A Nekrasov-Okounkov formula in type $\tilde{C}$

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ICJ

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Plan

1. A Nekrasov-Okounkov formula in type $\tilde{A}$
2. A Nekrasov-Okounkov formula in type $\tilde{C}$

Partitions

A partition λ of n is a decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. We represent a partition by its Ferrers diagram.

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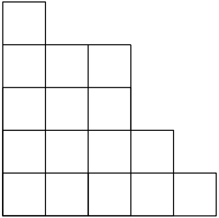

Figure: The Ferrers diagram of λ=(5,4,3,3,1)

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4	2	1		
5	3	2		
7	5	4	1	
9	7	6	3	1

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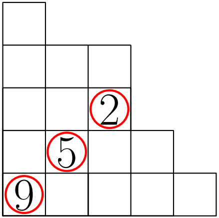

Figure: The Ferrers diagram of λ=(5,4,3,3,1) and its principal hook lengths

t-cores

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Macdonald formula in type $\tilde{A}$

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Theorem (Macdonald, 1972)

For any odd integer t, we have:

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0, v_1, \ldots, v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2 + v_1^2 + \cdots + v_{t-1}^2)/(2t)}, \qquad (1)$$

where the sum ranges over certain t-tuples of integers, satisfying some congruence condition.

Nekrasov-Okounkov formula in type $\tilde{A}$

Theorem (Nekrasov-Okounkov, 2003; Han, 2009)

For any complex number z we have

$$\prod_{k\geq 1}(1-x^k)^{z-1}=\sum_{\lambda\in\mathcal{P}}x^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{z}{h^2}\right).$$

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- conclude for any complex by polynomiality.

Macdonald in type $\tilde{C}$

Theorem (Macdonald, 1972)

For any integer t, we have:

$$\eta(X)^{2t^2+t} = c_1 \sum \prod_i v_i \prod_{i<j} ({v_i}^2 - v_j^2) X^{\|v\|^2/4(t+1)},$$

where the sum ranges over $(v_1, \ldots, v_t) \in \mathbb{Z}$ *such that* $v_i \equiv i \mod 2t+2$.

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We write $v_i = (2t+2)n_i + i$.

Self-conjugate and doubled distinct partitions

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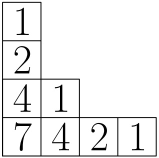

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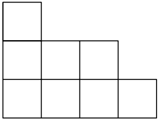

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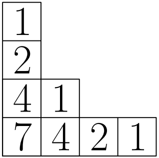

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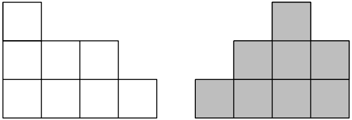

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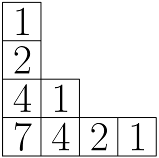

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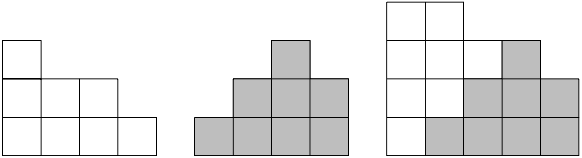

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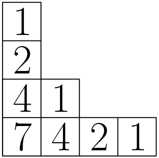

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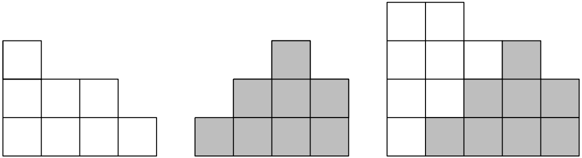

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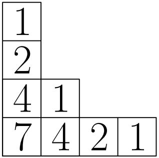

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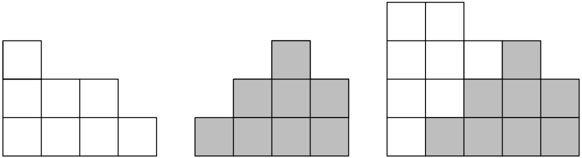

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Theorem (P., 2014)

The generating function for pairs of self-conjugate and doubled distinct t-cores is:

$$\sum_{(\lambda,\mu)\in S_c(t)\times DD(t)} q^{|\lambda|+|\mu|} = \frac{(q^2;q^2)_\infty (q^t;q^t)_\infty ((q^{2t-1};q^{2t-1})_\infty)^{t-2}}{(q;q)_\infty}$$

Some properties

Let λ be a self-conjugate (resp. doubled distinct) (t+1)-core, and h be one of its principal hook length.

- If $h > 2t+2$, then h-2t-2 is also a principal hook length
- If $h \equiv i \mod 2t+2$, for $1 \leq i \leq t$, then no principal hook length will be congruent to $-i \mod 2t+2$.

