Pfaffian Formulae and Their Applications to Symmetric Function Identities

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Schur-type Pfaffians

Pfaffian

Let $A = (a_{ij})_{1 \le i, j \le 2m}$ be a $2m \times 2m$ skew-symmetric matrix. The Pfaffian of A is defined by

$$\operatorname{Pf} A = \sum_{\pi \in \mathfrak{F}_{2m}} \operatorname{sgn}(\pi) a_{\pi(1),\pi(2)} a_{\pi(3),\pi(4)} \cdots a_{\pi(2m-1),\pi(2m)},$$

where $\mathfrak{F}_{2m}$ is the subset of the symmetric group $\mathfrak{S}_{2m}$ given by

$$\mathfrak{F}_{2m} = \left\{ \pi \in \mathfrak{S}_{2m} : \begin{array}{ccccccc} \pi(1) & < & \pi(3) & < & \cdots & < & \pi(2m-1) \\ \wedge & & \wedge & & & & \wedge \\ \pi(2) & & \pi(4) & & & & \pi(2m) \end{array} \right\},$$

and $\operatorname{sgn}(\pi)$ denotes the signature of π.

For example, if $2m = 4$, then

$$\operatorname{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Schur-type Pfaffians

Schur (1911)

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j + x_i}\right)_{1\le i,j\le n} = \prod_{1\le i<j\le n} \frac{x_j - x_i}{x_j + x_i}.$$

Laksov–Lascoux–Thorup (1989), Stembridge (1990)

$$\mathrm{Pf}\left(\frac{x_j - x_i}{1 - x_i x_j}\right)_{1\le i,j\le n} = \prod_{1\le i<j\le n} \frac{x_j - x_i}{1 - x_i x_j}.$$

Knuth (1996)

$$\mathrm{Pf}\left(\frac{x_j - x_i}{c + b(x_i + x_j) + a x_i x_j}\right)_{1\le i,j\le n} = \prod_{1\le i<j\le n} \frac{x_j - x_i}{c + b(x_i + x_j) + a x_i x_j},$$

where $b^2 = ac \pm 1$.

Sundquist (1996), Okada (1998)

$$\mathrm{Pf}\left(\frac{(a_j - a_i)(b_j - b_i)}{x_j - x_i}\right)_{1 \le i, j \le n} = \frac{1}{\prod_{1 \le i < j \le n}(x_j - x_i)} \det \widetilde{V}^{n/2,n/2}(\boldsymbol{x}; \boldsymbol{a}) \det \widetilde{V}^{n/2,n/2}(\boldsymbol{x}; \boldsymbol{b}),$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{a} = (a_1, \cdots, a_n)$, $\boldsymbol{b} = (b_1, \cdots, b_n)$ and

$$\widetilde{V}_n^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = \Big(\underbrace{1, x_i, x_i^2, \cdots, x_i^{p-1}}_{p}, \underbrace{a_i, a_i x_i, a_i x_i^2, \cdots, a_i x_i^{q-1}}_{q}\Big)_{1 \le i \le n}.$$

If we replace

$$x_i \text{ by } x_i^2, \quad a_i \text{ by } x_i, \quad \text{and} \quad b_i \text{ by } x_i,$$

then we recover Schur's Pfaffian.

Theorem A If $n+p+q=2m$ is even and $n \geq p+q$, then we have

$$\mathrm{Pf}\begin{pmatrix} \widetilde{S}_n(\boldsymbol{x};\boldsymbol{a},\boldsymbol{b}) & \widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{b}) \\ -{}^t\widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{b}) & O \end{pmatrix}$$

$$= \frac{(-1)^{\binom{p-q}{2}+(m-p)q}}{\prod_{1\leq i<j\leq n}(x_j-x_i)} \det \widetilde{V}_n^{m,m-p-q}(\boldsymbol{x};\boldsymbol{a}) \det \widetilde{V}_n^{m-q,m-p}(\boldsymbol{x};\boldsymbol{b}),$$

where

$$\widetilde{S}_n(\boldsymbol{x};\boldsymbol{a},\boldsymbol{b}) = \left(\frac{(a_j-a_i)(b_j-b_i)}{x_j-x_i}\right)_{1\leq i,j\leq n},$$

$$\widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{a}) = \Big(\underbrace{1, x_i, x_i^2, \cdots, x_i^{p-1}}_{p}, \underbrace{a_i, a_i x_i, a_i x_i^2, \cdots, a_i x_i^{q-1}}_{q}\Big)_{1\leq i\leq n}.$$

