Séminaire Lotharingien de Combinatoire **74** (2016), Article B74c

ENUMERATING EXCEDANCES WITH LINEAR CHARACTERS IN CLASSICAL WEYL GROUPS

SIVARAMAKRISHNAN SIVASUBRAMANIAN

Department of Mathematics
Indian Institute of Technology, Bombay
email: krishnan@math.iitb.ac.in

ABSTRACT. Several signed excedance-type statistics have nice formulae when summed over the symmetric group and over the hyperoctahedral group. Motivated by these, we consider sums of the form $f_{\chi,n}(q) = \sum_{w \in W} \chi(w) q^{\mathsf{exc}(w)}$ where W is a classical Weyl group of rank n, χ is a non-trivial one-dimensional character of W, and $\mathsf{exc}(w)$ is the excedance statistic of w.

We give formulae for these sums in a more general multivariate setting. We sharpen existing results for types A and B and give new results for classical Weyl groups of type B and type D.

1. INTRODUCTION

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations on $[n]$. Let $\mathfrak{B}_n$ be the set of permutations σ of $\{-n, -(n-1), \ldots, -1, 1, 2, \ldots n\}$ satisfying $\sigma(-i) = -\sigma(i)$. Clearly any such σ is well defined when given $\sigma(i)$ for $i \in [n]$. $\mathfrak{B}_n$ is referred to as the hyperoctahedral group or the group of signed permutations on $[n]$. Clearly, $|\mathfrak{S}_n| = n!$ and $|\mathfrak{B}_n| = 2^n n!$. For $\sigma \in \mathfrak{B}_n$ we alternatively denote $\sigma(i)$ as σ_i. For $1 \leq k \leq n$, we also denote $-k$ alternatively as $\overline{k}$. For $\sigma \in \mathfrak{B}_n$, define $\mathsf{NegSet}(\sigma) = \{\sigma_i : i > 0, \sigma_i < 0\}$ be the set of elements which occur with a negative sign. Define $\mathfrak{D}_n = \{\sigma \in \mathfrak{B}_n : |\mathsf{NegSet}(\sigma)| \text{ is even}\}$. That is, $\mathfrak{D}_n$ consists of the elements of $\mathfrak{B}_n$ with an even number of negative elements. The three families of groups $\mathfrak{S}_n$, $\mathfrak{B}_n$ and $\mathfrak{D}_n$ are the classical Weyl groups of type A, type B and type D respectively.

Let $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$ be the one-line representation of π. Define its excedance set $\mathsf{ExcSet}_\mathsf{A}(\pi)$ as $\{i \in [n] : \pi_i > i\}$ and its number of excedances as $\mathsf{exc}(\pi) = |\mathsf{ExcSet}_\mathsf{A}(\pi)|$. For $\pi \in \mathfrak{S}_n$, define its number of inversions as $\mathsf{inv}_\mathsf{A}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|$.

For a positive integer $n \geq 1$, define $\mathrm{SgnExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}_\mathsf{A}(\pi)} q^{\mathsf{exc}(\pi)}$ as the signed excedance enumerator of $\mathfrak{S}_n$. Let $\mathfrak{S}\mathfrak{D}_n$ be the set of derangements on $[n]$. We recall that, for a non-negative integer i, its q-analogue is defined as $[i]_q = 1+q+q^2+\cdots+q^{i-1}$, where q is an indeterminate and $[0]_q = 0$. For $n \geq 1$, define $\mathrm{DerSgnExc}_n(q) = \sum_{\pi \in \mathfrak{S}\mathfrak{D}_n} (-1)^{\mathsf{inv}_\mathsf{A}(\pi)} q^{\mathsf{exc}(\pi)}$ as the signed excedance enumerator over derangements. The motivation for this work stems from the following two attractive results of Mantaci [6] and of Mantaci and Rakotondrajao [7].

Theorem 1 (MANTACI)**.** *For $n \geq 1$, $\mathrm{SgnExc}_n(q) = (1-q)^{n-1}$.*

Theorem 2 (MANTACI AND RAKOTONDRAJAO). *Let $n \geq 1$ be a positive integer. Then,*

$$\mathrm{DerSgnExc}_n(q) = (-1)^{n-1} q \cdot [n-1]_q.$$

The original proofs of both results used sign reversing involutions and both papers prove more detailed results as well. An alternate proof of both these results was given by Sivasubramanian in [9]. Later, in [10], Sivasubramanian extended these results to $\mathfrak{B}_n$, in a more general bivariate setting.

For $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n \in \mathfrak{B}_n$, Brenti [4] defined its number of excedances as $\mathsf{exc}(\sigma) = |\{i \in [n] : \sigma_{|\sigma_i|} > \sigma_i\}| + |\{i \in [n] : \sigma_i = -i\}|$. This definition of Brenti was used by Sivasubramanian in [10]. For a multivariable generalisation, one definition of the set of excedances could thus be $\mathsf{ExcSet}(\sigma) = \{i \in [n] : \sigma_{|\sigma_i|} > \sigma_i\} \cup \{i \in [n] : \sigma_i = -i\}$. We have a different definition, which we give next.

For $\sigma \in \mathfrak{B}_n$, we define $\mathsf{ExcSet_A}(\sigma) = \{|\sigma_i| : \sigma_{|\sigma_i|} > \sigma_i\}$, $\mathsf{NegFixPtSet} = \{i \in [n] : \sigma_i = -i\}$ and $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet_A}(\sigma) \cup \mathsf{NegFixPtSet}(\sigma)$. Note that the union above is a disjoint union as $\mathsf{ExcSet_A}(\sigma) \cap \mathsf{NegFixPtSet}(\sigma) = \emptyset$. This definition of $\mathsf{ExcSet}_B(\sigma)$ differs slightly from $\mathsf{ExcSet}(\sigma)$ given in the previous paragraph, though we still have $\mathsf{exc}(\sigma) = |\mathsf{ExcSet}_B(\sigma)|$. We mention our reason for this change. One way to think of this definition of excedances is to use the cycles of σ which we now discuss. Our definition of cycles of σ differs from the standard definition and so we define what we mean by cycles of σ. For $\sigma \in \mathfrak{B}_n$, let $|\sigma| \in \mathfrak{S}_n$ be the permutation obtained by converting all negative entries $\overline{i}$ for each $i \in \mathsf{NegSet}(\sigma)$ to i. Let C be a cycle of $|\sigma|$ and in C change back each element $i \in \mathsf{NegSet}(\sigma)$ to $\overline{i}$ and do this for all cycles C of σ. This gives the cycles of $\sigma \in \mathfrak{B}_n$.

When we write $\sigma \in \mathfrak{B}_n$ or $\pi \in \mathfrak{S}_n$ in one-line notation, then we do not place a parenthesis bordering its elements, but we use parenthesis to write the cycles of σ or π as follows. For example, if $\sigma = 4, \overline{2}, 3, 5, \overline{1}$, then $|\sigma| = 4, 2, 3, 5, 1$. The cycles of $|\sigma|$ are $C = (1, 4, 5)$, $D = (2)$ and $E = (3)$. Hence, the cycles of σ are $C' = (\overline{1}, 4, 5)$, $D' = (\overline{2})$ and $E' = (3)$ and $\mathsf{ExcSet}_B(\sigma) = \{1, 2, 4\}$. For $1 \leq i \leq n$, to get σ_i, we locate $\pm i$ among the cycles of σ and define σ_i as the successor of $\pm i$ in that cycle. Our slightly modified definition of $\mathsf{ExcSet}_B(\sigma)$ is motivated by the $\mathfrak{S}_n$ case where we say position i is an index of excedance if, in the cycle containing i, its successor is larger than i. Zhao in [11] also defines the excedance set $\mathsf{ExcSet}_B(\sigma)$ as we have done, and uses it to define $\mathsf{exc}(\sigma) = |\mathsf{ExcSet}_B(\sigma)|$, but does not enumerate excedances with several variables as we do.

