Schubert calculus and hook formula

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at Strobl

–We use

equivariant cohomology theory and excited Young diagram

to give

a new skew shape hook formula and a generalization.

–We also give K-theory analogue of the formula.

–Finally we propose a further generalization as a conjecture and give a relation to the representation theory of p-adic groups. (This part is j/w M.Nakasuji)

Let $\lambda = (\lambda_1, \cdots, \lambda_d) \supset \mu = (\mu_1, \mu_2, \cdots, \mu_d)$ be partitions.
$STab(\lambda/\mu)$:The set of standard tableaux of skew shape λ/μ.
Theorem(H.Schubert 1891)

$$\#STab(\lambda/\mu) = |\lambda/\mu|! \times \det\left(z_{i,j}\right)_{d\times d}$$

where $z_{i,j} = \begin{cases} \frac{1}{(\lambda_j - \mu_i - j + i)!} & \text{if } \lambda_j - \mu_i - j + i \geq 0 \\ 0 & \text{otherwise} \end{cases}.$

Example $\lambda = (4,3)$, $\mu = (2,0)$.

$$\begin{array}{cccc} & & 1 & 2 \\ 3 & 4 & 5 & \end{array}, \quad \begin{array}{cccc} & & 1 & 3 \\ 2 & 4 & 5 & \end{array}, \quad \begin{array}{cccc} & & 1 & 4 \\ 2 & 3 & 5 & \end{array}, \quad \begin{array}{cccc} & & 1 & 5 \\ 2 & 3 & 4 & \end{array}, \quad \begin{array}{cccc} & & 2 & 3 \\ 1 & 4 & 5 & \end{array},$$

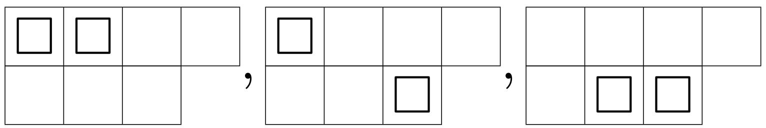
$$\begin{array}{cccc} & & 2 & 4 \\ 1 & 3 & 5 & \end{array}, \quad \begin{array}{cccc} & & 2 & 5 \\ 1 & 3 & 4 & \end{array}, \quad \begin{array}{cccc} & & 3 & 4 \\ 1 & 2 & 5 & \end{array}, \quad \begin{array}{cccc} & & 3 & 5 \\ 1 & 2 & 4 & \end{array} \qquad 5! \times \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{5!} & \frac{1}{3!} \end{vmatrix} = 9$$

Theorem(Skew shape hook formula) For $\lambda \supset \mu$:partitions,

$$\#STab(\lambda/\mu) = \frac{|\lambda/\mu|!}{\prod_{(i,j)\in\lambda} h_{i,j}} \times \left(\sum_{C\in\mathcal{E}(\mu,\lambda)} \prod_{(p,q)\in C} h_{p,q} \right)$$

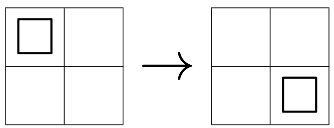
where $\mathcal{E}(\mu,\lambda)$ is the set of Excited Young diagrams of μ inside λ.

Example $\lambda = (4,3)$, $\mu = (2,0)$.

$\mathcal{E}(\mu,\lambda) = \Big\{$  $\Big\}$ hook length:

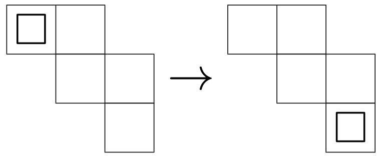
5	4	3	1
3	2	1	

$$\#STab(\lambda/\mu) = \frac{5!}{5\cdot4\cdot3\cdot1\cdot3\cdot2\cdot1} \times (5\cdot4 + 5\cdot1 + 2\cdot1) = \frac{27}{3} = 9$$

