OCCUPIED CORNERS IN TREE-LIKE TABLEAUX

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ABSTRACT. Tree-like tableaux are combinatorial objects that appear in a combinatorial understanding of the PASEP model from statistical mechanics. In this understanding, the corners of the southeast border correspond to the locations where a particle may jump to the right. Such corners may be of two types: either empty or occupied. Our main result is the following: on average, there is one occupied corner per tree-like tableau. We give two proofs of this result, a short one which gives us a polynomial version of the result, and another one using a bijection between tree-like tableaux and permutations which gives us additional information. Moreover, we obtain the same result for symmetric tree-like tableaux, and we refine our main result to an equivalence class. Finally we present a conjecture concerning the enumeration of corners, and we explain its consequences for the PASEP.

1. INTRODUCTION

Tree-like tableaux are certain fillings of Young diagrams that were introduced by Aval, Boussicault and Nadeau in [ABN13]. They are combinatorially equivalent to permutation tableaux and alternative tableaux [SW07, Vie08] which are counted by $n!$. Over the last years, these objects have been the subject of many papers, see [Nad11, Bur07, CN09]. The equilibrium state of the PASEP, an important model from statistical mechanics, can be described using these objects as shown in the papers [CW07a, CW07b, CW10]. This model is a Markov chain, where particles jump stochastically to the left or to the right by one site in a one-dimensional lattice of n sites. We recall briefly its definition.

The states of the PASEP are encoded by words in $\bullet$ and $\circ$ of length n, where $\circ$ corresponds to an empty site and $\bullet$ to a site with a particle. To perform a transition from a state, first choose uniformly at random one of the $n + 1$ locations which are: at the left of the word, at the right of the word, and between two sites. Then, with a probability defined beforehand, make a jump of particle if it is possible. Figure 1 gives an illustration of the PASEP model. The link with tree-like tableaux appears when
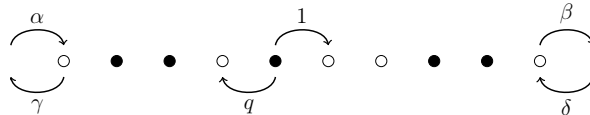

FIGURE 1. The PASEP model.

$\gamma = \delta = 0$. There is a Markov chain on tree-like tableaux that projects to the PASEP [CW07a]. The state to which a tree-like tableau projects is given by his southeast border as follows: we travel along this border starting from the southwest and without taking into account the first and the last step; an East step corresponds to $\bullet$ and a

North step to ∘. In particular, the locations where a particle may jump to the right, i.e., the patterns $X \bullet \circ Y$, $\circ Y$, and $X \bullet$, correspond to *corners*, while the locations where a particle may jump to the left, i.e., the patterns $X \circ \bullet Y$, since $\gamma = \delta = 0$, correspond to the *inner corners*. The number of locations where a particle may jump to the right or to the left does not seem to have been studied yet.

In this paper, we explain in which way the knowledge of the weighted average of the number of corners in tree-like tableaux of size n gives us the weighted average of the number of locations where a particle may jump to the right or to the left in a state of the PASEP. Moreover, we conjecture that there are $\frac{n+4}{6}$ corners per tree-like tableau, which implies $\frac{n+2}{3}$ jumping locations per PASEP state in the case $\alpha = \beta = q = 1$. Quite naturally we made a computer exploration of the average number of corners in symmetric tree-like tableaux of size $2n + 1$: it seems to be $\frac{4n+13}{12}$. We can distinguish two types of corners, the occupied corners which are filled with a point and the empty corners (cf. Definition 3.1). It turns out that in average there is one occupied corner per tree-like tableau. We present two proofs of this result, a short one which gives us also a polynomial version, and another one using the bijection between tree-like tableaux and permutations that sends crossings to occurrences of the generalized pattern 2-31 (cf. [ABN13, Section 4.1]). The additional information given by the second proof is the proportion of occupied corners numbered k, where the numbering is given by the insertion algorithm. Using the same idea as in the first proof, we show also that there is an occupied corner per symmetric tree-like tableaux. Moreover we can define an equivalence relation on tree-like tableaux (Subsection 3.5) based on the position of the points in the diagram. We prove that in each equivalence class there is one occupied corner per tree-like tableau, a refinement of the previous result.