We next give the definition of inversions in $\mathfrak{B}_n$. For $\sigma \in \mathfrak{B}_n$, recall that $\mathsf{NegSet}(\sigma) = \{\sigma_i : \sigma_i < 0\}$ is the set of negative values taken by σ, and let $\mathsf{nsum}(\sigma) = -\sum_{i \in \mathsf{NegSet}(\sigma)} i$ be the absolute value of the sum of the negative elements that occur in σ. Define the number of type-A inversions of σ as before as $\mathsf{inv_A}(\sigma) = |\{1 \leq i < j \leq n : \sigma_i > \sigma_j\}|$. Here, comparison is done with respect to the standard order on $\mathbb{Z}$. Define the number of inversions of $\sigma \in \mathfrak{B}_n$ as $\mathsf{inv_B}(\sigma) = \mathsf{nsum}(\sigma) + \mathsf{inv_A}(\sigma)$. This combinatorial definition of inversions in $\mathfrak{B}_n$ is also due to Brenti (see [4, Proposition 3.1]). If $\sigma \in \mathfrak{B}_n$, define $\mathsf{pos\_n}(\sigma)$ as the index $i \in [n]$ such that $|\sigma_i| = n$. Define

$$\mathrm{BSgnExc}_n(q, t) = \sum_{\sigma \in \mathfrak{B}_n} (-1)^{\mathsf{inv_B}(\sigma)} q^{\mathsf{exc}(\sigma)} t^{\mathsf{pos\_n}(\sigma)}.$$

Let $\mathfrak{B}\mathfrak{D}_n = \{\sigma \in \mathfrak{B}_n : \sigma_i \neq i, \text{ for all } i \in [n]\}$ be the set of type-B derangements. Define

$$\mathrm{BDerSgnExc}_n(q, t) = \sum_{\sigma \in \mathfrak{B}\mathfrak{D}_n} (-1)^{\mathsf{inv_B}(\sigma)} q^{\mathsf{exc}(\sigma)} t^{\mathsf{pos\_n}(\sigma)}.$$

The following was shown by Sivasubramanian [10, Theorems 8, 22].

Theorem 3 (Sivasubramanian). *For $n \geq 1$, $\mathrm{BSgnExc}_n(q,t) = t^n(1-q)^n$ and, for $n \geq 1$, $\mathrm{BDerSgnExc}_n(q,t) = (-qt)^n$.*

More general results with the exponent of t being the position of i were also given in [10]. Inspired by this result, Sivasubramanian proved the following bivariate generalisation of Theorem 1 and Theorem 2 for the type-A case. For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{S}_n$, define $\mathsf{pos\_n}(\pi)$ as the index i such that $\pi_i = n$ and define $\mathrm{SgnExc}_n(q,t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv_A}(\pi)} q^{\mathsf{exc}(\pi)} t^{\mathsf{pos\_n}(\pi)}$. Similarly, for a positive integer n, define

$$\mathrm{DerSgnExc}_n(q,t) = \sum_{\pi \in \mathfrak{SD}_n} (-1)^{\mathsf{inv_A}(\pi)} q^{\mathsf{exc}(\pi)} t^{\mathsf{pos\_n}(\pi)}.$$

Sivasubramanian showed the following in [10, Theorems 10, 24].

Theorem 4 (Sivasubramanian). *For $n \geq 1$, $\mathrm{SgnExc}_n(q,t) = t^{n-1}(1-q)^{n-2}(t-q)$ and for $n \geq 1$, $\mathrm{DerSgnExc}_n(q,t) = (-1)^{n-1} qt[n-1]_{qt}$.*

Note that setting $t = 1$ in Theorem 4 gives us Theorem 1 and Theorem 2. In this paper, we give multivariate generalisations of both results given in Theorem 4 (see Theorem 6 and Theorem 7) and, more generally, give similar multivariate results for classical Weyl groups when a non-trivial linear character of the Weyl group is used. Among other results, we prove the following more general type-B generalisation of Theorem 3. For $\sigma \in \mathfrak{B}_n$, define the monomial $m_\sigma = \prod_{i \in \mathsf{ExcSet}_B(\sigma)} q_i$, where the q_i's are commuting variables, and define the multivariate polynomial $\mathrm{BSgnMultiExc}_n(q_1, q_2, \dots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_n} (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma$. In Section 4, we prove the following.

Theorem 5. *For $n \geq 1$, we have*

$$\mathrm{BSgnMultiExc}_n(q_1, q_2, \dots, q_n, t) = t^n \prod_{i=1}^{n} (1 - q_i).$$

Note that setting $q_i = q$ for all $i \in [n]$ results in $m_\sigma = q^{\mathsf{exc}(\sigma)}$. A similar multivariate result when enumeration is done over $\mathfrak{SD}_n$ is given in Theorem 13. Thus, Theorem 3 is a special case of Theorem 5 and Theorem 13. The proof in this paper is entirely different from the proof of Theorem 3 given in [10] and shows somewhat surprisingly, that when we fix S, the set of negative elements that $\sigma \in \mathfrak{B}_n$ can take, then there are only four choices for S that result in non-zero enumerators (see Theorem 11). A similar four-choice based result is true when the term $(-1)^{\mathsf{inv_B}(\sigma)}$ is replaced by another non-trivial linear character (see Theorem 14).

Further, as a simple corollary of our proofs, we get similar signed excedance enumerators for type D Weyl groups. Recall that $\mathfrak{D}_n = \{\sigma \in \mathfrak{B}_n : |\mathsf{NegSet}(\sigma)| \text{ is even}\}$. The definition of excedance for elements $\sigma \in \mathfrak{D}_n$ is identical to the definition when considered as an element of $\mathfrak{B}_n$. For $\sigma \in \mathfrak{D}_n$, the minimum number of generators required to write σ as a word for a suitable choice of generators (that is, its length in the Coxeter sense) is defined as $\mathsf{inv_D}(\sigma)$, see Section 5. With these generators, it is known that $\mathsf{inv_D}(\sigma) = \mathsf{inv_B}(\sigma) - |\mathsf{NegSet}(\sigma)|$ (see Brenti [4, Equation 45]). Thus, for $\sigma \in \mathfrak{D}_n$, we have $(-1)^{\mathsf{inv_D}(\sigma)} \equiv (-1)^{\mathsf{inv_B}(\sigma)} \pmod 2$. Our main result about excedance enumeration in type-D Weyl groups is Theorem 18.

Reiner [8] enumerated descents in classical Weyl groups with respect to a linear character, and our work can be thought of as a counterpart of his, with excedances replacing descents.

For the symmetric group $\mathfrak{S}_n$ it is well known that excedances and descents are identically distributed and that both are enumerated by the Eulerian polynomial (see MacMahon [5, vol. I, p. 186]). A comparison of Reiner's results with ours reveals striking similarities when we enumerate descents and excedances along with non-trivial linear characters. This is true in the univariate case and true most of the times in the multivariate setting.

Signed enumeration of Mahonian statistics is also a well studied area. Adin, Gessel and Roichman gave formulae enumerating signed major indices in classical Weyl groups [1]. Later, Biagioli [2] gave formulae enumerating major indices twisted by a linear character in classical Weyl groups. Here we also give similar formulae, though we enumerate Eulerian statistics twisted by linear characters in classical Weyl groups.

2. Preliminaries on linear characters

We mention very briefly the relevant background on linear characters in this section. For a classical Weyl group W, a linear character is a group homomorphism $\chi : W \to \mathbb{C}^*$ where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. It is known that $\mathfrak{S}_n$ has two linear characters, denoted triv and sign. For all positive integers n and $\pi \in \mathfrak{S}_n$, we have

1. the trivial character $\mathsf{triv}(\pi) = 1$ and 2. the sign character $\mathsf{sign}(\pi) = (-1)^{\mathsf{inv_A}(\pi)}$.