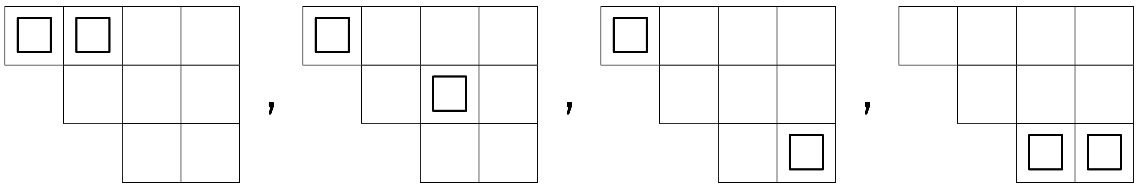
elementary excitation : 

Theorem (skew Shifted hook formula)
type D: For $\lambda \supset \mu$: strict partitions,

$$\#STab(S(\lambda/\mu)) = \frac{|\lambda/\mu|!}{\prod_{(i,j)\in\lambda} h^D_{i,j}} \times \left(\sum_{C \in \mathcal{E}_D(\mu,\lambda)} \prod_{(p,q)\in C} h^D_{p,q} \right)$$

where $\mathcal{E}_D(\mu,\lambda)$ is the set of type D Excited Young diagrams of $S(\mu)$ inside $S(\lambda)$. elementary excitation for diagonal 

Example $\lambda = (4,3,2), \mu = (2)$ $\frac{7!}{7\cdot6\cdot4\cdot3\cdot5\cdot3\cdot2\cdot2\cdot1} \times (7\cdot6+7\cdot3+7\cdot1+2\cdot1) = 12$

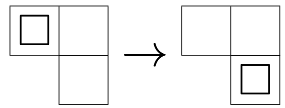


h^D :

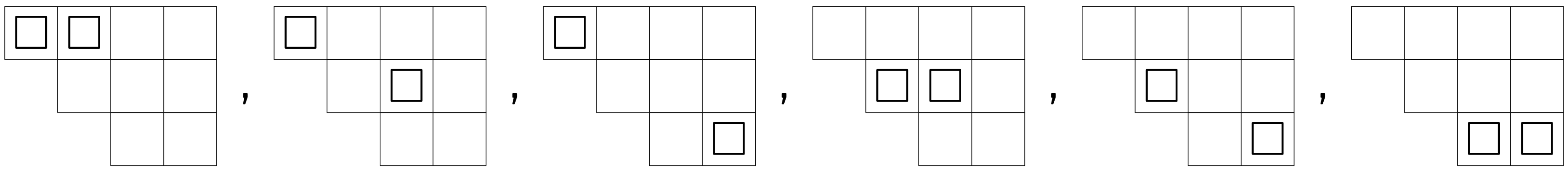
7	6	4	3
	5	3	2
		2	1

type B:

$$\#STab(\lambda/\mu) = \frac{|\lambda/\mu|!}{\prod_{(i,j)\in\lambda} h^B_{i,j}} \times \left(\sum_{C\in\mathcal{E}_B(\mu,\lambda)} \prod_{(p,q)\in C} h^B_{p,q} \right)$$

where $\mathcal{E}_B(\mu,\lambda)$ is the set of type B Excited Young diagrams of $S(\mu)$ inside $S(\lambda)$. elementary excitation for type B diagonal 

Example $\lambda = (4,3,2), \mu = (2)$



h^B :

4	7	6	3
	3	5	2
		2	1

$\frac{7!}{7\cdot6\cdot4\cdot3\cdot5\cdot3\cdot2\cdot2\cdot1} \times (4\cdot7+4\cdot5+4\cdot1+3\cdot5+3\cdot1+2\cdot1) = \frac{72}{6} = 12$

Excited Young diagram (defined by Ikeda-Naruse 2009,2013) can calculate many objects by weight sum type formula $\sum_{C \in \mathcal{E}} Wt(C)$.

