Selberg integrals and evaluations of Pfaffians

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Abstract

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In our previous works "Pfaffian decomposition and a Pfaffian analogue of q-Catalan Hankel determinants" (by M.Ishikawa, H. Tagawa and J. Zeng, *J. Combin. Theory Ser. A*, **120**, 2013, 1263–1284) we have proposed several ways to evaluate certain Catalan-Hankel Pffafians and also formulated several conjectures. In this work we propose a new approach to compute these Catalan-Hankel Pffafians using Selberg's integral as well as their q-analogues. In particular, this approach permits us to settle most of the conjectures in our previous paper.

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1. Backgrounds
2. Pfaffian
3. A Catalan Hankel Pfaffian
4. De Bruijn's formula
5. Selberg-Askey integral
6. Baker & Forrester Integral
7. Combinatorial Numbers
8. An open problem

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1. Masao Ishikawa, Hiroyuki Tagawa, and Jiang Zeng, "Pfaffian decomposition and a Pfaffian analogue of q-Catalan Hankel determinants", *Journal of Combinatorial Theory, Series A* **120** (2013), 1263–1284.
2. Masao Ishikawa and Jiang Zeng, "Selberg integrals and Catalan-Pfaffian Hankel determinants", *DMTCS proc. of FPSAC 2014, Chicago, USA* (2014), 547–558.

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Backgrounds

Catalan numbers

Definition

For $n = 0, 1, 2, \ldots$, The Catalan number C_n is defined to be

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan number C_n counts the Dyck paths from $(0,0)$ to $(2n,0)$.

Example

The generating function for the Catalan numbers is given by

$$\frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} C_n t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots.$$

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Catalan Hankel determinants

Theorem (Desainte-Catherine-Viennot 1986)

The Catalan Hankel determinants have a nice product formula as follows:

$$\det(C_{i+j+r})_{0\le i,j\le n-1} = \prod_{0\le i\le j\le r-1} \frac{i+j+2n}{i+j}.$$

holds for $r, n \ge 0$.

Theorem (Krattenthaler 2007)

$$\det\left(C_{k_{i+1}+j}\right)_{0\le i,j\le n-1} = \prod_{1\le i<j\le n} (k_i - k_j) \prod_{i=1}^{n} \frac{(i+n)!(2k_i)!}{(2i)!k_i!(k_i+n)!}$$

for a positive integer n and non-negative integers $k_1, k_2, \ldots, k_n$.

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for a positive integer n and non-negative integers $k_1, k_2, \ldots, k_n$.

Notation

q-shifted factorials

We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

for $n = 0, 1, 2, \ldots$. $(a;q)_n$ is called the *q-shifted factorial*.
Frequently used compact natation:

$$(a_1, a_2, \ldots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n$$

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Moments

Here we consider the series

$$\mu_n = \frac{(aq;q)_n}{(abq^2;q)_n} \quad (n = 0, 1, 2, \dots).$$

Specializations

If we put $a = q^\alpha$, $b = q^\beta$ and let $q \to 1$, then

$$\mu_n \to \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Note that

$$\frac{\left(\frac{1}{2}\right)_n}{(2)_n} = \frac{C_n}{2^{2n}}, \quad \frac{\left(\frac{1}{2}\right)_n}{(1)_n} = \frac{1}{2^{2n}}\binom{2n}{n}, \quad \frac{\left(\frac{3}{2}\right)_n}{(2)_n} = \frac{1}{2^{2n}}\binom{2n+1}{n}.$$

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q-Catalan Hankel determinants

Theorem [I, Tagawa and Zeng'09]

Let n be a positive integer and t non-negative integer. Then

$$\det\left(\mu_{i+j+r-2}\right)_{1\le i,j\le n} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \left\{\frac{(aq;q)_r}{(abq^2;q)_r}\right\}^n \times \prod_{k=1}^{n} \frac{(q,aq^{r+1},bq;q)_{n-k}}{(abq^{n-k+r+1};q)_{n-k}(abq^{r+2};q)_{2(n-k)}}.$$

Theorem [I, Tagawa and Zeng'09]

Let n be a positive integer and k_1, $k_2,\dots,$ k_n non-negative integers.

$$\det\left(\mu_{k_i+j-2}\right)_{1\le i,j\le n1} = a^{\binom{n}{2}} q^{\binom{n+1}{3}} \prod_{i=1}^{n} \frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \times \prod_{1\le i<j\le n} (q^{k_i}-q^{k_j}) \prod_{i=1}^{n} (bq;q)_{n-i}.$$

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Pfaffian

Definition

Let $[n] = \{1, \dots, n\}$, and $\binom{[n]}{l} = \{\, (i_1, \dots, i_l) \mid 1 \le i_1 < \cdots < i_l \le n \,\}$ for positive integers n and l. Let $\mathfrak{S}_n$ denote the symetric group on $[n]$. For positive integers l and n, we set

$$\binom{[ln]}{\underbrace{l, \dots, l}_{n \text{ times}}} = \left\{ (i_1, \dots, i_{ln}) \in \mathfrak{S}_{ln} \mid i_{(k-1)l+1} < \cdots < i_{kl} \text{ for } 1 \le k \le n \right\},$$

$$\left\langle \underbrace{l, \dots, l}_{n \text{ times}}^{[ln]} \right\rangle = \left\{ (i_1, \dots, i_{ln}) \in \mathfrak{S}_{ln} \,\middle|\, \begin{array}{l} i_{(k-1)l+1} < \cdots < i_{kl} \text{ for } 1 \le k \le n, \\ i_{(k-1)l+1} < i_{kl+1} \text{ for } 1 \le k \le n-1. \end{array} \right\},$$

Definition

A permutation $\sigma \in \left\langle {[2n] \atop 2,\dots,2} \right\rangle$ is called a *perfect matching* or a *1-factor*.