$\mathfrak{B}_n$ has four linear characters denoted triv_B, sign_B, neg_B and prod_B defined as follows. For all $n \geq 1$, and $\sigma \in \mathfrak{B}_n$, we have

(1) the trivial character $\mathsf{triv}_B(\sigma) = 1$,
(2) the sign character $\mathsf{sign}_B(\sigma) = (-1)^{\mathsf{inv_B}(\sigma)}$,
(3) the negative character $\mathsf{neg}_B(\sigma) = (-1)^{|\mathsf{NegSet}(\sigma)|}$, and
(4) the product character $\mathsf{prod}_B(\sigma) = (-1)^{\mathsf{inv_B}(\sigma)+|\mathsf{NegSet}(\sigma)|}$. It can be checked that $\mathsf{prod}_B(\sigma) = \mathsf{sign}(|\sigma|)$.

$\mathfrak{D}_n$ has two linear characters, denoted triv_D and sign_D. For all $n \geq 1$ and $\sigma \in \mathfrak{D}_n$, we have

1. the trivial character $\mathsf{triv}_D(\sigma) = 1$ and 2. the sign character $\mathsf{sign}_D(\sigma) = (-1)^{\mathsf{inv_D}(\sigma)}$.

It can again be checked that $\mathsf{sign}_D(\sigma) = \mathsf{sign}(|\sigma|)$. We refer the reader to Reiner [8] for a proof that these are the only linear-characters for W. Since enumerating excedance with the trivial character gives us Eulerian polynomials of types A, B and D respectively, we focus in this paper on enumeration with non-trivial linear characters.

3. Type A Weyl Groups

In this section, we present Theorem 6, which is a multivariate generalization of a part of Theorem 4.

For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, recall that $\mathsf{ExcSet_A}(\pi) = \{i \in [n] : \pi_i > i\}$. Let $\mathsf{pos\_n}(\pi)$ be the index i such that $\pi_i = n$. Furthermore, let $m_\pi = \prod_{i \in \mathsf{ExcSet_A}(\pi)} q_i$, where the q_i's are commuting variables. Define

$$\mathrm{SgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv_A}(\pi)} t^{\mathsf{pos\_n}(\pi)} m_\pi.$$

Note that when $q_i = q$ for all i, then $\mathrm{SgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \mathrm{SgnExc}_n(q, t)$.

Define the $n \times n$ matrix $M_n = (m_{i,j})_{1 \leq i,j \leq n}$ as follows. If $j \neq n$ and if $i < j$ set $m_{i,j} = q_i$. If $j \neq n$ and if $i \geq j$, set $m_{i,j} = 1$. If $j = n$ and if $i < n$, set $m_{i,n} = q_i t^i$ and set $m_{n,n} = t^n$.

In other words,

$$M_n = \begin{pmatrix} 1 & q_1 & q_1 & \cdots & q_1 t \\ 1 & 1 & q_2 & \cdots & q_2 t^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & t^n \end{pmatrix}.$$

Theorem 6. *For $n \geq 2$, we have*

$$\mathrm{SgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = t^{n-1}(t - q_{n-1}) \prod_{i=1}^{n-2} (1 - q_i).$$

Proof. We only sketch the proof as one part of it is identical to the proof of [10, Theorems 10, 24]. The identical part of the proof is to show $\mathrm{SgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \det M_n$. We then need to evaluate $\det M_n$. For this, we will induct on n. The base case when $n = 2$ is clear. For $1 \leq i \leq n$, let C_i denote the i-th column of M_n. Performing the column operation $C_1 = C_1 - C_2$ and then evaluating the determinant, we get $\det M_n = t(1 - q_1) \det M'_{n-1}$, where M'_{n-1} is a matrix almost identical to M_n, but with variables $q_2, q_3, \ldots, q_{n-1}$. Evaluating $\det M'_{n-1}$ by induction completes the proof. $\square$

3.1. **Enumeration over derangements.** In this subsection, we derive a multivariate generalization of a part of Theorem 4 on derangements. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}\mathfrak{D}_n$, let m_π be as defined above in Section 3. Define

$$\mathrm{DerSgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv_A}(\pi)} t^{\mathsf{pos\_n}(\pi)} m_\pi.$$

Note that, when $q_i = q$ for all i, then

$$\mathrm{DerSgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \mathrm{DerSgnExc}_n(q, t).$$

Define the $n \times n$ matrix $D_{M_n} = (m_{i,j})_{1 \leq i,j \leq n}$ as follows. First set $D_{M_n} = M_n$ and then reset all its diagonal elements to be zero. In other words,

$$D_{M_n} = \begin{pmatrix} 0 & q_1 & q_1 & \cdots & q_1 t \\ 1 & 0 & q_2 & \cdots & q_2 t^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Theorem 7. *For $n \geq 2$, we have*

$$\mathrm{DerSgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = (-1)^{n-1} \left\{ \sum_{i=1}^{n-1} t^i \prod_{j=1}^{i} q_j \right\}.$$

We again omit our proof as it is identical to the proof of Theorem 6. We only mention that we show $\mathrm{DerSgnMultiExc}_n(q_1, q_2, \ldots, q_{n-1}, t) = \det D_{M_n}$ and we then evaluate $\det D_{M_n}$.

4. TYPE B WEYL GROUPS

In this section, we prove results for excedance enumeration in $\mathfrak{B}_n$. There are three main results, one for each non-trivial linear character. In each case, we enumerate multivariate excedance with a different non-trivial linear character. In Subsection 4.1 we prove Theorem 11, in Subsection 4.2 we prove Theorem 14 and in Subsection 4.3 we prove Theorem 16. Further, we show results when summation is over the set of derangements $\mathfrak{BD}_n$ for two linear characters. These results are given in Subsections 4.1.1 and 4.3.1.

4.1. The linear character sign_B.

For a subset $S \subseteq [n]$, define

$$\mathfrak{B}_{n,S} = \{\sigma \in \mathfrak{B}_n : \mathsf{NegSet}(\sigma) = S\}$$

to be the set of signed permutations in $\mathfrak{B}_n$ with S as its set of negative elements. Clearly, for all $S \subseteq [n]$, we have $|\mathfrak{B}_{n,S}| = n!$. Define

$$\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_{n,S}} (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

We will need the following refinement of $\mathfrak{B}_{n,S}$. Let $n \geq 1$ and let $S \subseteq [n]$. For $1 \leq k \leq n$, define $\mathfrak{B}_{n,S,k} = \{\sigma \in \mathfrak{B}_{n,S} : \mathsf{pos\_n}(\sigma) = k\}$. Define

$$\mathrm{BSgnMultiExc}_{n,S,k}(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_{n,S,k}} (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

Lemma 8. *For $n \geq 3$ and all $S \subseteq [n]$, $\mathrm{BSgnMultiExc}_{n,S,k}(q_1, q_2, \ldots, q_n, t) = 0$ for $k \leq n-2$.*

Proof. Let $\sigma \in \mathfrak{B}_{n,S,k}$. Clearly, $\mathsf{pos\_n}(\sigma) = k$ where $k \neq n, n-1$. Let $\sigma_{n-1} = \alpha$ and let $\sigma_n = \beta$. Since $\mathsf{pos\_n}(\sigma) \neq n, n-1$, we get $\alpha \neq n$ and $\beta \neq n$. Let ψ be obtained from σ by swapping α and β. That is $\psi = \sigma \circ s_{n-1}$, where s_{n-1} is the transposition $(n-1, n)$. Clearly $\psi \in \mathfrak{B}_{n,S,k}$.