The article is organised as follows. Section 2 gives the definition of tree-like tableaux and introduces the insertion algorithm, *Insertpoint*. Section 3 is the main part of this paper: we give the definition of occupied corners and state our main result. Moreover, after giving the two different proofs, we extend this result to symmetric tree-like tableaux. We finish this section by a refinement of the main result that can be restated in terms of lattice paths. Section 4 describes the link between tree-like tableaux and the PASEP when $\delta = \gamma = 0$ more precisely, and states the two conjectures on the enumeration of corners in tree-like tableaux and symmetric tree-like tableaux.

2. Tree-like tableaux

In this section we recall basic notions and tools about tree-like tableaux. All the details can be found in the article [ABN13].

Definition 2.1 (Tree-like tableau). A *tree-like tableau* is a filling of a Young diagram with points inside some cells, obeying three rules:
   (1) the top left cell has a point called the *root point*;
   (2) for each non-root point, there is a point above it in the same column or to its left in the same row, but not both at the same time;
   (3) there is no empty row or column.

The *size* of a tree-like tableau is its number of points. We denote the set of the tree-like tableaux of size n by $\mathcal{T}_n$. In a tree-like tableau, we call the edges of the southeast border *border edges*. A tree-like tableau of size n has $n + 1$ border edges. We index

them from 1 to $n+1$ as done in Figure 2. The border edge numbered by i in a tree-like tableau T is denoted by $e_i(T)$. In the rest of the article, when there is no ambiguity, the tableau T might be omitted in all the notations.
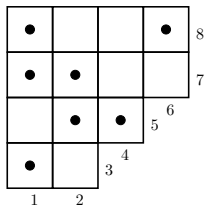

FIGURE 2. A tree-like tableau of size 7.

The set of tree-like tableaux has an inductive structure given by an insertion algorithm called *Insertpoint* which constructs a tree-like tableau of size $n + 1$ from a tree-like tableau of size n and the choice of one of its border edges. We briefly present the algorithm in order for the article to be self-contained. A more detailed presentation is given in [ABN13].

The *special point* plays an important role in *Insertpoint*. It is defined as the right-most point among those at the bottom of a column. We denote by $sp(T)$ the index of the horizontal border edge under the special point of a tree-like tableau T. In the figures of this article, the special point might be indicated by a square around it. The second notion we need is that of a *ribbon*, which is by definition a connected set of empty cells (with respect to adjacency) containing no 2×2 squares. Now we can introduce the insertion algorithm.

Definition 2.2 (Insertpoint). Let T be a tree-like tableau of size n and e_i one of its border edges. First, if e_i is horizontal (respectively vertical) we insert a row (respectively column) of empty cells, just below (respectively to the right of) e_i, starting from the left (respectively top) border of T and ending below (respectively to the right of) e_i. Moreover we put a point in the right-most (respectively bottom) cell. We obtain a tree-like tableau T' of size $n+1$. Then, depending on the relative ordering of i and $sp(T)$, we define $Insertpoint(T, e_i)$ as follows:

(1) If $i \geqslant sp$, then $Insertpoint(T, e_i) = T'$;
(2) otherwise, we add a ribbon along the border of T', from the new point to the special point of T. This new tree-like tableau is $Insertpoint(T, e_i)$.

An example of the two possible insertions is given in Figure 3, where the cells of the new row/column are shaded, and the cells of the ribbon are marked by crosses.

Remark 2.3. We notice that the new point is the special point of the new tableau. In addition, the index of the horizontal border edge under the new point is equal to the one we chose during the algorithm. In other words,

$$sp(Insertpoint(T, i)) = i.$$

3. ENUMERATION OF OCCUPIED CORNERS

As explained in the introduction, *corners* (see Definition 3.1) are interesting because in the PASEP model they correspond to the locations where a particle may jump to
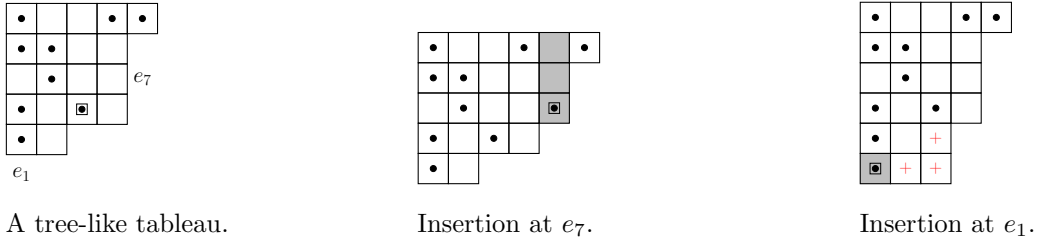
| A tree-like tableau. | Insertion at e_7. | Insertion at e_1. |