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$$\left\langle\begin{matrix}[ln]\\ \underbrace{l,\dots,l}_{n\text{ times}}\end{matrix}\right\rangle=\left\{(i_1,\dots,i_{ln})\in\mathfrak{S}_{ln}\,\middle|\,\begin{matrix}i_{(k-1)l+1}<\cdots<i_{kl}\text{ for }1\le k\le n,\\ i_{(k-1)l+1}<i_{kl+1}\text{ for }1\le k\le n-1.\end{matrix}\right\},$$

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Pfaffian

Example

$$\binom{[4]}{2} = \{12, 13, 14, 23, 24, 34\},$$

$$\binom{[4]}{2,2} = \{12\,34, 13\,24, 14\,23, 23\,14, 2413, 34\,12\},$$

$$\left\langle {[4] \atop 2,2} \right\rangle = \{12\,34, 13\,24, 14\,23\},$$

$$\left\langle {[6] \atop 2,2,2} \right\rangle = \{12\,34\,56, 12\,35\,46, 12\,36\,45, 13\,24\,56, 13\,25\,46,$$
$$13\,26\,35, 14\,23\,56, 14\,25\,36, 14\,26\,35, 15\,23\,46,$$
$$15\,24\,36, 15\,26\,34, 16\,23\,45, 16\,24\,35, 16\,25\,34\},$$

and $\binom{[6]}{2,2,2}$ has 90 elements.

Pfaffian

Definition

We say a matrix $A = (a_{ij})_{i,j\geq 1}$ (resp. $A = (a_{ij})_{1\leq i,j\leq n}$) is *skew-symmetric* if it satisfies $a_{ji} = -a_{ij}$ for $i, j \geq 1$ (resp. $1 \leq i, j \leq n$).

Definition

Let n be a positive integer. The *Pfaffian* of A is defined to be

$$\begin{aligned}\mathrm{Pf}(A) &= \frac{1}{n!} \sum_{\sigma \in \binom{[2n]}{2,\dots,2}} \mathrm{sgn}(\sigma)\, a_{\sigma_1,\sigma_2} \dots a_{\sigma_{n-1},\sigma_n} \\ &= \sum_{\sigma \in \left\langle {[2n] \atop 2,\dots,2} \right\rangle} \mathrm{sgn}(\sigma)\, a_{\sigma_1,\sigma_2} \dots a_{\sigma_{n-1},\sigma_n}.\end{aligned}$$

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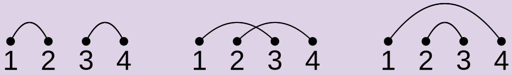
Pfaffians

Configurations arising from perfect matchings

We take $2n$ vertices in line on the x-axis, and connect σ_{2i-1} with σ_{2i}, $i = 1, \ldots, n$, by an arc in the upper half plane. Then we have

$$\operatorname{sgn}(\sigma) = (-1)^{\#\text{ crossings}}.$$



$\{\{1,2\},\{3,4\}\}$ $\{\{1,3\},\{2,4\}\}$ $\{\{1,4\},\{2,3\}\}$

Example

$$\operatorname{Pf}\begin{pmatrix} 0 & a_2^1 & a_3^1 & a_4^1 \\ -a_2^1 & 0 & a_3^2 & a_4^2 \\ -a_3^1 & -a_3^2 & 0 & a_4^1 \\ -a_4^1 & -a_4^2 & -a_4^3 & 0 \end{pmatrix} = a_2^1 a_4^3 - a_3^1 a_4^2 + a_4^1 a_3^2.$$

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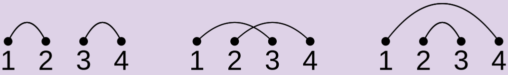

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A Catalan Hankel Pfaffian

Special cases

Theorem [I, Tagawa and Zeng'13]

Let $n \geq 1$ and $r \geq -1$ be integers.