- (skew) Schur functions, (skew) factorial Schur functions
- flagged Schur functions
- Vexillary double Schubert (Grothendieck) polynomials
- various determinant, Pfaffian formula
  (using lattice path uniformly)

Equivariant cohomology and localization

For flag manifold G/B or partial flag manifold G/P, we can consider T equivariant cohomology $H_T^*(G/B)$ or $H_T^*(G/P)$, where $T = (\mathbb{C}^*)^\ell$ is a maximal torus in G.

$H_T^*(G/B)$ and $H_T^*(G/P)$ are $H_T^*(pt) = \mathbb{Z}[t_1, \ldots, t_\ell]$ algebra.

Localization map

$$\Phi : H_T^*(G/B) \to \prod_{e_v \in (G/B)^T} H_T^*(e_v)$$

which is induced by the pullback $i_v^* : H_T^*(G/B) \to H_T^*(e_v)$ of the inclusion map $i_v : e_v \hookrightarrow G/B$ for each T-fixed point e_v. Φ is injective and we can describe the image using GKM-condition.

Schubert class and the structure constants

For each Schubert variety $X_w = \overline{B_- wB/B} \subset G/B$ of closure of an orbit of the opposite Borel B_- (codim $X_w = \ell(w)$), we can construct Schubert class $\sigma_w = [X_w] \in H_T^*(G/B)$, where w is an element in the Weyl group W of G.
These form a basis of $H_T^*(G/B)$ as $H_T^*(pt) = \mathbb{Z}[t_1, \ldots, t_\ell]$-module.
The structure constants $c_{w,v}^u \in H_T^*(pt)$ for the multiplication

$$\sigma_w \sigma_v = \sum_{u \in W} c_{w,v}^u \sigma_u$$

are called equivariant Littlewood-Richardson coefficients.

$\deg(c_{w,v}^u) = \ell(u) + \ell(v) - \ell(u)$ and $c_{w,v}^u \neq 0 \Longrightarrow w, v \leq u$.

For the special case of multiplication by σ_{s_i}, where s_i is a simple reflection is the equivariant Chevalley formula.

We will make a recurrence relation on the structure constants to prove a "generalization of hook formula".

Let Λ_{s_i} be the fundamental weight i.e. $< \Lambda_{s_i}, \alpha_j^\vee >= \delta_{i,j}$.

The equivariant Chevalley formula is

$$\sigma_{s_i}\sigma_w = (\Lambda_{s_i} - w\Lambda_{s_i})\sigma_w + \sum_{w \lessdot u} < \Lambda_{s_i}, \gamma^\vee > \sigma_u$$

where $w \lessdot u$ means that $\ell(u) = \ell(w) + 1$ and $u = ws_\gamma$ for some positive root γ.

Note that this formula can be extended to arbitrary Coxeter group. (We can define "equivariant Schubert class" without geometry)

Example (of equivariant Chevalley formula) of type A.

$$\begin{aligned}\sigma_{s_1}\sigma_{s_1s_2} &= (\Lambda_{s_1} - s_1s_2\Lambda_{s_1})\sigma_{s_1s_2} + <\Lambda_{s_1}, \alpha_1^\vee> \sigma_{s_1s_2s_1} \\ &= (t_2 - t_1)\sigma_{s_1s_2} + \sigma_{s_1s_2s_1}\end{aligned}$$

We utilize the associativity relation of the multiplication

$$(\sigma_{s_i}\sigma_w)\sigma_v = \sigma_{s_i}(\sigma_w\sigma_v)$$

to get a <u>recurrence relation</u> among $c^u_{w,v}$.

Assume $w \leq v$ and take the coefficients of σ_v.