FIGURE 3. Insertion algorithm applied to T at e_1 and e_7.

the right. Our main result is that, in $\mathcal{T}_n$, on average there is one *occupied corner* per tree-like tableaux. It is a nice and surprising property, and a first step in the study of unrestricted corners. In this section, we show this new result in two[1] different ways. The first proof (Subsection 3.2) is the shortest and gives us a polynomial version of the result. The second proof (Subsection 3.3) is interesting because it tells us how the point number k is in a corner, where the numbering is induced by the insertion algorithm. In Subsection 3.4, we extend the result to symmetric tree-like tableaux, again with a polynomial version. Finally, in the last subsection, we define an equivalence relation on tree-like tableaux, and we show that in each equivalence class there is one occupied corner per tree-like tableaux.

3.1. **Main result.** First of all, let us define occupied corners.

Definition 3.1 (Occupied corner). In a tree-like tableau T, the *corners* are the cells for which the bottom and the right edges are border edges. We say that a corner is *occupied*, if it contains a point. We denote by $oc(T)$ the number of occupied corners of T, and we extend this notation to a subset X of $\mathcal{T}_n$ by $oc(X) = \sum_{T \in X} oc(T)$.

Figure 4 gives us an example of a tree-like tableau T for which $oc(T) = 2$.
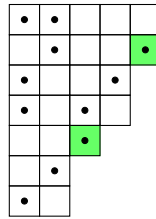

FIGURE 4. A tree-like tableau with 4 corners, 2 of which are occupied.

Theorem 3.2. *The number of occupied corners in the set of tree-like tableaux of size n is given by*

$$oc(\mathcal{T}_n) = n!.$$

In other words, on average there is one occupied corner per tree-like tableau. The reader may check Theorem 3.2 for the case $n = 3$ using Figure 5.

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[1] In the unpublished note [LZ15], we present a third proof based on two mirror insertion algorithms.

$oc = 2$ $oc = 1$ $oc = 1$ $oc = 1$ $oc = 1$ $oc = 0$
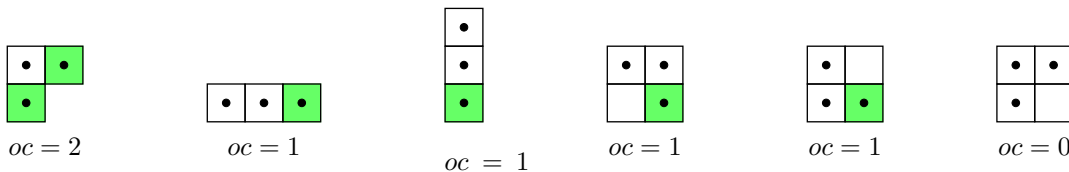
FIGURE 5. Theorem 3.2 for $n = 3$.

3.2. **Short proof.** Theorem 3.2 is a direct consequence of the following bijection.

Proposition 3.3. *The set of occupied corners in $\mathcal{T}_n$ is in bijection with the set of pairs consisting of a tree-like tableau of size $n-1$ and an integer of $[\![1, n]\!]$.*

Proof. Let OC be an occupied corner in $\mathcal{T}_n$, let T be the tree-like tableau corresponding to OC, and let i be the index of the horizontal border edge of OC. Among the row and the column containing OC there is one which has no other point than the one of OC. In the case where it is the row (respectively column), we remove this row (respectively column), and we call T' the tree-like tableau of size $n-1$ we obtain. In T' the border edge of index i is horizontal (respectively vertical). From the pair (T', i) we can construct back OC by inserting a row (respectively a column) at $e_i(T')$ and putting a point in the right-most (respectively the bottom) cell. Figure 6 gives us an illustration of the bijection. $\square$

A tree-like tableau with two corners c_1 and c_2.

The pair associated to c_1.
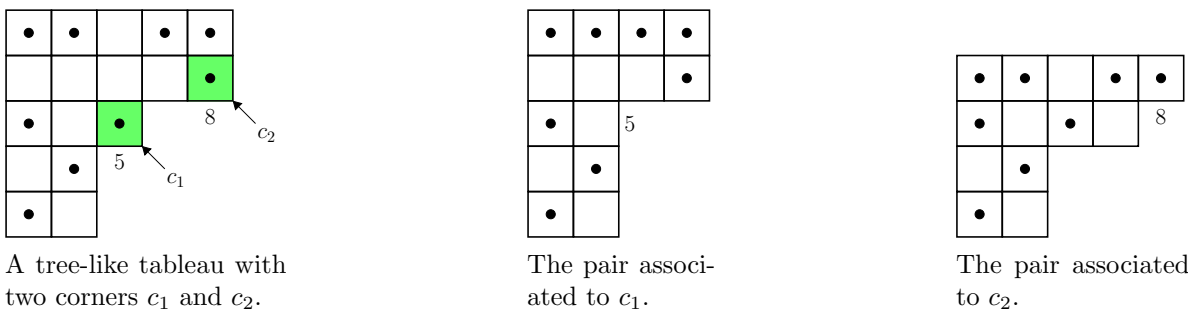
The pair associated to c_2.

