New Foundations of Combinatorial Theory

Part 1. What is a formula?

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Foundations of Combinatorial Theory

Part I: *Theory of Möbius function* (1964)

by **Gian-Carlo Rota**

Main Result:

COMBINATORICS $\hookrightarrow$ MATHEMATICS

Our Goal:

Clarify what it means to succeed in Combinatorics.

- *What is a formula?*
- *What is a combinatorial interpretation?*
- *What is a nice bijection?*

What is a formula?

Two types of answer:

(1) *The most satisfactory form of $f(n)$ is a completely explicit closed formula involving only well-known functions, and free from summation symbols. Only in rare cases will such a formula exist. As formulas for $f(n)$ become more complicated, our willingness to accept them as "determinations" of $f(n)$ decreases.*

We will be concerned almost exclusively with enumerative problems that admit solutions that are more concrete than an algorithm.

Richard Stanley, EC1

(2) Formula $=$ Algorithm working in time $o(f(n))$.

Herb Wilf, *What is an answer?* (1982)

Complexity approach

Let's count the number of rooted labeled trees:

$$f(n) = n^{n-1}$$

$$f(n) = n \cdot \sum_{T \subset K_n} 1$$

Formally, both are formulas! Indeed,

$$\log n \cdot n^{n-2} = o\left(n^{n-1}\right)$$

Moral: Not a good qualitative difference!

More examples

$D(n) := \#\,\text{Derangements}(n) = \left[n!/e\right] \quad \leftarrow$ No combinatorial meaning!

But the following *non-positive* formula does:

$$D(n) = \sum_{k=0}^{n} (-1)^k \, \frac{n!}{k!}$$

Same for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\phi^n + \phi^{-n}\right) \qquad \text{where} \qquad \phi = \frac{\sqrt{5}+1}{2}$$

Again, no combinatorial meaning!

Ménage Problem

From `Wikipedia`:

A_n = number of different ways in which it is possible to seat a set of male-female couples at a dining table so that men and women alternate and nobody sits next to his or her partner.

$$A_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

$$A_n = nA_{n-1} + 2A_{n-2} - (n-4)A_{n-3} - A_{n-4}$$

(cf. Zeilberger's rant on YouTube, you must be 18+)

Abstract approach

Formula for $\{a_n\}$ is a formula for a sequence!

NOT for a family of combinatorial objects.

$$D(n) = nD(n-1) + (-1)^n$$

$$F_n = F_{n-1} + F_{n-2}$$

Note that these are better formulas.

But the first one is still *non-positive*!

Moral: We want nice *positive* combinatorial formulas!

New philosophy

Nice combinatorial interpretation is a formula in itself!

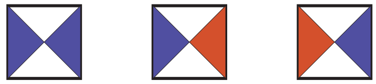
Here *nice* = algorithmically efficient.

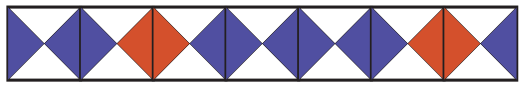
Efficient can mean *time* OR *space* OR further
restrictions on the standard computational model.

Explanation: When such combinatorial interpretation is found, it in itself can lead to better understanding AND new algorithmic solutions.

Counting with Wang tiles

Fibonacci numbers:





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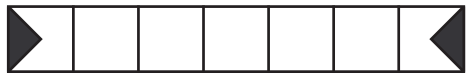
Wang tilings of a rectangle

T



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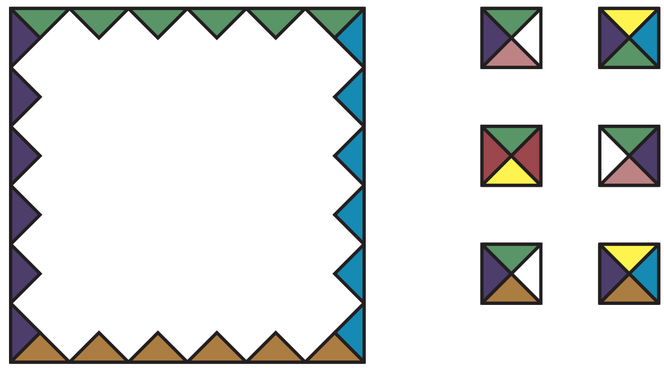
Let $a_n(T) =$ the number of tilings of $[1 \times n]$ with T.