1. Let C_n denote the Catalan numbers.

$$\mathrm{Pf}\left((j-i)C_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1}\frac{(4k+1)!}{(2k)!}\prod_{k=1}^{n}\frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-2\}!},$$

2. Let $D_n = \binom{2n}{n}$ denote the central binomial coefficients.

$$\mathrm{Pf}\left((j-i)D_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1}\frac{(4k)!}{(2k)!}\prod_{k=1}^{n}\frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-3\}!}.$$

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General case

Theorem [I, Tagawa and Zeng'13]

Let $n \geq 1$ and $r \geq -1$ be integers. Then we have

$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\leq i,j\leq 2n}$$
$$= a^{n(n-1)}q^{n(n-1)(4n+1)/3+n(n-1)r}$$
$$\times\prod_{k=1}^{n-1}(bq;q)_{2k}\prod_{k=1}^{n}\frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

In I, Tagawa and Zeng'13, we gave an algebraic proof of this theorem using Jackson's formula for ${}_6\phi_5$, which was relatively long. The purpose of this talk is to give another proof of this theorem using Selberg's integral, and settle the most of the conjectures described in I, Tagawa and Zeng'13.

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Hyperpfaffians and De Bruijn's formula

References (hyperpfaffians)

1. Alexander I. Barvinok., “New algorithms for linear k-matroid intersection and matroid *k*-parity problems”, *JMathematical Programming*, **69** (1995), 449–470.
2. Jean-Gabriel Luque and Jean-Yves Thibon, “Pfaffian and Hafnian identities in shuffle algebras”, *Advances in Applied Mathematics* **29** (2002), 620–646.
3. Sho Matsumoto, “Hyperdeterminantal expressions for Jack functions of rectangular shapes”, *Journal of Algebra* **320** (2008) 612–632.

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Definition 1 (Barvinok's and Luque-Thibon's hyperpfaffian)

Let l be an even integer, n a postive integer, and let $B = (B(\boldsymbol{i}))_{\boldsymbol{i}\in\binom{[2n]}{l}}$ be a l-demensional array. The *hyperpfaffian* $\mathrm{Pf}^{[l,1]}(B)$ *of B* is defined by Barvinok as

$$\mathrm{Pf}^{[l,1]}(B) = \frac{1}{n!}\sum_{\sigma\in\binom{[ln]}{l,\dots,l}} \mathrm{sgn}(\sigma)\prod_{i=1}^{n} B(\sigma(l(i-1)+1),\dots,\sigma(li)).$$

Let l, n postive integers, and let $B = (B(\boldsymbol{i}))_{\boldsymbol{i}\in\binom{[2n]}{l}}$ be as above. The *hyperpfaffian* $\mathrm{Pf}^{[l,1]}(B)$ *of B* is defined by Luque and Thibon as

$$\mathrm{pf}^{[l,1]}(B) = \sum_{\sigma\in\left\langle {[ln] \atop l,\dots,l} \right\rangle} \mathrm{sgn}(\sigma)\prod_{i=1}^{n} B(\sigma(l(i-1)+1),\dots,\sigma(li)),$$

which generalize Barvinok's definition to the case where l is odd.

Hyperpfaffian

Definition 2 (Matsumoto's hyperpfaffian)

Let m and n be postive integers, and let $B = (B(\boldsymbol{i}_1, \dots, \boldsymbol{i}_m))_{\boldsymbol{i}_1,\dots,\boldsymbol{i}_m \in \binom{[2n]}{2,\dots,2}}$ be a $2m$-demensional array The *hyperpfaffian* $\mathrm{Pf}^{[2,m]}(B)$ is defined by Matsumoto as

$$\mathrm{Pf}^{[2,m]}(B) = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \binom{[2n]}{2,\dots,2}} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \times \prod_{i=1}^{n} B(\sigma_1(2i-1), \sigma_1(2i), \cdots, \sigma_m(2i-1), \sigma_m(2i)).$$

Definition 3 (hyperpfaffian)

Let l, m and n be postive integers, and let $B = (B(\boldsymbol{i}_1, \dots, \boldsymbol{i}_m))_{\boldsymbol{i}_1,\dots,\boldsymbol{i}_m \in \binom{[ln]}{l,\dots,l}}$ be a lm-demensional array. We adopt the following definition:

$$\mathrm{Pf}^{[l,m]}(B) = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \binom{[ln]}{l,\dots,l}} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \times \prod_{i=1}^{n} B(\sigma_1(l(i-1)+1), \cdots, \sigma_1(li), \cdots, \sigma_m(l(i-1)+1), \cdots, \sigma_m(li)),$$

$$\mathrm{pf}^{[l,m]}(B) = \sum_{\sigma_1,\dots,\sigma_m \in \left\langle {[ln] \atop l,\dots,l} \right\rangle} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \times \prod_{i=1}^{n} B(\sigma_1(l(i-1)+1), \cdots, \sigma_1(li), \cdots, \sigma_m(l(i-1)+1), \cdots, \sigma_m(li)).$$

Hyperafnians

Properties of hyperpfaffians

1. $\mathrm{Pf}^{[l,m]}(B)$ is defined only when l is even.
2. Generally the two definitions do not agree. But if B satisfies
$$B(\boldsymbol{i}_{\tau(1)},\dots,\boldsymbol{i}_{\tau(m)})=B(\boldsymbol{i}_1,\dots,\boldsymbol{i}_m)\qquad({}^{\forall}\tau\in\mathfrak{S}_m)$$
then we have $\mathrm{Pf}^{[l,m]}(B)=\mathrm{pf}^{[l,m]}(B)$.
3. Hafnian $\mathrm{Hf}^{[l,m]}(B)$, $\mathrm{hf}^{[l,m]}(B)$ are defined similarly.