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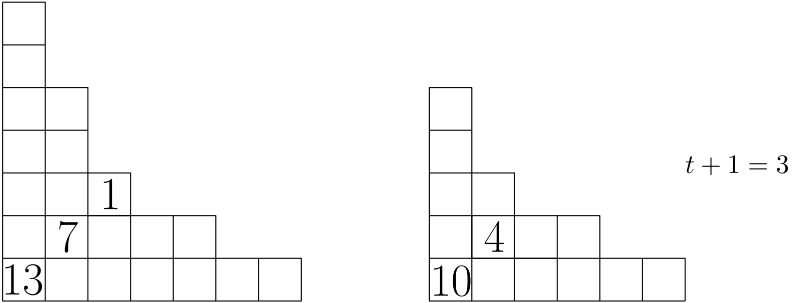

A bijection

Theorem (P., 2014)

Let t an integer ≥ 2.
There exists a bijection $\phi \; : \; S_c(t+1) \times DD(t+1) \to \mathbb{Z}$ *such that:*

$$(\lambda, \mu) \mapsto (n_1, \ldots, n_t)$$

- $$|\lambda| + |\mu| = (t+1)\sum_{i=1}^{t} n_i^2 + \sum_{i=1}^{t} i n_i$$

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- $|\lambda| + |\mu| = (t+1)\sum_{i=1}^{t} n_i^2 + \sum_{i=1}^{t} i n_i$
- $\prod_i [(2t+2)n_i + i] \prod_{i<j} \left[((2t+2)n_i + i)^2 - ((2t+2)n_j + j)^2\right] =$

$$\frac{\delta_\lambda \delta_\mu}{c_1} \prod_{h_{ii}} \left(1 - \frac{2t+2}{h_{ii}}\right)\left(1 - \frac{t+1}{h_{ii}}\right) \prod_{j=1}^{h_{ii}-1} \left(1 - \left(\frac{2t+2}{h_{ii} + \epsilon_j j}\right)^2\right)$$

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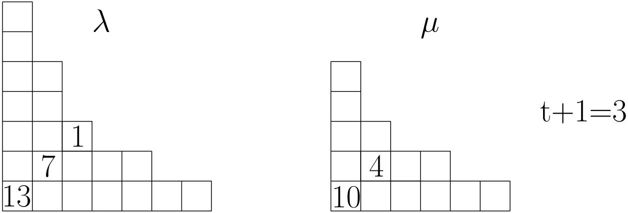

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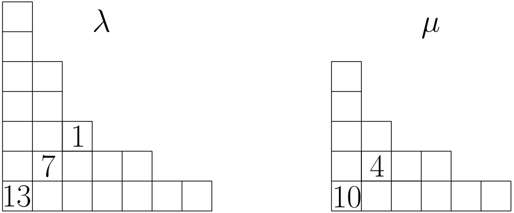

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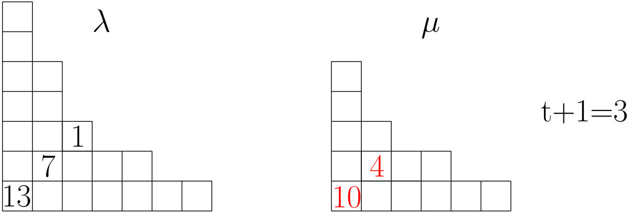

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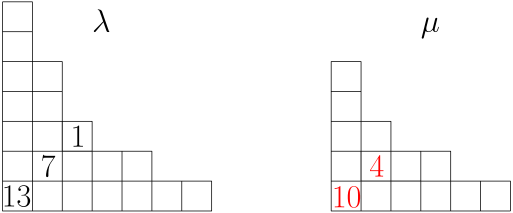

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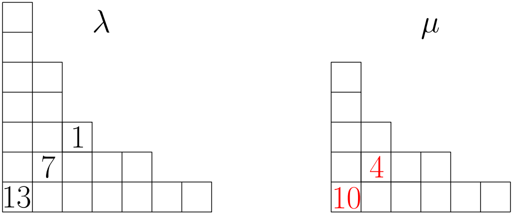

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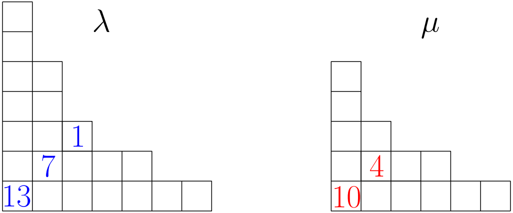

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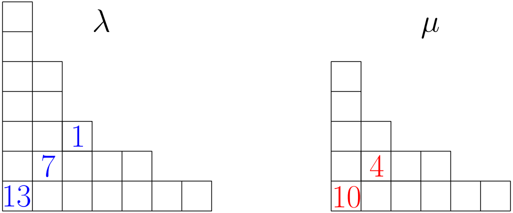

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$\Rightarrow n_2 = \frac{-(3+\Delta_2)-2}{6} = -3$

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Theorem (P., 2014)

For any complex number z we have

$$\prod_{k\geq 1}(1-x^k)^{2z^2+z} = \sum_{(\lambda,\mu)\in\mathcal{S}_c\times DD} \delta_\lambda\delta_\mu x^{|\lambda|+|\mu|}$$

$$\times \prod_{h_{ii}} \left(1-\frac{2z+2}{h_{ii}}\right)\left(1-\frac{z+1}{h_{ii}}\right)\prod_{j=1}^{h_{ii}-1}\left(1-\left(\frac{2z+2}{h_{ii}+\epsilon_j j}\right)^2\right)$$

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- Replace the previous sum by a sum over all partitions in $S_c \times DD$
- Check that coefficents of x^n on both sides are polynomials in t, and conclude that the formula is true for any complexe number z

Applications and future work

- z= -1 : expansion of $\prod_{i\geq 1}(1-x^i)$

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- generalization with extra parameters
- hook type formula $f^\lambda = \dfrac{n!}{\prod_h h}$

Thank you for your attention