The case $p=q=0$ is the Sundquist–Okada Pfaffian.

Theorem B If $n + r = 2m$ is even and $n \geq r$, then we have

$$
\operatorname{Pf}\left(\begin{array}{c|c}
\left(\dfrac{a_j - a_i}{1 - x_i x_j}\right)_{1 \leq i, j \leq n} & \left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_{1 \leq i \leq n} \\
\hline
-{}^t\left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_{1 \leq i \leq n} & O
\end{array}\right)
$$

$$
= \frac{(-1)^{\binom{m}{2} + \binom{r}{2}}}{\prod_{1 \leq i < j \leq n} (1 - x_i x_j)}
$$

$$
\times \det \Big(\underbrace{x_i^{m-1}, x_i^m + x_i^{m-2}, x_i^{m+1} + x_i^{m-3}, \cdots, x_i^{2m-2} + 1}_{m},
$$

$$
\underbrace{a_i x_i^{m-1}, a_i\left(x_i^m + x_i^{m-2}\right), \cdots, a_i\left(x_i^{n-2} + x_i^r\right)}_{m-r} \Big)_{1 \leq i \leq n}.
$$

If $r = 0$ and $a_i = x_i$ $(1 \leq i \leq n)$, then this reduces to the LLTS Pfaffian.

Idea of the Proof

Theorem B follows from Theorem A with $p = q$ or $p = q + 1$ by substituting

$$x_i \text{ by } x_i + x_i^{-1}, \quad \text{and} \quad b_i \text{ by } x_i.$$

Theorem A is proved by comparing the coefficient of $\boldsymbol{b}^I = \prod_{i \in I} b_i$ on the both sides, and the proof is reduced to showing

$$\det \left(\begin{array}{c|c} \left(\dfrac{c_i - d_j}{z_i - w_j} \right)_{1 \le i \le r, 1 \le j \le s} & \left(1, z_i, z_i^2, \cdots, z_i^{l-1} \right)_{1 \le i \le r} \\ \hline -{}^t\left(1, w_j, w_j^2, \cdots, w_j^{k-1} \right)_{1 \le j \le s} & O \end{array} \right)$$

$$= \frac{(-1)^{s + \binom{r-k}{2}}}{\prod_{i=1}^{r} \prod_{j=1}^{s} (z_i - w_j)} \det \widetilde{V}_n^{s+l, r-l}(\boldsymbol{z} \cup \boldsymbol{w}; \boldsymbol{c} \cup \boldsymbol{d}).$$

Applications to Symmetric Function Identities

1. Theorem A $\longrightarrow$ Schur's P-functions.
2. Thoerem B $\longrightarrow$ restricted Littlewood's formula.

Applications to Schur's P-functions

Schur's P-functions $P_\lambda(\boldsymbol{x})$ (or Q-functions $Q_\lambda(\boldsymbol{x})$) are symmetric functions, which play a fundamental role in the theory of projective representations of the symmetric groups, similar to that of Schur functions $s_\lambda(\boldsymbol{x})$ in the theory of linear representations.

Nimmo gave a formula for $P_\lambda(x_1, \cdots, x_n)$ in terms of a Pfaffian. Let λ be a strict partition of length l, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. If $n+l$ is even, then we have

$$P_\lambda(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{x_i + x_j}{x_i - x_j} \cdot \mathrm{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_i - x_j}{x_i + x_j} \right)_{1 \le i, j \le n} & \left(x_i^{\lambda_l}, x_i^{\lambda_{l-1}}, \cdots, x_i^{\lambda_1} \right)_{1 \le i \le n} \\ \hline * & O \end{array} \right).$$

A similar formula holds in the case where $n+l$ is odd.