Definition

Let $B=(B(\boldsymbol{i}_1,\dots,\boldsymbol{i}_m))_{\boldsymbol{i}_1,\dots,\boldsymbol{i}_m\in\binom{[N]}{l,\dots,l}}$ be a lm-demensional array of size N. If $K_1,\dots,K_m\in\binom{[N]}{ln}$, then we write $B_{K_1,\dots,K_m}$ for the lm-demensional array $(B(\boldsymbol{i}_1,\dots,\boldsymbol{i}_m))_{\boldsymbol{i}_1\in K_1,\dots,\boldsymbol{i}_m\in K_m}$ of size ln.

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Notation

Definition

Let $A = (a_{i,j})_{i,j\geq 1}$ be a matrix (of finite or infinite row/column length). If $I = \{i_1, \dots, i_r\}$ (resp. $J = \{j_1, \dots, j_r\}$) are a set of row (resp. column) indices, then we write $A_J^I = A_{j_1,\dots,j_r}^{i_1,\dots,i_r}$ for the $r \times r$ submatrix obtained from A by choosing the rows indexed by I and columns indexed by J.

Example

For $A = (a_j^i)_{i,j\geq 1}$ we use the notation

$$A_{2,4,5}^{1,3,5} = \begin{pmatrix} a_2^1 & a_4^1 & a_5^1 \\ a_2^3 & a_4^3 & a_5^3 \\ a_2^5 & a_4^5 & a_5^5 \end{pmatrix}, \qquad A_{2,4,5}^{[3]} = \begin{pmatrix} a_2^1 & a_4^1 & a_5^1 \\ a_2^2 & a_4^2 & a_5^2 \\ a_2^3 & a_4^3 & a_5^3 \end{pmatrix}.$$

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Minor summation formula of hyperpfaffians

Theorem (Minor summation formula of hyperpfaffians)

Let l, n, N and p be positive integers such that $ln \leq N$. Let $H(s) = (h_{ij}(s))_{1\leq i\leq ln, 1\leq j\leq N}$ be $ln \times N$ rectangular matrices for $1 \leq s \leq p$, and let $A = (A(\boldsymbol{i}^{(1)}, \cdots, \boldsymbol{i}^{(p)}))_{\boldsymbol{i}^{(1)},\cdots,\boldsymbol{i}^{(p)}\in\binom{[N]}{l}}$ be an lp-demensional array of size N. Then we have

$$\sum_{K^{(1)},\dots,K^{(p)}\in\binom{[N]}{ln}} \mathrm{Pf}^{[l,p]}(A_{K^{(1)},\dots,K^{(p)}}) \prod_{s=1}^{p} \det H(s)_{[ln],K^{(s)}} = \mathrm{Pf}^{[l,p]}(Q),$$

where $Q = (Q(i_1, \dots, i_{lp}))_{1\leq i_1,\dots,i_{lp}\leq ln}$ is the l-alternating lp-dimensional tensor of size ln defined by

$$Q(\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(p)}) = \sum_{K^{(1)},\dots,K^{(p)}\in\binom{[N]}{l}} A(K^{(1)}, \dots, K^{(p)}) \prod_{s=1}^{p} \det(H(s)_{\boldsymbol{i}^{(s)},K_s}).$$

Definition (q-Jackson integral)

The *q-Jackson integral* from a to b is defined by

$$\int_a^b f(x)\,d_qx = (1-q)\sum_{n=0}^{\infty} f(a+(b-a)q^n)(b-a)q^n.$$

provided the sum converges absolutely.

Definition (q-Jackson integral)

Assume we are given a weight function w on $[a,b]$. We write $d_q\omega(x) = w(x)d_qx$, i,e,

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De Bruijn's formula (hyperpfaffian version)

Theorem (De Bruijn's formula)

Let l, m and n be positive integers, and let $\omega(d_qx) = w(x)d_qx$ be a measure on $[a,b]$, Let $\phi_{i,j}^{(s)}(x)$ be a function on $[a,b]$ for $i \in [ln]$ and $j \in [l]$. Then we have

$$\int_{a\le x_1^{(1)}<\cdots<x_n^{(1)}\le b}\cdots\int_{a\le x_1^{(p)}<\cdots<x_n^{(p)}\le b}\prod_{s=1}^{p}\det\left(\phi_{i,1}^{(s)}(x_j^{(s)})\Big|\cdots\Big|\phi_{i,l}^{(s)}(x_j^{(s)})\right)_{i\in[ln],j\in[n]}$$
$$\times d_q\omega(\boldsymbol{x}^{(1)})\cdots d_q\omega(\boldsymbol{x}^{(p)}) = \mathrm{Pf}^{[l,p]}\left(Q(\boldsymbol{i}^{(1)},\dots \boldsymbol{i}^{(p)})\right)_{\boldsymbol{i}^{(1)},\dots,\boldsymbol{i}^{(p)}\in\binom{[ln]}{l}},$$