We claim that we have $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2), $\mathsf{pos\_n}(\pi) = \mathsf{pos\_n}(\sigma)$ and $m_\sigma = m_\psi$. Since $\psi = \sigma \circ s_{n-1}$, clearly $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2). Further, since $\mathsf{pos\_n}(\sigma) \notin \{n-1, n\}$, we get $\mathsf{pos\_n}(\sigma) = \mathsf{pos\_n}(\psi)$. Let C_1 be the cycle of σ that contains the element $\pm n$ with the sign of n as it occurs in σ and likewise let cycle C_2 contain $\pm(n-1)$. Since $\sigma_{n-1} = \alpha$ and $\sigma_n = \beta$, in C_1, β will be the successor of n and, in C_2, α will be the successor of $n-1$. Clearly, swapping α and β will not have any effect on elements $i \in \mathsf{ExcSet_B}$ with $|i| < n-1$. If $i = n-1$, then $i \in \mathsf{ExcSet_B}(\sigma)$ if and only if $n-1$ occurs in σ with negative sign and this is independent of α and β. A similar argument is true for $i = n$ as well. Thus, when $i = n-1$ or $i = n$, $i \in \mathsf{ExcSet_B}(\sigma)$ if and only if $i \in \mathsf{ExcSet_B}(\psi)$. Thus, $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi)$ and hence $m_\sigma = m_\pi$.

As we have paired up elements σ and ψ of $\mathfrak{B}_{n,S,k}$ such that $m_\sigma = m_\psi$, $\mathsf{pos\_n}(\sigma) = \mathsf{pos\_n}(\psi)$ and $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2), the sum over elements of $\mathfrak{B}_{n,S,k}$ is zero, completing the proof. $\square$

Thus, for all $S \subseteq [n]$, to calculate $\mathrm{BSgnExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$, we only need to sum over those $\sigma \in \mathfrak{B}_{n,S}$ with $\mathsf{pos\_n}(\sigma) \in \{n-1, n\}$. We next show that both of these can be done by induction on n. We need the following lemma when $n = 2$ for various subsets $S \subseteq [n]$.

Lemma 9. *Fix $n = 2$. Then,*

$$\mathrm{BSgnMultiExc}_{2,S}(q_1, q_2, t) = \begin{cases} t^2 - tq_1, & \text{if } S = \emptyset, \\ q_1(t - t^2), & \text{if } S = \{1\}, \\ q_2(t - t^2), & \text{if } S = \{2\}, \\ t^2 q_1 q_2 - tq_2, & \text{if } S = \{1, 2\}. \end{cases}$$

Proof. A simple calculation completes the proof. □

A simple consequence of Lemma 9 is obtained in the case $|S| = 1$, upon setting $t = 1$.

Corollary 10. *For any $S \subseteq [2]$ with $|S| = 1$, $\mathrm{BSgnMultiExc}_{2,S}(q_1, q_2, 1) = 0$.*

Our next result is the main result of this subsection. It shows, surprisingly, that only four of the 2^n choices for S give non-zero enumerator values.

Theorem 11. *Let $n \geq 2$ and $S \subseteq [n]$. Then $\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$ takes the following values:*

S	$\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$
$\emptyset$	$t^{n-1}(t - q_{n-1}) \prod_{i=1}^{n-2}(1 - q_i)$
$[n] - \{n\}$	$(t^{n-1} - t^n)q_{n-1} \prod_{i=1}^{n-2}(1 - q_i)$
$\{n\}$	$(t^{n-1} - t^n)q_n \prod_{i=1}^{n-2}(1 - q_i)$
$[n]$	$-t^{n-1}q_n(1 - tq_{n-1}) \prod_{i=1}^{n-2}(1 - q_i)$
otherwise	0

Proof. Recall that S is the set of negative elements for all the σ that we consider. When $n = 2$, the proof follows from Lemma 9 and so we assume $n \geq 3$. When $\mathsf{NegSet}(\sigma) = S = \emptyset$, there are no negative elements. When there are no negative elements, such $\sigma \in \mathfrak{B}_n$ are actually elements of $\mathfrak{S}_n$. Considering $\pi \in \mathfrak{S}_n$ as an element of $\mathfrak{B}_n$, from the definition, we clearly have $\mathsf{inv_A}(\pi) = \mathsf{inv_B}(\pi)$. Thus, this case follows from Theorem 6. Hence, assume that $S \neq \emptyset$. We split the rest of the proof into two cases.

Case 1 (when $n \notin S$): Thus n occurs for all such σ as $+n$ and $S \subseteq [n-1]$. By Lemma 8, we only need to sum over $\sigma \in \mathfrak{B}_{n,S,k}$ for $k = n - 1$ and $k = n$.

First, consider the case when $k = n$. Recall that $\mathfrak{B}_{n,S,n} = \{\sigma \in \mathfrak{B}_{n,S} : \mathsf{pos\_n}(\sigma) = n\}$, and recall that

$$\mathrm{BSgnMultiExc}_{n,S,n}(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_{n,S,n}} (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

For $\sigma \in \mathfrak{B}_{n,S,n}$, since $\mathsf{pos\_n}(\sigma) = n$ we get a common factor t^n from all terms in the summation. Since $n \notin S$, we must have $\sigma_n = +n$ for all $\sigma \in \mathfrak{B}_{n,S,n}$. Let ψ be the restriction of σ to indices $1, 2, \ldots, n-1$. Clearly $\psi \in \mathfrak{B}_{n-1,S}$ and all elements of $\mathfrak{B}_{n-1,S}$ arise in this manner. Clearly, $\mathsf{inv_B}(\sigma) = \mathsf{inv_B}(\psi)$. Further, as n occurs as $+n$ in σ, $n \notin \mathsf{ExcSet_B}(\sigma)$ and as other indices of excedance are identical, $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi)$. Thus, $\mathrm{BSgnMultiExc}_{n,S,n}(q_1, q_2, \ldots, q_n, t)$ is t^n times $\mathrm{BSgnMultiExc}_{n-1,S}(q_1, q_2, \ldots, q_{n-1}, t)$ with t set to 1. That is,

$$\mathrm{BSgnMultiExc}_{n,S,n}(q_1, q_2, \ldots, q_n, t) = t^n \mathrm{BSgnMultiExc}_{n-1,S}(q_1, q_2, \ldots, q_{n-1}, t = 1).$$

Next, consider $\sigma \in \mathfrak{B}_{n,S}$ with $\mathsf{pos\_n}(\sigma) = n - 1$. Recall that

$$\mathfrak{B}_{n,S,n-1} = \{\sigma \in \mathfrak{B}_{n,S} : \mathsf{pos\_n}(\sigma) = n - 1\},$$

and recall that

$$\mathrm{BSgnMultiExc}_{n,S,n-1}(q_1,q_2,\ldots,q_n,t) = \sum_{\sigma \in \mathfrak{B}_{n,S,n-1}} (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

Again, for all $\sigma \in \mathfrak{B}_{n,S,n-1}$, since $\mathsf{pos\_n}(\sigma) = n-1$ we get a common factor of t^{n-1} from all terms. As before, since $n \notin S$, we must have $\sigma_{n-1} = +n$. Writing σ in cycle notation, we see that in the cycle containing n, the predecessor of n exists and is either $n-1$ if $n-1 \notin S$ or $\overline{n-1}$ if $n-1 \in S$. Let ψ be obtained from σ by deleting the occurrence of $+n$ from the cycles of σ. This means in cycle notation, we short circuit $+n$ and set the successor of $\pm(n-1)$ in ψ to be the successor of $+n$ in σ. Clearly $\psi \in \mathfrak{B}_{n-1,S}$ and all elements of $\mathfrak{B}_{n-1,S}$ arise in this manner. We claim that $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2). To see this, define $\psi' \in \mathfrak{B}_{n,S}$ by $\psi'(i) = \psi(i)$ for $1 \leq i \leq n-1$ and $\psi'(n) = n$. Then it is easy to see that $\mathsf{inv_B}(\psi) \equiv \mathsf{inv_B}(\psi')$ (mod 2). Since $\psi' = \sigma \circ s_{n-1}$, we have $\mathsf{inv_B}(\psi') \not\equiv \mathsf{inv_B}(\sigma)$ (mod 2). Thus, $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2). The relation between $\mathsf{ExcSet_B}(\sigma)$ and $\mathsf{ExcSet_B}(\psi)$ needs two more cases.