On the other hand, by replacing x_i by x_i^2, a_i by x_i, and b_i by x_i, the left hand side of the Pfaffian formula in Theorem A reads

$$\operatorname{Pf}\left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i}\right)_{1 \le i, j \le n} & \left(1, x_i^2, x_i^4, \cdots, x_i^{2(p-1)}, x_i, x_i^3, x_i^5, \cdots, x_i^{2(q-1)+1}\right)_{1 \le i \le n} \\ \hline * & O \end{array}\right).$$

Comparing this with Nimmo's formula, we obtain an algebraic proof of

Theorem (Worley; Conj. by Stanley) We put

$$\rho_k = (k, k-1, \cdots, 2, 1).$$

Then we have

$$P_{\rho_k + \rho_l}(\boldsymbol{x}) = s_{\rho_k}(\boldsymbol{x}) s_{\rho_l}(\boldsymbol{x}).$$

In particular, we have

$$P_{\rho_k}(\boldsymbol{x}) = s_{\rho_k}(\boldsymbol{x}).$$

Similarly, by specializing

$$x_i \longleftarrow x_i^2, \quad a_i \longleftarrow \frac{x_i}{1+tx_i}, \quad b_i \longleftarrow x_i$$

in Theorem A, and equating the coefficients of t^l, we can prove

Theorem (Worley) We put

$$\rho_k = (k, k-1, \cdots, 2, 1), \quad \text{and} \quad (1^l) = (\underbrace{1, \cdots, 1}_{l}).$$

If $0 \leq l \leq k+1$, then we have

$$P_{\rho_k + (1^l)}(\boldsymbol{x}) = \sum_{\lambda} s_\lambda(\boldsymbol{x}),$$

where λ runs over all partitions satisfying $\rho_k \subset \lambda \subset \rho_{k+1}$ and $|\lambda| - |\rho_k| = l$.

Applications to restricted Littlewood's formulae

Theorem (Schur, Littlewood)

$$\sum_{\lambda} s_\lambda(x_1, \cdots, x_n) = \frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)},$$

where λ runs over all partitions.

Theorem (King; Conj. by Lievens–Stoilova–Van der Jeugt)

$$\sum_{l(\lambda)\le l} s_\lambda(x_1, \dots, x_n) = \frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)} \times \frac{\det\left(x_i^{n-j} - (-1)^l \chi[j>l] x_i^{n-l+j-1}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}},$$

where λ runs over all partitions of length $\le l$, and $\chi[j>l] = 1$ if $j > l$ and 0 otherwise.

We can give another proof to King's formula by using the minor-summation formula (Ishikawa–Wakayama) and Schur-type Pfaffian formula (Theorem B).

Theorem (Ishikawa–Wakayama) Suppose that $n+r$ is even and $0 \leq n-r \leq N$. For a matrix

$$T = (t_{ij})_{1 \leq i \leq n,\, 1 \leq j \leq r+N} = \begin{pmatrix} \overset{[1,r]}{H} & \overset{[r+1,r+N]}{K} \end{pmatrix}$$

and a skew-symmetric matrix $A = (a_{ij})_{r+1 \leq i,\, j \leq r+N}$, we have

$$\sum_{J} \operatorname{Pf} A(J) \cdot \det \begin{pmatrix} H & K[J] \end{pmatrix} = (-1)^{r(r-1)/2} \operatorname{Pf} \begin{pmatrix} K A {}^t K & H \\ -{}^t H & O \end{pmatrix},$$

where $J = \{j_1 < \cdots < j_{n-r}\}$ runs over all $(n-r)$-element subsets of $[r+1, r+N]$ and

$$A(J) = (a_{j_p, j_q})_{1 \leq p,\, q \leq n-r}, \quad K[J] = (t_{p, j_q})_{1 \leq p \leq n,\, 1 \leq q \leq n-r}.$$