where

$$Q(\boldsymbol{i}^{(1)},\dots \boldsymbol{i}^{(p)}) = \int_a^b\cdots\int_a^b\prod_{s=1}^{p}\det(\phi_{i_\lambda^{(s)},\mu}^{(s)}(x^{(s)}))_{1\le\lambda,\mu\le l}\ d_q\omega(x^{(1)})\cdots d_q\omega(x^{(p)})$$

for $\boldsymbol{i}^{(1)},\dots,\boldsymbol{i}^{(p)} \in \binom{[ln]}{l}$. Here $(\phi_{i,1}(x_j)|\cdots|\phi_{i,l}(x_j))_{i\in[ln],j\in[n]}$ stands for the $ln \times ln$ matrix whose ith row is given by

$$(\phi_{i,1}(x_1),\cdots,\phi_{i,l}(x_1),\dots,\phi_{i,1}(x_n),\cdots,\phi_{i,l}(x_n)).$$

Hyperpfaffians and integrals

Theorem

Let $d\omega(x) = w(x)dx$ be a measure on an interval $[a, b]$, and let $\mu_i = \int_0^a x^i \, d\omega(x)$ denote the ith moment. Then we have

$$\mathrm{Pf}^{[2,m]}\left(\prod_{s=1}^{m}(i_{2s} - i_{2s-1}) \cdot \mu_{i_1+\cdots+i_{2m}-2m+mr}\right)_{1\le i_1,\dots,i_{2m}\le 2n} = \frac{1}{n!}\int_{[0,a]^n}\prod_i x_i^{m(r+1)}\prod_{i<j}(x_i - x_j)^{4m}\,d\omega(\boldsymbol{x}).$$

Corollary (A special case $m = 1$)

Let ω and μ_i be as above, Then we have

$$\mathrm{Pf}\left((j-i)\,\mu_{i+j+r-2}\right)_{1\le i<j\le 2n} = \frac{1}{n!}\int_{[0,a]^n}\prod_i x_i^{r+1}\prod_{i<j}(x_i - x_j)^4\,d\omega(\boldsymbol{x}).$$

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Theorem

Let $d_q\omega(x) = w(x)d_qx$ be a measure on $[0, a]$, and let $\mu_i = \int_0^a x^i \, d_q\omega(x)$ be the ith moment of ω. Then we have

$$\mathrm{Pf}^{[2,m]}\Big(\prod_{s=1}^{m}(q^{i_{2s-1}-1} - q^{i_{2s}-1}) \cdot \mu_{i_1+\cdots+i_{2m}-2m+mr}\Big)_{1\le i_1,\dots,i_{2m}\le 2n}$$

$$= \frac{q^{m\binom{n}{2}}(1-q)^{mn}}{n!} \int_{[0,a]^n} \prod_i x_i^{m(r+1)} \prod_{i<j}(x_i - x_j)^{2m}$$

$$\times \prod_{i<j}(qx_i - x_j)^m (x_i - qx_j)^m \, \omega(d_q\boldsymbol{x}).$$

Hyperpfaffians and integrals (a q-analogue)

Corollary (A special case $m = 1$)

Let $d_q\omega(x) = w(x)d_qx$ be a measure on $[0, a]$, and let $\mu_i = \int_0^a x^i \, d_q\omega(x)$ be the ith moment of ω. Then we have

$$\begin{aligned}
&\operatorname{Pf}\left((q^{i-1} - q^{j-1}) \cdot \mu_{i+j-2+r}\right)_{1 \le i,j \le 2n} \\
&= \frac{q^{\binom{n}{2}}(1-q)^n}{n!} \int_{[0,a]^n} \prod_i x_i^{r+1} \prod_{i<j} (x_i - x_j)^2 \\
&\qquad \times \prod_{i<j} (qx_i - x_j)(x_i - qx_j) \, d_q\omega(\boldsymbol{x}).
\end{aligned}$$

We put $l = 2$, $\phi_{i,1}(x) = q^{i-1}x^{i-1}$, $\phi_{i,2}(x) = x^{i-1}$, then use the Vandermonde determinant.

Selberg-Askey integral

Little q-Jacobi type pfaffian

Theorem [I, Tagawa and Zeng'13]

Let $n \geq 1$ and $r \geq -1$ be integers. Then we have

$$\operatorname{Pf}\left((q^{i-1} - q^{j-1}) \frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}} \right)_{1 \leq i,j \leq 2n}$$
$$= a^{n(n-1)} q^{n(n-1)(4n+1)/3 + n(n-1)r}$$
$$\times \prod_{k=1}^{n-1} (bq;q)_{2k} \prod_{k=1}^{n} \frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

Another proof is obtained by taking the weight function

$$w(x) = \frac{1}{1-q} \cdot \frac{(aq, bq; q)_\infty}{(abq^2, q; q)_\infty} \cdot \frac{(qx;q)_\infty}{(bqx;q)_\infty} x^{\alpha+1},$$

and use De Bruijn's formula.

