial of degree less than or equal to $k+1$ such that $D(0) = 1$ and $G(x)$ is a power series.

Step 4 [Define A^*, B^*, C^*]. Let

$$A^*(x) = \left(-D^2 A / A_k + BDx^k - CA_k x^{2k}\right) / x^{2k+2};$$
$$B^*(x) = 2AD/(A_k x^k) - B;$$
$$C^*(x) = -Ax^2 / A_k.$$

Lemma

Let $A(x), B(x), C(x) \in \mathbb{F}[x]$ be three polynomials such that $B(0) = 1, C(0) = 0, C(x) \neq 0, A(x) \neq 0$ and

$$(A^*, B^*, C^*; k, A_k, D) = \texttt{NextABC}(A, B, C).$$

If $F(x)$ is the power series defined by

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

Then, $F(x)$ can be written as

$$F(x) = \frac{-A_k x^k}{D(x) - x^{k+2}G(x)}$$

where $G(x)$ is a power series satisfying

$$A^*(x) + B^*(x)G(x) + C^*(x)G(x)^2 = 0.$$

./..

Lemma (continued)

Furthermore, $A^*(x), B^*(x), C^*(x)$ are three polynomials in $\mathbb{F}[x]$ such that $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$ and

$$\deg(A^*) \leq d;\ \deg(B^*) \leq d+1;\ \deg(C^*) \leq d+2,$$

where

$$d = d(A,B,C) = \max(\deg(A), \deg(B) - 1, \deg(C) - 2).$$

Algorithm HFrac

Prototype: $(a_k, d_k, D_k)_{k=0,1,\ldots} = \texttt{HFrac}(A, B, C; p)$

Input: p a prime number;

$A(x), B(x), C(x) \in \mathbb{F}_p[x]$ three polynomials such that $B(0) = 1$, $C(0) = 0$ and $C(x) \neq 0$;

Output: a finite or periodic sequence $(a_k, d_k, D_k)_{k=0,1,\ldots}$

Step 1. $j := 0$, $A^{(j)} := A$, $B^{(j)} := B$, $C^{(j)} := C$.

Step 2. If $A^{(j)} = 0$, then return the finite sequence

$$(a_k, d_k, D_k)_{k=0,1,\ldots,j-1}.$$

The algorithm terminates.

Step 3. If $A^{(j)} \neq 0$, then let

$$(A^{(j+1)}, B^{(j+1)}, C^{(j+1)}; d_j, a_j, D_j) := \text{NextABC}(A^{(j)}, B^{(j)}, C^{(j)})).$$

Let $j := j + 1$.

Step 4. If there exits $0 \leq i < j$ such that

$$(A^{(i)}, B^{(i)}, C^{(i)}) = (A^{(j)}, B^{(j)}, C^{(j)}),$$

then return the infinite sequence

$$((a_k, d_k, D_k)_{k=0,1,\ldots,i-1}, (a_k, d_k, D_k)^*_{k=i,i+1,\ldots,j-1}).$$

The algorithm terminates. Else, go to Step 2.

Remark 1.
The loop Steps 2-4 will be broken at Step 2 or Step 4, since the degrees of the polynomials $A^{(i)}, B^{(i)}, C^{(i)}$ are bounded, and the coefficients are taken from $\mathbb{F}_p$. The number of different triplets $(A^{(i)}, B^{(i)}, C^{(i)})$ is finite.

Remark 2.

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

$F(x)$ is well defined with the condition

$B(0) = 1$, $C(0) = 0$ and $C(x) \neq 0$

In fact,

$$F(x) = \frac{-A(x) - C(x)F(x)^2}{B(x)}$$

Proof of the Main Theorem

(i) If $B(0)=1, C(0)=0, C(x)\neq 0$, let

$$(a_k, d_k, D_k)_{k=0,1,\ldots} = \text{HFrac}(A, B, C; p).$$

By Lemma,

$$F(x) = \cfrac{-a_0 x^{d_0}}{D_0(x) + \cfrac{a_1 x^{d_0+d_1+2}}{D_1(x) + \cfrac{a_2 x^{d_1+d_2+2}}{D_2(x) + \cfrac{a_3 x^{d_2+d_3+2}}{\ddots}}}}$$

and the above H-fraction is ultimately periodic.

(ii) $B(0) = 1,\ C(x) = 0$;

(iii) $B(0) = 1,\ C(0) \neq 0,\ A(0) = 0$;

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

Using (i) with some modifications.