If $n-1 \in S$, then, we claim $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi)$. To see this last equality, since $n-1 \in S$, all σ have only occurrences of $\overline{n-1}$. Thus, $n-1 \in \mathsf{ExcSet_B}(\psi)$ as $\overline{n-1}$ is the minimum element and all successors of $\overline{n-1}$ in the cycle of ψ containing it will be equal or larger. In σ, the successor of $\overline{n-1}$ is $+n$ and so $n-1 \in \mathsf{ExcSet_B}(\sigma)$ and similarly, as n is the largest element, $n \notin \mathsf{ExcSet_B}(\sigma)$. Thus, in this case,

$$\mathrm{BSgnMultiExc}_{n,S,n-1}(q_1,q_2,\ldots,q_n,t) = -t^{n-1}\mathrm{BSgnMultiExc}_{n-1,S}(q_1,q_2,\ldots,q_{n-1},1).$$

If $n-1 \notin S$, then, we claim $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi) \cup \{n-1\}$. To see this, note that, as $n-1 \notin S$, and as $(n-1)$ is the largest element of ψ, $n-1 \notin \mathsf{ExcSet_B}(\psi)$. In σ, since the successor of $n-1$ is $+n$, we have $n-1 \in \mathsf{ExcSet_B}(\sigma)$ and, clearly, other excedance indices remain identical. Thus, in this case, we get

$$\begin{aligned}\mathrm{BSgnMultiExc}_{n,S,n-1}&(q_1,q_2,\ldots,q_n,t)\\ &= -t^{n-1}q_{n-1}\mathrm{BSgnMultiExc}_{n-1,S}(q_1,q_2,\ldots,q_{n-1},1).\end{aligned}$$

Summing the relevant cases, if $n \notin S$, we get

$$\begin{aligned}(1)\quad &\mathrm{BSgnMultiExc}_{n,S}(q_1,q_2,\ldots,q_n,t)\\ &= \begin{cases}(t^n - t^{n-1}q_{n-1})\mathrm{BSgnMultiExc}_{n-1,S}(q_1,q_2,\ldots,q_{n-1},1), & \text{if } n-1 \notin S.\\ (t^n - t^{n-1})\mathrm{BSgnMultiExc}_{n-1,S}(q_1,q_2,\ldots,q_{n-1},1), & \text{if } n-1 \in S.\end{cases}\end{aligned}$$

Case 2 (when $n \in S$): Thus, $\overline{n}$ appears in σ. The arguments are similar in this case. As before, we just need to sum the two cases when $\mathsf{pos\_n}(\sigma) = n$ and when $\mathsf{pos\_n}(\sigma) = n-1$. Recall the definitions of $\mathfrak{B}_{n,S,n}$ and $\mathfrak{B}_{n,S,n-1}$ and let $T = S - \{n\}$. When $\mathsf{pos\_n}(\sigma) = n$, we claim that $\mathrm{BSgnExc}_{n,S,n}(q_1,q_2,\ldots,q_n,t) = -t^n q_n \mathrm{BSgnExc}_{n-1,T}(q_1,q_2,\ldots,q_{n-1},1)$. To see this, restricting $\sigma \in \mathfrak{B}_{n,S,n}$ to indices in $[n-1]$ gives us ψ where $\psi \in \mathfrak{B}_{n-1,T}$ and every $\psi \in \mathfrak{B}_{n-1,T}$ appears in this manner. Under this restriction, clearly $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi) \cup \{n\}$ as $\sigma_n = \overline{n}$. Moreover, it is easy to see that $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi)$ (mod 2). Thus, we get

$$\mathrm{BSgnMultiExc}_{n,S,n}(q_1,q_2,\ldots,q_n,t) = -t^n q_n \mathrm{BSgnMultiExc}_{n-1,T}(q_1,q_2,\ldots,q_{n-1},1).$$

If $\mathsf{pos\_n}(\sigma) = n-1$, and if $n-1 \notin S$, then $n-1$ occurs as $+(n-1)$. Hence, when viewed as cycles, $n-1$ is succeeded by $\overline{n}$. We claim that $\mathrm{BSgnMultiExc}_{n,S,n-1}(q_1, q_2, \ldots, q_n, t) = t^{n-1} q_n \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1)$. To see this, note that $\mathsf{pos\_n}(\sigma) = n-1$, and so we get a common factor of t^{n-1} from all such terms. Further, we have $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi) \cup \{n\}$. To see this, as $\overline{n}$ is the least element of σ, it will be smaller than its successor and so $n \in \mathsf{ExcSet_B}(\sigma)$. Clearly, the other excedance indices are identical. Further, it is easy to see that $\mathsf{inv_B}(\sigma) \equiv \mathsf{inv_B}(\psi)$ (mod 2). Thus, we get

$$\mathrm{BSgnMultiExc}_{n,S,n-1}(q_1, q_2, \ldots, q_n, t) = t^{n-1} q_n \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1).$$

If $\mathsf{pos\_n}(\sigma) = n-1$, and if $n-1 \in S$, then $n-1$ occurs as $\overline{n-1}$. In this case, we claim that

$$\mathrm{BSgnMultiExc}_{n,S,n-1}(q_1, q_2, \ldots, q_n, t) \\ = t^{n-1} q_n / q_{n-1} \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1).$$

To see this, let $\psi \in \mathfrak{B}_{n-1,T}$ be obtained from $\sigma \in \mathfrak{B}_{n,S}$ by short-circuiting n. Under this correspondence, it is easy to see that $\mathsf{inv_B}(\sigma) \equiv \mathsf{inv_B}(\psi)$ (mod 2). Further, we claim that $\mathsf{ExcSet_B}(\sigma) = \mathsf{ExcSet_B}(\psi) - \{n-1\} \cup \{n\}$. To see this, note that, as $\overline{n-1}$ is the smallest element of ψ, $n-1 \in \mathsf{ExcSet_B}(\psi)$. In σ, however, since $\overline{n}$ is the successor of $\overline{n-1}$, $n-1 \notin \mathsf{ExcSet_B}(\sigma)$, while $n \in \mathsf{ExcSet_B}(\sigma)$ as $\overline{n}$ is the smallest element. As the other indices of excedance are identical, we get

$$\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) \\ = t^{n-1} q_n / q_{n-1} \cdot \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1).$$

Adding the relevant cases, if $n \in S$, recalling $T = S - \{n\}$, we get

$$\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) \\ = \begin{cases} (t^{n-1} - t^n) q_n \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1), & \text{if } n-1 \notin S, \\ (t^{n-1}/q_{n-1} - t^n) q_n \mathrm{BSgnMultiExc}_{n-1,T}(q_1, q_2, \ldots, q_{n-1}, 1), & \text{if } n-1 \in S. \end{cases} \tag{2}$$

We next show by induction on n that $\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, 1) \neq 0$ only when $S = \emptyset$ and $S = [n]$. The base case is when $n = 2$ and follows from Corollary 10. Assume that the statement is true for all positive integers up to $n-1$ and for all $S \subseteq [n-1]$. Consider the next positive integer n. If $\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, 1) \neq 0$, and if $n \notin S$, by reverse induction on n, using (1) we get that $S = \emptyset$ is the only possibility. Similarly, if $n \in S$, using (2) we get that $S = [n]$ is the only possibility.