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Selberg-Askey integral

Theorem (Askey-Habsieger-Kadell)

$$\int_{[0,1]^n} \prod_{i<j} t_i^{2k} \left(q^{1-k} t_j/t_i; q\right)_{2k} \prod_{i=1}^{n} t_i^{x-1} \frac{(t_i q; q)_\infty}{(t_i q^y; q)_\infty} d_q \boldsymbol{t}$$

$$= q^{kx\binom{n}{2}+2k^2\binom{n}{3}} A_n(x, y; q),$$

where

$$A_n(x, y; q) = \prod_{j=1}^{n} \frac{\Gamma_q(x+(j-1)k)\Gamma_q(y+(j-1)k)\Gamma_q(jk+1)}{\Gamma_q(x+y+(n+j-2)k)\Gamma_q(k+1)}.$$

Here the q-gamma fonction is defined on $\mathbb{C} \setminus \mathbb{Z}^-$ by

$$\Gamma_q(a) = \frac{(q; q)_\infty}{(q^a; q)_\infty}(1-q)^{1-a}.$$

Reference (Askey-Habsieger-Kadell)

1. R. Askey, “Some basic hypergeometric extenqions of integrals of Selberg and Andrews”, *SIAM J. Math. Anal.*, **11**, 1980, 203–951.
2. L. Habsieger, Une q-intégrale de Selberg et Askey, SIAM J. Math. Anal. 19 (1988), no. 6, 1475–1489
3. K. W. J. Kadell, A proof of Askey’s conjectured q-analogue of Selberg’s integral and a conjecture of Morris, SIAM J. Math. Anal. 19 (1988), 969–986.

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Baker & Forrester Integral

Al-Salam and Carlitz type Pfaffian 1

Theorem

We have

$$\mathrm{Pf}\left((q^{i-1}-q^{j-1})F_{i+j-3}(a;q)\right)_{1\le i,j\le 2n} = a^{n(n-1)}q^{\frac{1}{6}n(n-1)(4n-5)}\prod_{k=1}^{n}(q;q)_{2k-1},$$

$$\mathrm{Pf}\left((q^{i-1}-q^{j-1})F_{i+j-2}(a;q)\right)_{1\le i,j\le 2n} = a^{n(n-1)}q^{\frac{1}{6}n(n-1)(4n+1)}\prod_{k=1}^{n}(q;q)_{2k-1}\sum_{k=0}^{n}q^{(n-k)(n-k-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} a^k,$$

where $$F_n^{(a)}(a;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q a^k.$$

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$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})G_{i+j-3}(a;q)\right)_{1\le i,j\le 2n}$$
$$= a^{n(n-1)}q^{-n(n-1)(4n-5)/3}\prod_{k=1}^{n}(q;q)_{2k-1},$$
$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})G_{i+j-2}(a;q)\right)_{1\le i,j\le 2n}$$
$$= a^{n(n-1)}q^{-\frac{2}{3}n(n-1)(2n-1)}\prod_{k=1}^{n}(q;q)_{2k-1}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} a^k,$$

where $$G_n^{(a)}(a;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q a^{k(k-n)}.$$

Al-Salam and Carlitz I, II polynomials

Definition (Al-Salam and Carlitz I, II polynomials)

Al-Salam and Carlitz defined the sequences $\{U_n^{(a)}(y;q)\}$ ($a < 0$) and $\{V_n^{(a)}(x;q)\}$ of orthogonal polynomials satisfying

$$\int_a^1 U_m^{(a)}(x;q)U_n^{(a)}(x;q)w_U^{(a)}(x;q)\,d_qx = (1-q)(-a)^n q^{\frac{n(n-1)}{2}}(q;q)_n\delta_{m,n},$$

$$\int_1^\infty V_m^{(a)}(x;q)V_n^{(a)}(x;q)w_V^{(a)}(x;q)\,d_qx = (1-q)a^n q^{-n^2}(q;q)_n\delta_{m,n},$$

where the weight functions $w_U^{(a)}(x;q)$ and $w_V^{(a)}(x;q)$ are given by

$$w_U^{(a)}(x;q) = (qx;q)_\infty\left(\frac{qx}{a};q\right)_\infty \Big/ (q;q)_\infty\,(aq;q)_\infty\left(\frac{q}{a};q\right)_\infty,$$

$$w_V^{(a)}(x;q) = (q;q)_\infty\,(aq;q)_\infty\left(\frac{q}{a};q\right)_\infty \Big/ (x;q)'_\infty\left(\frac{x}{a};q\right)_\infty.$$

Here $(x;q)'_\infty$ denotes the product except the term which equals 0.