For example,

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

$F(x)$ exists:

$$F(x) = \sqrt{\frac{-A(x)}{C(x)}} = \sqrt{\frac{(a_k x^k)^2 + \cdots}{C(x)}} = a_k x^k \sqrt{\frac{1 + \cdots}{C(x)}}$$

Let

$$F(x) = \frac{a_k x^k}{D(x) - x^{k+2} G(x)}.$$

Then, $G(x)$ satisfies

$$A^*(x) + B^*(x)G(x) + C^*(x)G(x)^2 = 0$$

with A^*, B^*, C^* defined by:

$$A^*(x) = (D^2 A + C a_k^2 x^{2k})/x^{3k+2};$$
$$B^*(x) = -2ADx^{k+2}/x^{3k+2};$$
$$C^*(x) = Ax^{2k+4}/x^{3k+2}.$$

If $p \neq 2$, then A^*, B^*, C^* are polynomials such that

$$B^*(0) \neq 0, C^*(0) = 0, C^*(x) \neq 0.$$

Apply (i) for (A^*, B^*, C^*).

Lemma

If the H-fraction expansion of a power series F is ultimately periodic, then the Hankel determinant sequece $H(F)$ is ultimately periodic.

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Proof. By the Fundamental Theorem

$$H_{s_j}(F(x)) = (-1)^{\epsilon} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}},$$

where $\epsilon = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 0$.

$\mathbf{s}, \mathbf{v}$ periodic implies H_n periodic.

Notation

$$\frac{v_0}{u_1} + \frac{v_1}{u_2} + \frac{v_2}{u_3} + \frac{v_3}{u_4} + \cdots = \cfrac{v_0}{u_1 + \cfrac{v_1}{u_2 + \cfrac{v_2}{u_3 + \cfrac{v_3}{u_4 + \ddots}}}}.$$

Example 1

Let $p = 5$ and

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_5[[x]]$$

or

$$-1 + (1 - x^4)F + (-x + x^5)F^2 = 0.$$

$$A := -1; \quad B := 1 - x^4; \quad C := -x + x^5;$$

$$B(0) = 1, \quad C(0) = 0, \quad C(x) \neq 0$$

By Algorithm HFrac, F has the following H-fraction expansion

$$\frac{1}{1+4x} + \Big(\frac{4x^2}{1+3x} + \frac{3x^2}{1+x} + \frac{4x^3}{1+3x+2x^2} + \frac{4x^3}{1+x} + \frac{3x^2}{1+3x} + \frac{4x^2}{1+3x} + \frac{4x^2}{1+3x} + \Big)^*.$$

$$H(g) = (1,1,1,2,0,2,4,1,4,1,4,2,0,2,1,1)^*.$$

Example 2

Same F as Example 1, but with $p = 2$

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_2[[x]]$$

$$F = \frac{1}{1+x} + \left(\frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \right)^*.$$

$H(F) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*.$

Example 3

Let $p = 2$ and $G = xF$ where F is defined in Example 2.

$$G = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2} \in \mathbb{F}_2[[x]]$$

$$-x + (1 - x^4)G + (-1 + x^4)G^2 = 0 \quad \text{with} \quad G(0) = 0.$$

Since $C(x) = (-1 + x^4), \quad C(0) = 1$, we cannot apply Algorithm HFrac directly Let

$$G = \frac{x}{1 + x + x^2 + x^3 G_1}.$$

$$x^3 + (1 + x^4)G_1 + x^3 G_1^2 = 0.$$

By Algorithm HFrac, we get the following H-fraction expansion

$$G_1 = \frac{x^3}{1+x^4} + \left(\frac{x^6}{1} + \frac{x^4}{1+x^2} + \frac{x^4}{1} + \frac{x^6}{1+x^4} + \right)^*.$$

Hence

$$G = \frac{x}{1+x+x^2} + \left(\frac{x^6}{1+x^4} + \frac{x^6}{1} + \frac{x^4}{1+x^2} + \frac{x^4}{1} + \right)^*.$$

By Examples 2 and 3 ($xF = G$)

$$x\left(\frac{1}{1+x} + \left(\frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \right)^*\right)$$

$$= \frac{x}{1+x+x^2} + \left(\frac{x^6}{1+x^4} + \frac{x^6}{1} + \frac{x^4}{1+x^2} + \frac{x^4}{1} + \right)^*.$$

Remark. The H-fraction of xF is G, but not

$$\frac{x}{1+x} + \left(\frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \right)^*$$

Super δ-fraction

Definition.