Outline of the Proof of King's formula

For simplicity, we assume that l is even. We put $r = n - l$.
First we apply the minor-summation formula to the matrices

$$T = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots \end{pmatrix} \end{array}, \quad A = \begin{array}{c} \begin{matrix} r & r+1 & r+2 & r+3 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ -1 & 0 & 1 & 1 & \cdots \\ -1 & -1 & 0 & 1 & \cdots \\ -1 & -1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

with $H = \left(x_i^j\right)_{1 \le i \le n,\, 0 \le j \le r-1}$ and $K = \left(x_i^j\right)_{1 \le i \le n,\, j \ge r}$. Then we can express the summation

$$\sum_{l(\lambda) \le l} s_\lambda(x_1, \dots, x_n)$$

in terms of a Pfaffian.

Partitions of length $\leq n$ are in bijection with n-element subsets of nonnegative integers, via

$$\lambda \longleftrightarrow I_n(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_{n-1} + 1, \lambda_n\}.$$

Then we have

$$l(\lambda) \leq l \Longleftrightarrow [0, n - l - 1] \subset I_n(\lambda).$$

If $l(\lambda) \leq l$ and $J = I_n(\lambda) \setminus [0, n - l - 1]$, then

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det \begin{pmatrix} H & K[J] \end{pmatrix}}{\prod_{1 \leq i < j \leq n}(x_j - x_i)}, \quad \operatorname{Pf} A(J) = 1.$$

Hence, by applying the minor-summation formula, we have

$$\sum_{l(\lambda) \leq l} s_\lambda(x_1, \cdots, x_n) = \frac{(-1)^{r(n-r)}}{\prod_{1 \leq i < j \leq n}(x_j - x_i)} \operatorname{Pf} \begin{pmatrix} K A {}^t K & H \\ -{}^t H & O \end{pmatrix}.$$

The (i,j) entry of KA^tK is given by

$$\frac{x_i^r x_j^r (x_j - x_i)}{(1-x_i)(1-x_j)(1-x_ix_j)} = \frac{x_i^r x_j^r}{1-x_ix_j}\left(\frac{x_j}{1-x_j} - \frac{x_i}{1-x_i}\right).$$

Hence we have

$$\sum_{l(\lambda)\le l} s_\lambda(x_1,\cdots,x_n) = \frac{(-1)^{r(n-r)}}{\prod_{1\le i<j\le n}(x_j-x_i)}$$

$$\times \operatorname{Pf}\left(\begin{array}{c|c} \left(\dfrac{x_i^r x_j^r (a_j - a_i)}{1-x_ix_j}\right)_{1\le i,j\le n} & \left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_{1\le i\le n} \\ \hline -{}^t\left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_{1\le i\le n} & O \end{array}\right),$$

where $a_i = x_i/(1-x_i)$. Now we can use Theorem B to convert the Pfaffian into a determinant.

Variations

By applying the minor-summation formula to appropriate matrices T and A, and using Theorem B, we have

Theorem (King)

$$\sum_{\lambda \in \mathcal{E},\, l(\lambda) \le l} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{n}(1-x_i^2)\prod_{1\le i<j\le n}(1-x_ix_j)} \times \frac{\det\left(x_i^{n-j} - \chi[j \ge l] x_i^{n-l+j}\right) + \det\left(x_i^{n-j} + \chi[j \ge l] x_i^{n-l+j}\right)}{2\det\left(x_i^{n-j}\right)},$$

$$\sum_{{}^t\lambda \in \mathcal{E},\, l(\lambda) \le 2l} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{1\le i<j\le n}(1-x_ix_j)} \cdot \frac{\det\left(x_i^{n-j} + \chi[j > 2l+1] x_i^{n-2l+j-2}\right)}{\det\left(x_i^{n-j}\right)},$$

where $\mathcal{E}$ is the set of even partitions, i.e., partitions with only even parts.