When $S = \emptyset$, it is easy to see from (1) that

$$\mathrm{BSgnMultiExc}_{n,\emptyset}(q_1, q_2, \ldots, q_n, 1) = \prod_{i=1}^{n-1} (1 - q_i).$$

This also follows from the type-A result, that is, from Theorem 6. When $S = [n]$, we claim that

$$\mathrm{BSgnMultiExc}_{n,[n]}(q_1, q_2, \ldots, q_n, 1) = -q_n \prod_{i=1}^{n-1} (1 - q_i).$$

This can be seen as follows. Let $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$ and let $\overline{\pi} \in \mathfrak{B}_n$ be obtained from π by negating π_i for all $i \in [n]$. It is easy to see when π and $\overline{\pi}$ are viewed as elements of $\mathfrak{B}_n$, that $\mathsf{inv_B}(\pi) = (-1)^n \cdot \mathsf{inv_B}(\overline{\pi})$ and that $\mathsf{ExcSet_B}(\pi) = [n] - \mathsf{ExcSet_B}(\overline{\pi})$. Coupling these observations with Theorem 6 gives us

$$\mathrm{BSgnExc}_{n,[n]}(q_1, q_2, \ldots, q_n, 1) = -q_n \prod_{i=1}^{n-1} (1 - q_i).$$

Hence, we infer that $\mathrm{BSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) \neq 0$ for only four values of S. These four values are given in the statement of the theorem. Further, it is simple to compute this value for each of the four choices of S and to see that they agree with the given expression. This completes the proof of the theorem. □

A simple corollary of Theorem 11 is Theorem 5.

Proof of Theorem 5. $\mathrm{BSgnMultiExc}_n(q_1, q_2, \ldots, q_n, t) = t^n \prod_{i=1}^n (1-q_i)$ can be obtained by summing over the four choices of S given in Theorem 11, completing the proof. □

4.1.1. *Enumeration over Derangements.* Recall that $\mathfrak{BD}_n \subseteq \mathfrak{B}_n$ is the set of type B derangements. Here, we present results when enumeration is done over $\mathfrak{BD}_n$. However, we are unable to find results when the set S of negative elements is fixed. Hence, our proof is somewhat different.

Before we state Theorem 13, which is our main result, we need a preliminary lemma. At the heart, our proof is identical to the proof of [10, Theorem 8]. However, since the definition of $\mathsf{ExcSet}_B(\sigma)$ is different in this paper when compared to that given in [10], we present a proof.

The involution τ_r: For $r \in [n]$, define an involution $\tau_r : \mathfrak{B}_n \to \mathfrak{B}_n$ as follows. We let $\tau_r(\sigma) = \psi_1\psi_2 \ldots \psi_n$, where $\psi_i = \sigma_i$ if $\psi_i \neq \pm r$, and $\psi_i = -\sigma_i$ otherwise. We prove the following property of τ_r.

Lemma 12. *Let $\sigma \in \mathfrak{B}_n$ and $m > 0$ be the minimal element in absolute value in a cycle of σ of length at least two. That is, $m \neq |\sigma_m|$ and $m = \min(m, |\sigma(m)|, |\sigma^2(m)|, \ldots)$. Furthermore, let $\psi = \tau_{|m|}(\sigma)$. Then $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet}_B(\psi)$.*

Proof. Let C be a cycle of σ with length larger than two and let $m = \min\{|x| : x \in C\}$. Assume that m appears with positive sign in σ and $\overline{m}$ appears in ψ (otherwise, change the names of σ and ψ). Thus, for this proof we may assume $m > 0$. As C has length larger than 2, denote the successor of m by $\mathsf{succ}(m)$ and the predecessor of m by $\mathsf{pred}(m)$. The cycles of ψ are identical to those of σ, with the only difference being the sign of m. Since m changes sign, $\mathsf{ExcSet_A}(\sigma) - \mathsf{ExcSet_A}(\psi) \subseteq \{m, \mathsf{pred}(m)\}$. We claim that $\mathsf{ExcSet_A}(\sigma) = \mathsf{ExcSet_A}(\psi)$.

Since $\mathsf{ExcSet_A}(\sigma) = \{|\sigma_i| : \sigma_{|\sigma_i|} > \sigma_i\}$, if $m \in \mathsf{ExcSet_A}(\sigma)$, then $m < \mathsf{succ}(m)$. Thus $-m < \mathsf{succ}(m)$ and $m \in \mathsf{ExcSet_A}(\psi)$. Similarly, if $m \notin \mathsf{ExcSet_A}(\sigma)$, then by definition $m > \mathsf{succ}(m)$. As m has least $|x|$ value over $x \in C$, we must have $\mathsf{succ}(m) < 0$. Hence $\overline{m} > \mathsf{succ}(m)$ and so $m \notin \mathsf{ExcSet_A}(\psi)$. Thus, $m \in \mathsf{ExcSet_A}(\sigma)$ if and only if $m \in \mathsf{ExcSet_A}(\psi)$.

If $\mathsf{pred}(m) \in \mathsf{ExcSet_A}(\sigma)$, then we must have $\mathsf{pred}(m) < m$. Since $m > 0$ and m has least absolute value, $\mathsf{pred}(m) < 0$. Thus, in our notation, $|\mathsf{pred}(m)| \in \mathsf{ExcSet_A}(\sigma)$ and we clearly have $\mathsf{pred}(m) < \overline{m}$ and so $|\mathsf{pred}(m)| \in \mathsf{ExcSet_A}(\psi)$. Likewise, if $\mathsf{pred}(m) \notin \mathsf{ExcSet_A}(\sigma)$, then, we must have $\mathsf{pred}(m) > m$. Since $m > 0$, and m has least absolute value, $\mathsf{pred}(m) > m > 0$. Thus, $\mathsf{pred}(m) > \overline{m}$ and thus $\mathsf{pred}(m) \notin \mathsf{ExcSet_A}(\psi)$. Thus, $\mathsf{pred}(m) \in \mathsf{ExcSet_A}(\sigma)$ if and only if $\mathsf{pred}(m) \in \mathsf{ExcSet_A}(\psi)$. Since $\mathsf{NegFixPtSet}(\sigma) = \mathsf{NegFixPtSet}(\psi)$, we have $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet}_B(\psi)$. □

For $n \geq 1$, define

$$\mathrm{BDerSgnMultiExc}_n(q_1, q_2, \ldots, q_n, t) = (-1)^{\mathsf{inv_B}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

Our main result is the following generalization of a part of Theorem 3.

Theorem 13. *For $n \geq 1$, we have*

$$\mathrm{BDerSgnMultiExc}_n(q_1, q_2, \ldots, q_n, t) = (-t)^n \prod_{i=1}^{n} q_i.$$

Proof. For $\sigma \in \mathfrak{BD}_n$, let $\ell(\sigma)$ be the largest index $i \in [n]$ such that $\sigma_i \neq -i$. As $\sigma \in \mathfrak{BD}_n$, $\sigma_i \neq i$ for all i, and thus $\ell(\sigma)$ is not defined if and only if $|\sigma|$ is the identity permutation id.

For $\sigma \in \mathfrak{BD}_n$ with $|\sigma| \neq \mathsf{id}$, let m be the smallest element in absolute value such that $\sigma(m) \neq -m$ and let $\psi = \tau_{|m|}(\sigma)$. Clearly, $\psi \in \mathfrak{BD}_n$. By Lemma 12, $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet}_B(\psi)$. By [10, Lemma 3], $\mathsf{inv_B}(\sigma) \not\equiv \mathsf{inv_B}(\psi) \pmod 2$. Thus, the multivariate polynomial $\mathrm{BDerSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$ will not get any contribution from any such σ.