FIGURE 6. Bijective proof of Theorem 3.2

We can obtain a polynomial version of the main result. Let us define

$$P_n(x) = \sum_{T \in \mathcal{T}_n} x^{oc(T)} = \sum_{k \geqslant 0} a_{n,k} x^k,$$

where the coefficient $a_{n,k}$ equals the number of tree-like tableaux of size n with k occupied corners. In order to determine a recurrence relation defining these polynomials, we start by finding a recurrence relation for the coefficients $a_{n,k}$. We do this by analysing how the number of occupied corners varies in the previous bijection during the addition of an occupied corner. Let T' be a tree-like tableau of size $n-1$. We denote the number of occupied corners of T' by k. Applying the previous bijection, if we choose one of the $2k$ border edges of the occupied corners of T', we obtain a tree-like tableau T with k occupied corners, otherwise T has $k+1$ occupied corners. By applying the bijection to

all the pairs of $\mathcal{T}_{n-1} \times [\![1, n]\!]$, a tree-like tableau T of size n is obtained $oc(T)$ times. For $n \geqslant 2$ and $k \geqslant 1$, we obtain the following recurrence relation:

$$ka_{n,k} = 2ka_{n-1,k} + (n - 2(k-1))a_{n-1,k-1}.$$

This can be translated into a recurrence relation for the polynomials P_n,

$$xP'_n = 2xP'_{n-1} + nxP_{n-1} - 2x^2 P'_{n-1},$$

which is equivalent to

$$P'_n = nP_{n-1} + 2(1-x)P'_{n-1}. \tag{3.1}$$

If we set $P_0 = 1$, Equation (3.1) is also true for $n = 1$. The sequence of polynomials $(P_n)_{n \geqslant 0}$ is uniquely defined by

$$\begin{cases} \text{for all } n \geqslant 1, \ P'_n = nP_{n-1} + 2(1-x)P'_{n-1} \text{ and } P_n(1) = n!; \\ P_0 = 1. \end{cases}$$

We recover the main result by observing that the number of occupied corners in $\mathcal{T}_n$ is equal to $P'_n(1)$. By Equation (3.1), $P'_n(1) = nP_{n-1}(1) = n!$ since $P_{n-1}(1) = |\mathcal{T}_{n-1}| = (n-1)!$. To give an illustration, we present the first terms of the sequence $(P_n)_{n \geqslant 0}$:

$$P_0 = 1,$$
$$P_1 = x,$$
$$P_2 = 2x,$$
$$P_3 = x^2 + 4x + 1,$$
$$P_4 = 6x^2 + 12x + 6,$$
$$\vdots$$
$$P_{10} = 720x^5 + 21600x^4 + 188640x^3 + 748800x^2 + 1475280x + 1193760.$$

Remark 3.4. The number of tree-like tableaux of size n with 0 occupied corners is given by $P_n(0)$. By computer exploration using the open-software `Sage` [S$^+$15], we found that the sequence $(P_n(0))_{n \geqslant 0}$ might be equal to the sequence [Slo, A184185], which counts the number of permutations of $\{1, \ldots, n\}$ having no cycles of the form

$$(i, i+1, i+2, \ldots, i+j-1)$$

with $j \geqslant 1$. At this point, we are not able to prove this fact.

In order to estimate the dispersion of the statistic of occupied corners, we compute the variance σ^2. By definition, we have

$$\sigma^2 = \frac{1}{n!} \left(\sum_{T \in \mathcal{T}_n} oc(T)^2 \right) - 1,$$

hence,

$$
\begin{aligned}
\sigma^2 &= \frac{1}{n!}\left(P_n''(1) + P_n'(1)\right) - 1 \\
&= \frac{1}{n!}\left((n-2)P_{n-1}'(1) + P_n'(1)\right) - 1 \\
&= \frac{1}{n!}\left((n-2)(n-1)! + n!\right) - 1 \\
&= \frac{n-2}{n}.
\end{aligned}
$$

Consequently, the variance tends to 1 as n tends to infinity.