Baker & Forrester Integral

Theorem [Baker and Forrester(2000)]

Baker and Forrester(2000) prove

$$\int_{[a,1]^n} \Delta_k^2(\boldsymbol{x}) \prod_{i=1}^{n} w_U^{(a)}(x_i;q) d_q \boldsymbol{x}$$

$$= (1-q)^n (-a)^{\frac{kn(n-1)}{2}} q^{k^2\binom{n}{3} - \frac{k(k-1)}{2}\binom{n}{2}} \prod_{i=1}^{n} \frac{(q;q)_{ki}}{(q;q)_k},$$

$$\int_{[1,\infty]^n} \Delta_k^2(\boldsymbol{x}) \prod_{i=1}^{n} w_V^{(a)}(x_i;q)\, d_q\boldsymbol{x}$$

$$= (1-q)^n a^{\frac{kn(n-1)}{2}} q^{-2k^2\binom{n}{3} - k^2\binom{n}{2}} \prod_{i=1}^{n} \frac{(q;q)_{ki}}{(q;q)_k},$$

where $\quad \Delta_k^2(\boldsymbol{x}) = \prod_{i<j} \prod_{l=-k+1}^{k} (x_i - q^l x_j).$

Reference (Baker & Forrester Integral)

1. T. H. Baker and P. J. Forrester, “Multivariable Al-Salam & Carlitz polynomials associated with the type A q-Dunkl kernel”, *Math. Nachr.*, **212 (2000):5–35.**
2. W. Al-Salam and L. Carlitz, “Some orthogonal q-polynomials”, *Math. Nachr.*, **30** (1965), 47–61.

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Combinatorial Numbers

Definition (Motzkin, Delannoy, Schröder and Narayana numbers)

Let $M_n = \sum_{k=0}^{n} \binom{n}{2k} C_k$ denote the *Motzkin numbers*, $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ the *central Delannoy numbers*, and $S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k$ *Schröder numbers*. The *Narayana numbers* $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, $n = 1, 2, 3 \ldots, 1 \leq k \leq n$, gives the *Narayana polynomials*

$$N_n(a) = \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} a^k,$$

which is the moment sequece of a generalized Thebyshev polynomials of the first kind.

Several numbers counting paths

The numbers

The first few terms of Motzkin, Delannoy, Schröder and Narayana numbers are as follows.

$$\{M_n\}_{n\geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, 323, \dots$$

$$\{D_n\}_{n\geq 0} = 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, \dots$$

$$\{S_n\}_{n\geq 0} = 1, 2, 6, 22, 90, 394, 1806, 8558, 41586, \dots$$

$$\{N_n(a)\}_{n\geq 0} = a, a^2 + a, a^3 + 3a^2 + a, a^4 + 6a^3 + 6a^2 + a, \dots$$

Theorem

Let $n \geq 1$ be an integers. Then the following identities would hold:

$$\operatorname{Pf}\left((j-i)M_{i+j-3}\right)_{1\leq i,j\leq 2n} = \prod_{k=0}^{n-1}(4k+1),$$

$$\operatorname{Pf}\left((j-i)D_{i+j-3}\right)_{1\leq i,j\leq 2n} = 2^{n^2-1}(2n-1)\prod_{k=1}^{n-1}(4k-1),$$

$$\operatorname{Pf}\left((j-i)S_{i+j-2}\right)_{1\leq i,j\leq 2n} = 2^{n^2}\prod_{k=0}^{n-1}(4k+1),$$

$$\operatorname{Pf}\left((j-i)N_{i+j-2}(a)\right)_{1\leq i,j\leq 2n} = a^{n^2}\prod_{k=0}^{n-1}(4k+1).$$

1 M. Ishikawa and C. Koutschan. "Zeilberger's holonomic ansatz for Pfaffians", *CoRR abs*, 1201.5253, 2012,

Conjectured Identities

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Selberg's Integral

Theorem (Atle Selberg 1944)

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma}\, d\boldsymbol{t}$$
$$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\,\Gamma(\beta+j\gamma)\,\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\,\Gamma(1+\gamma)}$$

1. Selberg A., "Remarks on a multiple integral", *Norsk Mat. Tidsskr.* **26** (1944), 71–78.

Definition

We write the right-hand side product as $S_n(\alpha,\beta,\gamma)$.

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Definition

We write the right-hand side product as $S_n(\alpha, \beta, \gamma)$.

Theorem (Hankel hyperpfaffian of Motzkin numbers)

$$\mathrm{Pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot M_{i_1+\cdots+i_{2m}-3m}\right)_{1\le i_1,\dots,i_{2m}\le 2n}$$
$$=\frac{1}{(2\pi)^n n!}\int_{[-1,\,3]^n}\prod_{i<j}(x_i-x_j)^{4m}\prod_{i}\sqrt{(x_i+1)(3-x_i)}\,d\boldsymbol{x}$$
$$=\frac{4^{2mn(n-1)+2n}}{(2\pi)^n n!}S\left(\frac{3}{2},\frac{3}{2},2m\right),$$

Theorem (Hankel hyperpfaffian of central Delannoy numbers)

$$\mathrm{Pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot D_{i_1+\cdots+i_{2m}-3m}\right)_{1\le i_1,\dots,i_{2m}\le 2n}$$

$$=\frac{1}{\pi^n n!}\int_{[3-2\sqrt{2},\,3+2\sqrt{2}]^n}\frac{\prod_{i<j}(x_i-x_j)^{4m}}{\prod_i\sqrt{6x_i-x_i^2-1}}\,d\boldsymbol{x}$$

$$=\frac{(4\sqrt{2})^{2mn(n-1)}}{(2\pi)^n n!}S\left(\frac{1}{2},\frac{1}{2},2m\right),$$