Hence, the only σ that contribute are those with $\ell(\sigma)$ undefined. Stated differently, only σ such that $|\sigma| = \mathsf{id}$ survive the cancellations. Thus, only one element $\psi = -\mathsf{id} \in \mathfrak{BD}_n$ with $\psi_i = -i$ for $i \in [n]$ contributes. Clearly, $\mathsf{pos\_n}(\psi) = n$, $\mathsf{sign}_B(\psi) = (-1)^n$ and since $\mathsf{ExcSet}_B(\psi) = [n]$, $m_\psi = \prod_{i=1}^n q_i$. That is, $\mathrm{BDerSgnMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) = (-t)^n \prod_{i=1}^n q_i$, completing the proof. $\square$

4.2. **The linear character prod_B.** Recall that $\mathsf{prod}_B(\sigma) = (-1)^{\mathsf{inv_B}(\sigma) + |\mathsf{NegSet}(\sigma)|}$ for $\sigma \in \mathfrak{B}_n$. For a positive integer $n \geq 1$, define

$$\mathrm{BProdMultiExc}_n(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_n} \mathsf{prod}_B(\sigma) t^{\mathsf{pos\_n}(\sigma)} m_\sigma$$

As before, define

$$\mathrm{BProdMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_{n,S}} \mathsf{prod}_B(\sigma) t^{\mathsf{pos\_n}(\sigma)} m_\sigma$$

for $S \subseteq [n]$. The main result of this section is the following counterpart of Theorem 11.

Theorem 14. *Let $n \geq 2$. Then $\mathrm{BProdMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$ takes the following values:*

S	$\mathrm{BProdMultiExc}_{n,S}(q_1, q_2, \ldots, q_n, t)$
$\emptyset$	$t^{n-1}(t - q_{n-1}) \prod_{i=1}^{n-2}(1 - q_i)$
$[n] - \{n\}$	$(-1)^n (t^n - t^{n-1}) q_{n-1} \prod_{i=1}^{n-2}(1 - q_i)$
$\{n\}$	$(t^n - t^{n-1}) q_n \prod_{i=1}^{n-2}(1 - q_i)$
$[n]$	$(-t)^{n-1} q_n (1 - t q_{n-1}) \prod_{i=1}^{n-2}(1 - q_i)$
otherwise	0

Proof. The proof of this theorem works in the same way as the proof of Theorem 11, and hence we only sketch the salient points. To see that the earlier proof works, note that those terms which cancel due to Lemma 8 when the exponent of (-1) is $\mathsf{inv_B}(\sigma)$ also cancel when the exponent of (-1) is $\mathsf{inv_B}(\sigma) + |\mathsf{NegSet}(\sigma)|$, as we only switch the elements at the two positions $n-1, n$, and thus $\mathsf{NegSet}(\sigma) = \mathsf{NegSet}(\psi)$, where ψ is as defined in Lemma 8. Moreover, cancellations in the inductive step also work as we do not flip the sign of any element in the earlier proof. The only difference is that, since the exponent of (-1) now has an extra term of $|\mathsf{NegSet}(\sigma)|$, we multiply the enumerator in the case when $\mathsf{NegSet}(\sigma) = S$ by $|S|$. This change manifests itself in slightly modified enumerators. This completes the proof of the theorem. $\square$

Define without the term $t^{\mathsf{pos\_n}(\sigma)}$,

$$\mathrm{BProdMultiExc}_n(q_1, q_2, \ldots, q_n) = \sum_{\sigma \in \mathfrak{B}_n} \mathsf{prod}_B(\sigma) m_\sigma.$$

A simple corollary of Theorem 14 is the following, whose straightforward proof we omit.

Theorem 15. *For positive integers n, we have*

$$\begin{aligned}&\mathrm{BProdMultiExc}_n(q_1, q_2, \ldots, q_n, t)\\ &= \begin{cases} t^n(1+q_n)\prod_{i=1}^{n-1}(1-q_i), & \text{if } n \text{ is odd,}\\ \{t^n(1+q_{n-1}+q_n+q_nq_{n-1}) - 2t^{n-1}(q_{n-1}+q_n)\}\prod_{i=1}^{n-2}(1-q_i), & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

In particular, for positive integers n, we have

$$\mathrm{BProdMultiExc}_n(q_1, q_2, \ldots, q_n) = \begin{cases} (1+q_n)\prod_{i=1}^{n-1}(1-q_i), & \text{if } n \text{ is odd,}\\ \prod_{i=1}^{n}(1-q_i), & \text{if } n \text{ is even.}\end{cases}$$

4.3. **The linear character neg_B.** Recall that $\mathsf{neg}_B(\sigma) = (-1)^{|\mathsf{neg}(\sigma)|}$ for $\sigma \in \mathfrak{B}_n$. For a positive integer $n \geq 1$, define

$$\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{B}_n} \mathsf{neg}_B(\sigma) t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

For this enumerator, we are unable to get detailed information with respect to a given subset S of negative elements as in the previous two subsections. Nonetheless, we show the following about $\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t)$. Our proof needs both the involution τ_r defined in Subsubsection 4.1.1 and Lemma 12. Our main result is the following.

Theorem 16. *For $n \geq 1$, we have* $\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = t^n \prod_{i=1}^{n}(1-q_i)$.

Proof. For $\sigma \in \mathfrak{B}_n$, let $\ell(\sigma)$ be the largest index $i \in [n]$ such that $\sigma_i \neq \pm i$. Note that $\ell(\sigma)$ is not defined if and only if $|\sigma|$ is the identity permutation id. For $\sigma \in \mathfrak{B}_n$ with $|\sigma| \neq \mathsf{id}$, let m be the smallest element in absolute value such that $\sigma(m) \neq -m$ and let $\psi = \tau_{|m|}(\sigma)$. By Lemma 12, $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet}_B(\pi)$ and, since the element m has its sign changed,

$$|\mathsf{NegSet}(\sigma)| \not\equiv |\mathsf{NegSet}(\psi)| \text{ (mod 2)}.$$

Thus, all such σ do not contribute to $\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t)$.

Hence, only those σ obtained by changing the signs of some elements of the identity permutation id survive. All such permutations have $\mathsf{pos\_n}(\sigma) = n$ and any such σ with $\mathsf{NegSet}(\sigma) = S$ will contribute $(-1)^{|S|} \prod_{i \in S} q_i$. Since $S \subseteq [n]$, we have

$$\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = t^n \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \in S} q_i.$$

That is,

$$\mathrm{BNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = t^n \prod_{i=1}^{n} (1 - q_i),$$

completing the proof. □

4.3.1. *Enumeration over Derangements.* We prove an analogue of Theorem 13 for the linear character $\mathsf{neg}_B(\sigma)$. Define

$$\mathrm{BDerNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{BD}_n} \mathsf{neg}_B(\sigma) t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

For this enumerator, we show the following.

Theorem 17. *For $n \geq 1$, we have*

$$\mathrm{BDerNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = (-t)^n \prod_{i=1}^{n} q_i.$$

Proof. Arguments as in the proof of Theorem 13 show that only one $\sigma \in \mathfrak{B}_n$ survives the cancellations. This is the permutation $\psi = -\mathsf{id}$ with $\psi_i = -i$ for all $i \in [n]$. We clearly get

$$\mathrm{BDerNegMultiExc}_n(q_1, q_2, \ldots, q_n, t) = (-t)^n \prod_{i=1}^{n} q_i,$$

completing the proof. □

5. TYPE D WEYL GROUPS

Recall that $\mathfrak{D}_n = \{\sigma \in \mathfrak{B}_n : |\mathsf{NegSet}(\sigma)| = \text{ even}\}$. It is well known (see [3]) that $\mathfrak{D}_n$ is generated by $s_{\overline{1}}, s_1, s_2, \ldots, s_{n-1}$, where $s_{\overline{1}} = (-2, -1)$ and s_i for $1 \leq i < n$ is as in the type-A case. Define

$$\mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t) = \sum_{\sigma \in \mathfrak{D}_n} (-1)^{\mathsf{inv_D}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma.$$

The main result of this section is the following.