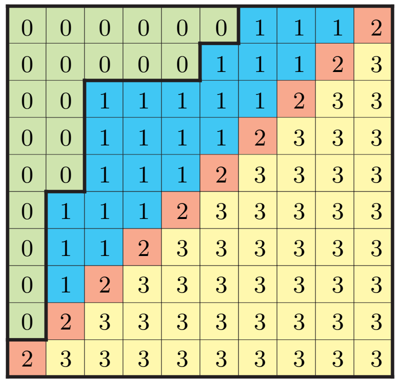
Transfer matrix method:

$$\mathcal{A}(t) \,=\, \sum_{n=0}^{\infty} a_n t^n \,=\, \frac{P(t)}{Q(t)}$$

Wang tilings of a square



Catalan numbers



An example Catalan number matrix, and the corresponding lattice path.

Main Theorem (Garrabrant, P.)

The following functions count Wang Tilings of a square:

(1) The number of integer partitions of n,
(2) The number of set partitions of an n element set (ordered Bell numbers),
(3) The Catalan number C_n,
(4) The Motzkin number M_n.
(5) The number of Gessel walks of length n,
(6) $n!$,
(7) The number of alternating permutations $Alt(n)$ of length n,
(8) The number of permutations of length n whose assents and descents follow a given periodic sequence,
(9) The number $D(n)$ of derangements of length n,
(10) The ménage numbers A_n,
(11) The Menger number $L(k, n)$ of n by k Latin squares for any fixed k,
(12) The number $Pat_k(n)$ of permutations of length n with no increasing subsequence of length k,
(13) The number $B(n)$ of Baxter permutations of length n,
(14) The number $Alt(n)$ of alternating sign matrices of size n,
(15) The number $G(n)$ of labeled connected graphs on n vertices,

Permutations

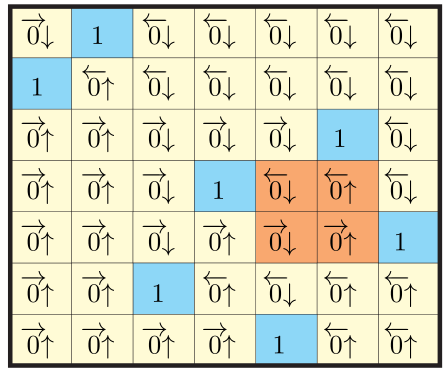
$\overrightarrow{0}\downarrow$	1	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	1	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\uparrow$	1	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	1
1	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\uparrow$
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	1	$\overleftarrow{0}\uparrow$

Baxter Permutations

Baxter permutations are permutations $\sigma \in S_n$ such that there are no indices $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j+1) > \sigma(i) > \sigma(k) > \sigma(j)$.

Observation: a given permutation matrix is a Baxter permutation is equivalent to ensuring that the two given 2×2 submatrices do not appear.

0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	1	0	0	0
0	0	0	0	0	1	0
0	0	1	0	0	0	0
0	0	0	0	0	0	1



Number of connected graphs $g(n)$ on $n+1$ vertices

Note the asymptotics: $g(n) \sim 2^{n(n+1)/2}$ (so, it barely fits).

Lemma:

$$g(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} (2^k - 1)\, g(k-1)\, g(n-k).$$

There is a way to realize this recurrence relation with Wang tiles.

This is used to prove part (15). Our construction requires over 10^7 tiles.

Thank you!