For each positive integer $\delta = 1, 2, 3, \ldots$, a *super continued fraction* associated with δ, called *super δ-fraction* for short, is defined to be a continued fraction of the following form

$$F(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+\delta}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+\delta}}{1 + u_3(x)x - \ddots}}}$$

where

- $v_j \neq 0$ are contants,
- k_j are nonnegative integers
- $u_j(x)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, 0 is of degree -1.

Special cases

- When $\delta = 1$ and all $k_j = 0$, the super δ-fraction is the traditional S-fraction.

- When $\delta = 2$ and all $k_j = 0$, the super δ-fraction is the traditional J-fraction.

- The super 2-fraction is *Hankel continued fraction*.

- When $\delta = 1$ and $u_j(x) = 0$, the super 1-fraction is a special C-fraction (set $b_j = k_0 + k_1 + \cdots k_{j-1} + \lfloor j/2 \rfloor$ in [Ci13]).

Theorem (super 1-fraction)

Let p be a prime number and $F(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[x]$ are three polynomials with one of the following conditions

(i) $B(0) = 1,\ C(0) = 0,\ C(x) \neq 0$;

(ii) $B(0) = 1,\ C(x) = 0$;

(iii) $B(0) = 1,\ C(0) \neq 0,\ A(0) = 0$;

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

Then, the super 1-fraction expansion of $F(x)$ exists and is ultimately periodic.

The super δ-fraction expansion of $F(x)$ exists and is ultimately periodic.

True for $\delta = 1, 2$
False for $\delta \geq 3$

Theorem [H, 2014].

For each pair of positive intergers a, b, let

$$G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}}.$$

Then $H(G_{a,b}) \pmod 2$ is periodic.

Proof

Let $f(x) = G_{a,b}(x) \in \mathbb{F}_2[[x]]$. Then

$$x^{2^a} f(x) = \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^{a+1}} f(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^a} f(x^2) = f(x) - \frac{1}{1 - x^{2^b}};$$

$$1 + (1 + x^{2^b}) f(x) + x(1 + x^{2^b}) x^{2^a - 1} f(x)^2 = 0.$$

By the Main Theorem, the Hankel determinant sequence $H(f)$ is ultimately periodic.
QED.

Stern sequence (1858)

$(a_n)_{n=0,1,\ldots}$ is defined by $a_0 = 0, a_1 = 1$ and for $n \geq 1$

$$a_{2n} = a_n, \qquad a_{2n+1} = a_n + a_{n+1}.$$

The generating function for Stern's sequence is denoted by

$$S(x) = \sum_{n=0}^{\infty} a_{n+1} x^n$$

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Theorem

$$H_n(S)/2^{n-2} \equiv (0,0,1,1)^* \pmod 2.$$

Proof

It is well known

$$S(x) = (1 + x + x^2)S(x^2) \in \mathbb{Q}[[x]].$$

Since $S(x) \pmod 2$ is rational, there exists a positive integer N such that $H_k(S) \equiv 0 \pmod 2$ for all $k \geq N$. We must use the grafting technique. First, the H-fraction of $S(x)$ is

$$S(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 + 2x + \cfrac{2x^2}{1 - \cfrac{2x^3}{1 - 3/2x + 11/4x^2 + \ddots}}}}.$$

The even number 2 occurs in the sequence (v_j), in particular at position v_2. Define $G(x)$ by

$$S(x) = \frac{1}{1 - x - \dfrac{x^2}{1 + 2x + 2x^2 G(x)}}.$$

The power series $G(x)$ satisfies the following relation

$$(1 + x + x^2) + (1 + x + x^2)G(x) + x^4 G(x^2) \equiv 0 \pmod 2.$$

By Algorithm HFrac, we get $H(G) \equiv (1, 1, 0, 0)^* \pmod 2$. Hence

$$H_n(S)/2^{n-2} \equiv (0, 0, 1, 1)^* \pmod 2.$$

QED

Conclusion

	Known	New
1.	Traditional result for real number	Result for formal power series
2.	Definition with exception	No exception
3.	Results obtained case by case	Unified result
4.	Lengthy human proof	Automatic computer proof

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