Theorem 18. *For even positive integers $n = 2k$, with $k \geq 1$, we have*

$$\mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t) = \left(\prod_{i=1}^{n-2} (1 - q_i) \right) [t^{n-1}(-q_{n-1} - q_n) + t^n(1 + q_{n-1}q_n)].$$

For odd positive integers $n = 2k+1$, we have $\mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t) = t^n \prod_{i=1}^{2k} (1 - q_i)$.

Proof. As mentioned in Section 1, though each $\sigma \in \mathfrak{D}_n$ has two statistics $\mathsf{inv_B}(\sigma)$ and $\mathsf{inv_D}(\sigma)$, we have $(-1)^{\mathsf{inv_B}(\sigma)} \equiv (-1)^{\mathsf{inv_D}(\sigma)} \pmod 2$. Thus, we have

$$\mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t) = \sum_{S \subseteq [n], |S| \text{ even}} \mathrm{BSgnExc}_{n,S}(q_1, q_2, \ldots, q_n, t).$$

Consider the case when n is an even positive integer first. For $n \geq 2$, from Theorem 11, we need to add the two cases when $S = \emptyset$ and when $S = [n]$. A simple addition proves the result. Next, consider the case when n is odd. Here, we need to add the cases when $S = \emptyset$ and $S = [n] - \{n\}$. Adding these two cases completes the proof. $\square$

We end this paper with a problem. The set T_n of $\sigma \in \mathfrak{D}_n$ which survive the cancellations that occur in the proof of Theorem 18 has an inductive structure. We have another candidate set Q_n with

$$\sum_{\sigma \in Q_n} (-1)^{\mathsf{inv_D}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma = \mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t).$$

However, we are unable to show a sign reversing involution on the elements $\sigma \in \mathfrak{D}_n - Q_n$. We define the set Q_n for even positive integers n first. For $n = 2$, $Q_2 = \mathfrak{D}_2$. For $n = 2(k+1)$, consider each $\sigma \in Q_{2k}$ and form four signed permutations ψ_r for $r = 1, 2, 3, 4$ as follows. For each $r = 1, 2, 3, 4$ and $i \in [2k]$, define $\psi_r(i) = \sigma(i)$. Further, define

$$\begin{aligned}
\psi_1(2k+1) = 2k+1 &\text{ and } \psi_1(2k+2) = 2k+2, \\
\psi_2(2k+1) = 2k+2 &\text{ and } \psi_2(2k+2) = 2k+1, \\
\psi_3(2k+1) = -(2k+1) &\text{ and } \psi_3(2k+2) = -(2k+2), \\
\psi_4(2k+1) = -(2k+2) &\text{ and } \psi_4(2k+2) = -(2k+1).
\end{aligned}$$

Thus, we get four signed permutations for each $\sigma \in Q_{2k}$ and denote by Q_{2k+2} the set of such signed permutations obtained. Clearly, for all k, $Q_{2k} \subseteq \mathfrak{D}_{2k}$ and $|Q_{2k}| = 2^{2k}$. As an example, we illustrate this procedure and get Q_4 from Q_2. We write $\sigma \in \mathfrak{D}_n$ consecutively for brevity. Since $Q_2 = \{12, 21, \overline{12}, \overline{21}\}$, we get

$$\begin{aligned}
Q_4 = \{1234, 1243, 12\overline{34}, 12\overline{43}\} \cup \{2134, 2143, 21\overline{34}, 21\overline{43}\} \\
\cup \{\overline{12}34, \overline{12}43, \overline{1234}, \overline{1243}\} \cup \{\overline{21}34, \overline{21}43, \overline{2134}, \overline{2143}\},
\end{aligned}$$

where the set of ψ_i's obtained from each $\sigma \in \mathfrak{D}_2$ is given separately for clarity. For odd positive integers $n = 2k+1$ define Q_n as follows. In this case, we obtain Q_{2k+1} from Q_{2k} by the following process. For each $\sigma \in Q_{2k}$, define ψ by $\psi(i) = \sigma(i)$ for $i \in [2k]$ and define $\psi(2k+1) = 2k+1$. Do this for each $\sigma \in Q_{2k}$ and let Q_{2k+1} be the set of ψ so obtained. Clearly, $Q_{2k+1} \subseteq \mathfrak{D}_{2k+1}$ and $|Q_{2k+1}| = 2^{2k}$. For example, $Q_3 = \{123, 213, \overline{12}3, \overline{21}3\}$. It is easy to check by induction on n that

$$\sum_{\sigma \in Q_n} (-1)^{\mathsf{inv_D}(\sigma)} t^{\mathsf{pos\_n}(\sigma)} m_\sigma = \mathrm{DSgnExc}_n(q_1, q_2, \ldots, q_n, t).$$

With this notation, we state our problem below.

Problem 19. *Show that for all n there is an involution $\phi : (\mathfrak{D}_n - Q_n) \rightarrow (\mathfrak{D}_n - Q_n)$ such that for all $\sigma \in \mathfrak{D}_n - Q_n$, $\mathsf{pos\_n}(\sigma) = \mathsf{pos\_n}(\phi(\sigma))$, $\mathsf{ExcSet}_B(\sigma) = \mathsf{ExcSet}_B(\phi(\sigma))$ and* $\mathsf{inv_D}(\sigma) \not\equiv \mathsf{inv_D}(\phi(\sigma)) \pmod 2$.

ACKNOWLEDGEMENT

I thank the referees for their valuable comments and helpful suggestions. Support from project grant 15IRCCFS003, given by IRCC, IIT Bombay is gratefully acknowledged.

REFERENCES

[1] ADIN, R. M., GESSEL, I. R., AND ROICHMAN, Y. Signed Mahonians. *Journal of Combin. Theory, Series A 109* (2005), 25–43. [4]

[2] BIAGIOLI, R. Signed Mahonian polynomials for classical Weyl groups. *European Journal of Combinatorics 27* (2006), 207–217. [4]

[3] BJÖRNER, A., AND BRENTI, F. *Combinatorics of Coxeter Groups*. GTM 231, Springer Verlag, 2005. [13]

[4] BRENTI, F. q-Eulerian polynomials arising from Coxeter groups. *European Journal of Combinatorics 15* (1994), 417–441. [2, 3]

[5] MACMAHON, P. A. *Combinatory Analysis*. Cambridge University Press, 1915-1916 (Reprinted by AMS Chelsea, 2000). [3]

[6] MANTACI, R. *Statistiques Eulériennes sur les Groupes de Permutation*. Ph.D. thesis, Université Paris, 1991. [1]

[7] MANTACI, R., AND RAKOTONDRAJAO, F. Exceedingly deranging! *Advances in Applied Mathematics 30* (2003), 177–188. [1]

[8] REINER, V. Descents and one-dimensional characters for classical Weyl groups. *Discrete Mathematics 140* (1995), 129–140. [3, 4]

[9] SIVASUBRAMANIAN, S. Signed excedance enumeration via determinants. *Advances in Applied Mathematics 47* (2011), 783–794. [2]

[10] SIVASUBRAMANIAN, S. Signed excedance enumeration in the hyperoctahedral group. *Electronic Journal of Combinatorics 21(2)* (2014), P2.10. [2, 3, 5, 10, 11]

[11] ZHAO, A. F. Excedance numbers for permutations of type B. *Electronic Journal of Combinatorics 20*(2) (2013), #P 28. [10]