Then we get

$$\sum_{w \leq z \leq v} c^z_{s_i,w} c^v_{z,v} = c^v_{s_i,v} c^v_{w,v}.$$

Therefore

$$\sum_{w < z \leq v} c^z_{s_i,w} c^v_{z,v} = c^v_{s_i,v} c^v_{w,v} - c^w_{s_i,w} c^v_{w,v}.$$

If $c^v_{s_i,v} - c^w_{s_i,w} \neq 0$, we can rewrite this as follows.

$$c^v_{w,v} = \sum_{w < z \leq v} \frac{c^z_{s_i,w}}{c^v_{s_i,v} - c^w_{s_i,w}} c^v_{z,v}.$$

$$c^v_{w,v} = \sum_{w<z\leq v} \frac{c^z_{s_i,w}}{c^v_{s_i,v} - c^w_{s_i,w}} c^v_{z,v}.$$

Continuing this process we get

$$\frac{c^v_{w,v}}{c^v_{v,v}} = \sum_{w=z_0<z_1<\cdots<z_r=v} \prod_{j=0}^{r-1} \frac{c^{z_{j+1}}_{f(z_j),w}}{c^v_{f(z_j),v} - c^{z_j}_{f(z_j),z_j}}$$

where $f : [w,v) \to S$ is an assignment of simple reflection to each $z \in [w,v) = \{z \in W | w \leq z < v\}$ such that $c^v_{f(z),v} - c^z_{f(z),z} \neq 0$.

For partial flag case G/P, we can choose $f : [w,v)_P \to S \backslash S_P$.

These arguments are essentially due to L.Mihalcea in his paper on equivariant quantum cohomology. But he did not mention the relation to hook formula.

Note that $c^v_{w,v} = i^*_{e_v}\sigma_w$ is the value of the localization and can be calculated by Billey's formula.

Fix a reduced expression $v = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of v and assume $w \le v$.

$$c^v_{w,v} = \sum_J \beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}$$

where $\beta_j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j})$ and $J = (j_1, j_2, \cdots j_r)$ runs over all subexpressions of the reduced expression of $v = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ such that $s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_r}} = w$ and $r = \ell(w)$.

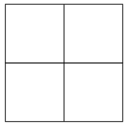
Example (type A) $v = s_2s_1s_3s_2$.

$$\begin{aligned}
&\beta_1 = t_3 - t_2\\
&\beta_2 = s_2(t_2 - t_1) = t_3 - t_1\\
&\beta_3 = s_2s_1(t_4 - t_3) = t_4 - t_2\\
&\beta_4 = s_2s_1s_3(t_3 - t_2) = t_4 - t_1
\end{aligned}$$

$c^v_{s_2,v} = (t_3 - t_2) + (t_4 - t_1)$, $c^v_{v,v} = (t_3 - t_2)(t_3 - t_1)(t_4 - t_2)(t_4 - t_1)$

$$\frac{c^v_{w,v}}{c^v_{v,v}} = \sum_{w=z_0<z_1<\cdots<z_r=v} \prod_{j=0}^{r-1} \frac{c^{z_{j+1}}_{f(z_j),w}}{c^v_{f(z_j),v} - c^{z_j}_{f(z_j),z_j}}$$

Type A Grassmannian case $G/P = Gr(d,n)$.
$d = 2, n = 4$ In this case $f(z) = s_2$ for all z.

Set $v = s_2 s_1 s_3 s_2$ i.e. .

$c^v_{e,v} = 1, c^v_{v,v} = (t_3 - t_2)(t_3 - t_1)(t_4 - t_2)(t_4 - t_1)$
There are two sequence satisfying the condition.
$e < s_2 < s_1 s_2 < s_3 s_1 s_2 < s_2 s_3 s_1 s_2 = v$ and
$e < s_2 < s_3 s_2 < s_1 s_3 s_2 < s_2 s_1 s_3 s_2 = v$.