Hyperpfaffian version 3

Theorem (Hankel hyperpfaffian of Schröder numbers)

$$\mathrm{Pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot S_{i_1+\cdots+i_{2m}-3m+1}\right)_{1\le i_1,\dots,i_{2m}\le 2n}$$
$$=\frac{1}{(2\pi)^n n!}\int_{[3-2\sqrt{2},3+2\sqrt{2}]^n}\prod_{i<j}(x_i-x_j)^{4m}\prod_{i}\sqrt{6x_i-x_i^2-1}\,d\boldsymbol{x}.$$
$$=\frac{(4\sqrt{2})^{2mn(n-1)+2n}}{(2\pi)^n n!}S\left(\frac{3}{2},\frac{3}{2},2m\right),$$

An Open Problem

Definition (Gessel and Xin numbers a_n)

Let $a_n = \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n}$. Gessel and Xin prove that

$$\det(a_{i+j-1})_{1\le i,j\le n} = \prod_{i=1}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i+1)!(4i)!},$$

which equals the number of $(2n+1)\times(2n+1)$ alternating sign matrices that are invariant under vertical reflection. Krattenthaler (Theorem 31) collects *19 Hankel determinants* of this type.

1. I. Gessel and G. Xin, “The Generating Function of Ternary Trees and Continued Fractions”, *Electron. J. Combin.* **13** (2006):R53.
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We made the following cnjectures for Pfaffian analogues:

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We made the following cnjectures for Pfaffian analogues:

Conjectured identities 1

Conjecture

1. If $a_n^{(1)} = \frac{1}{3n+1}\binom{3n+1}{n} = [x^n]\frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{1}{2};\frac{27}{4}x\right)}$, then

$$\operatorname{Pf}\left((j-i)a^{(1)}_{i+j-1}\right)_{1\le i,j\le 2n} = \frac{1}{2^n}\prod_{k=0}^{n-1}\frac{(12k+6)!(4k+1)!(3k+2)!}{(8k+2)!(8k+5)!(3k+1)!}$$

2. If $a_n^{(2)} = \frac{1}{3n+2}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{4}{3},\frac{2}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\operatorname{Pf}\left((j-i)a^{(2)}_{i+j-2}\right)_{1\le i,j\le 2n} = 12^{-n}\prod_{k=0}^{n-1}\frac{(12k+10)!(4k+2)!(4k+1)}{(8k+3)!(8k+7)!(3k+2)(12k+5)}.$$

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$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(1)}\right)_{1\le i,j\le 2n} = \frac{1}{2^n}\prod_{k=0}^{n-1}\frac{(12k+6)!(4k+1)!(3k+2)!}{(8k+2)!(8k+5)!(3k+1)!}.$$

2. If $a_n^{(2)} = \frac{1}{3n+2}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{4}{3},\frac{2}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-2}^{(2)}\right)_{1\le i,j\le 2n} = 12^{-n}\prod_{k=0}^{n-1}\frac{(12k+10)!(4k+2)!(4k+1)}{(8k+3)!(8k+7)!(3k+2)(12k+5)}.$$

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Conjectured identities 2

Conjecture

3. If $a_n^{(3)} = \frac{2}{3n+1}\binom{3n+1}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{7}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{5}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(3)}\right)_{1\le i,j\le 2n} = \left(\frac{4}{3}\right)^n \prod_{k=0}^{n-1} \frac{(12k+15)!(4k+5)!(2k+1)}{(8k+8)!(8k+11)!(12k+13)}.$$

4. If $a_n^{(4)} = \frac{2}{(3n+1)(3n+2)}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(4)}\right)_{1\le i,j\le 2n} = \left(\frac{2}{3}\right)^n (6n+1)! \prod_{k=0}^{n-1} \frac{(12k+6)!(4k+5)!(4k+3)}{(8k+5)!(8k+10)!(k+1)(3k+1)}.$$

Conjectured identities 2

Conjecture

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$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(4)}\right)_{1\le i,j\le 2n} = \left(\frac{2}{3}\right)^n (6n+1)! \prod_{k=0}^{n-1} \frac{(12k+6)!(4k+5)!(4k+3)}{(8k+5)!(8k+10)!(k+1)(3k+1)}.$$

Conjectured identities 3

Conjecture

5. If $a_n^{(5)} = \frac{9n+5}{(3n+1)(3n+2)}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-2}^{(5)}\right)_{1\le i,j\le 2n} = 3^{-n}\prod_{k=0}^{n-1}\frac{(6k+6)!(2k)!}{(4k+1)!(4k+4)!(3k+2)}.$$

Conjectured identities 3

Conjecture

5. If $a_n^{(5)} = \frac{9n+5}{(3n+1)(3n+2)}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\operatorname{Pf}\left((j-i)a_{i+j-2}^{(5)}\right)_{1\le i,j\le 2n} = 3^{-n}\prod_{k=0}^{n-1}\frac{(6k+6)!(2k)!}{(4k+1)!(4k+4)!(3k+2)}.$$

Thank you for your attention!