$$\frac{1}{(t_3-t_2)(t_3-t_1)(t_4-t_2)(t_4-t_1)} = \left(\frac{1}{(t_4-t_1)+(t_3-t_2)}\right)\left(\frac{1}{t_4-t_1}\right)\left(\frac{1}{t_4-t_2}\right)\left(\frac{1}{t_3-t_2}\right)+\left(\frac{1}{(t_4-t_1)+(t_3-t_2)}\right)\left(\frac{1}{t_4-t_1}\right)\left(\frac{1}{t_4-t_2}\right)\left(\frac{1}{t_3-t_2}\right)$$

We can specialize $t_i = i$ to get
$\frac{1}{1\cdot 2\cdot 2\cdot 3} = \frac{1}{4!} + \frac{1}{4!}$ i.e. $\frac{4!}{1\cdot 2\cdot 2\cdot 3} = 1 + 1 = 2$ the hook formula

$$\frac{c^v_{w,v}}{c^v_{v,v}} = \sum_{w=z_0<z_1<\cdots<z_r=v} \prod_{j=0}^{r-1} \frac{c^{z_{j+1}}_{f(z_j),w}}{c^v_{f(z_j),v} - c^{z_j}_{f(z_j),z_j}}$$

Theorem

$$\frac{c^v_{w,v}}{c^v_{v,v}} = \prod_{\alpha:\text{positive root},w\leq vs_\alpha<v} \frac{1}{\alpha} \iff X_w \text{ is smooth at } e_v$$

$X_e = G/B$ is smooth at every e_v $(v \in W)$

$c^v_{e,v} = 1$, $c^v_{v,v} = \prod_{\alpha>0,\leq vs_\alpha<v} \alpha$

In general $c^v_{w,v}$ is calculated using Excited Young diagram.

Equivariant Chevalley formula for K-theory
(Lenart-Postnikov 2007, Lenart-Shimozono 2012)
Let $\mathcal{O}_w$ be the structure sheaf of the Schubert variety X_w. We define affine hyperplane $H_{\alpha,k} := \{x \in \mathfrak{h}^*_{\mathbb{R}}; \langle x, \alpha^\vee \rangle = k\}$ for $k \in \mathbb{Z}$. $\underline{\Lambda_{s_i}\text{-chain}}$ is an ordered sequence of affine hyperplanes $H_{\alpha,k}$ corresponding to a reduced alcove path from the fundamental alcove A_0 to $A_0 - \Lambda_{s_i}$. $A_0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}}; 0 < \langle \lambda, \alpha^\vee \rangle < 1, \forall \alpha : \text{ positive root}\}$

$$[\mathcal{O}_{s_i}][\mathcal{O}_z] = E(\Lambda_{s_i} - z(\Lambda_{s_i}))[\mathcal{O}_z] + \sum_{\substack{\text{reverse subsequence} \\ h_1 > \cdots > h_q \\ \text{of } \Lambda_{s_i}\text{-chain s.t.} \\ z \lessdot z s_{h_1} \lessdot z s_{h_1} s_{h_2} \lessdot \cdots \lessdot z s_{h_1} s_{h_2} \cdots s_{h_q}}} (1 + tE(\Lambda_{s_i} - z\tilde{s}_{h_1} \cdots \tilde{s}_{h_q}(\Lambda_{s_i})))t^{q-1}[\mathcal{O}_{z s_{h_1} \cdots s_{h_q}}]$$

where $E(\alpha) := \dfrac{e^{t\alpha} - 1}{t}$ i.e. $1 + tE(\alpha) = e^{t\alpha}$. $(t = -1)$

Hecke algebra and Yang-Baxter basis

Let W be a Weyl group with simple reflections $S = \{s_1, \ldots, s_r\}$.

Hecke algebra associated to W is a non-commutative $\mathbb{Z}[q]$-algebra with

generators $t_1, t_2, ..., t_r$ and

relations $(t_i - q)(t_i + 1) = 0$, $t_i t_j t_i \cdots = t_j t_i t_j \cdots$ braid relation

$t_w := t_{i_1} \cdots t_{i_\ell}$ for $w = s_{i_1} \cdots s_{i_\ell} \in W$ a reduced expression.

$\{t_w\}_{w \in W}$ form a standard basis.
There is another basis called Yang-Baxter basis.

Yang-Baxter basis $\{Y_w\}_{w\in W}$ was defined by Lascoux-Leclerc-Thibon (1997) for the case of type A.

It is inductively defined by

$$Y_e = 1$$
$$Y_{ws_i} = Y_w\left(h_i + \frac{1}{E(w(\alpha_i))}\right) \text{ if } ws_i > w \ ,$$

where $h_i = \frac{t_i}{q}$ and $E(\alpha_i) = \dfrac{e^{t\alpha_i} - 1}{t}$ for $t = 1 - 1/q$.

This is well defined because of the Yang-Baxter relations.

For example, if $s_i s_j s_i = s_j s_i s_j$

$$(h_i+\tfrac{1}{E(x)})(h_j+\tfrac{1}{E(x+y)})(h_i+\tfrac{1}{E(y)}) = (h_j+\tfrac{1}{E(y)})(h_i+\tfrac{1}{E(x+y)})(h_j+\tfrac{1}{E(x)}).$$

We can define $p(w,v)$ and $\tilde{p}(w,v)$ as the coefficients of

$$Y_v = \sum_{w \le v} p(w,v) h_w \tag{1}$$

and

$$h_v = \sum_{w \le v} \tilde{p}(w,v) Y_w. \tag{2}$$

Theorem [Nakasuji-N.] Assume that W is a finite group and let w_0 be the longest element of W. Then we have, for $w \le v$,

$$\tilde{p}(w,v) = (-1)^{\ell(v)-\ell(w)} p(vw_0, ww_0).$$

For the case of type A was proved by Lascoux-Leclerc-Thibon.

Casselman's problem on Iwahori fixed vectors for unraified principalceries representation of a p-adic group is interpreted in Hecke algebra as follows.

natural basis $\phi(w) = t_w$

Casselman basis f_v is dual to the intertwining operator M_u.

Casselman's problem is to express f_v in terms of $\phi(w)$.

The answer is as follows.
Proposition[Nakasuji-N.]

$$\phi(w) = \sum_{w \le v} p(w^{-1}, v^{-1}) f_v$$

$$f_w = \sum_{w \le v} \tilde{p}(w^{-1}, v^{-1}) \phi(v)$$

We have a conjectural formula of $p(w,v)$ using λ-chain.

Conjecture 1

$$p(w,v) = \sum_{v=v_0 \overset{J_1}{\to} v_1 \overset{J_2}{\to} \cdots \overset{J_r}{\to} v_r = w} \prod_{i=1}^{r} wt_{J_i}(v_{i-1}, v_i)$$

where $w' \overset{J}{\to} w$ means that there is a (not necessary saturated) path $w' = z_0 > z_1 > \cdots > z_k = w$ with the property that $z_{i-1} s_{\gamma_{j_i}} = z_i$ for a subsequence $J = (j_1, j_2, \cdots, j_k)$ of a $\Lambda_{f(w)}$-chain $\gamma_1, \gamma_2, \cdots, \gamma_m$.

and $wt(w', w)_J = \frac{t^{a(J)}(1-t)^{b(J)}(1+tE(-w\tilde{s}_J^{-1}(0)))}{tE(w\Lambda_{f(w)} - v\Lambda_{f(w)})}$

$a(J) = |J|$ and $b(J) = \frac{\ell(w') - \ell(w) - |J|}{2}$.

Conjecture 2

$$X_w \text{ is smooth at } e_v \iff \prod_{w \leq s_\beta v < v; \beta > 0} \left(1 + \frac{1}{E(\beta)}\right) = \sum_{w \leq z \leq v} p(z,v)$$

When $w = e$ this conjecture holds.

We can prove $\Leftarrow$ using the criterion given by equivariant cohomology.

Thank you!