3.3. **Enumerative proof using a bijection with permutations.** Let T be a tree-like tableau of size $n \geqslant 2$. There is a unique way to obtain T by performing consecutive insertions starting with the tree-like tableau of size 1. Since T is of size n, we do $n-1$ insertions. Hence, we can uniquely encode $T \in \mathcal{T}_n$ by the sequence of the indices chosen during the insertion algorithm: $(m_2(T), \ldots, m_n(T))$ where $1 \leqslant m_k(T) \leqslant k$ for $k \in [\![2, n]\!]$. For $k \in [\![2, n]\!]$, we denote by $p_k(T)$ the point added during the $(k-1)$-st insertion. Let us recall that (cf. Remark 2.3) when p_k is inserted, the index of the border edge below it is m_k. By convention, $m_1(T) = 1$, and $p_1(T)$ corresponds to the point of the unique tree-like tableau of size 1. For example, the tree-like tableau of Figure 2 is encoded by $(1, 1, 3, 2, 2, 1, 4)$.

Observation 3.5. *Let $k \in [\![2, n]\!]$. The point p_k is in a corner if, and only if, the following two conditions are satisfied:*

- *$m_k \geqslant m_{k-1}$, i.e., if no ribbon is added during its insertion;*
- *$m_j > m_k + 1$ for all $k < j \leqslant n$, i.e., if after p_k all the other points are inserted to the northeast of the vertical border edge e_{m_k+1} (p_k is not covered by a ribbon).*

In order to get a statistic on permutations which is in bijection with the occupied corners and keep the information of the index k, we introduce the bijection ϕ, defined in [ABN13, Section 4.1], between tree-like tableaux and permutations. Let T be a tree-like tableau of size n and $(m_1, \ldots, m_n)$ its encoding. $\sigma = \phi(T)$ is the unique permutation with the non-inversion table equal to $(m_1 - 1, \ldots, m_n - 1)$. It can be computed algorithmically in the following way. First, set $\sigma(n)$ to m_n. Suppose that we have defined $\sigma(n), \ldots, \sigma(i+1)$. If we write

$$
\{x_1 < x_2 < \cdots < x_i\} = [\![1, n]\!] \setminus \{\sigma(i+1), \ldots, \sigma(n)\},
$$

we take $\sigma(i) = x_{m_i}$. In other words, consider the word $12 \cdots n$, remove the m_n-th letter, then the m_{n-1}-st one, and so on, until one gets the empty word. Then, concatenate the letters in the order they got removed, the first removed one being at the end of the word, and the last removed one at the beginning. In this way, we obtain σ as a word. To give an example of this bijection, consider the tree-like tableau T of Figure 2 encoded by $(1, 1, 3, 2, 2, 1, 4)$. We have $\phi(T) = 6275314$. Now we can translate Observation 3.5 into a result on permutations.

Proposition 3.6. *Let $T \in \mathcal{T}_n$ and $\sigma = \phi(T)$. Then, for $k \in [\![2, n]\!]$, p_k is in a corner of T if, and only if, the following two conditions are satisfied:*

(1) *$\sigma(k-1) = \sigma(k) + 1$ or $\sigma(k-1) < \sigma(k)$;*

(2) $\sigma(j) > \sigma(k) + 1$ *for all* $k < j \leqslant n$.

Proof. We will first prove that if p_k is in a corner of T, the index k satisfies the conditions (1) and (2). We use the characterisation of Observation 3.5 to prove the first implication. If $k = n$, we only need to prove condition (1). There are two possibilities: if $m_n = n$, then $\sigma(n) = n$ and $\sigma(n-1) < \sigma(n)$, otherwise, in the construction process of σ, we choose $\sigma(n) = m_n$, hence for $\sigma(n-1)$ we choose the m_{n-1}-th letter of the word

$$1 \cdots (m_n - 1)(m_n + 1) \cdots n,$$

and since $m_{n-1} \leqslant m_n$ we have $\sigma(n-1) \in \{1, \ldots, m_n-1, m_n+1\}$. In the case $k \neq n$, we know that $m_j > m_k + 1$ for all $k < j \leqslant n$, hence, during the construction of those $\sigma(j)$, the subword $1 \cdots m_k(m_k + 1)$ of the initial word $1 \cdots n$ is never modified. In particular, we have $\sigma(j) > m_k + 1$. Since $m_k = n$ implies $k = n$, the subword $1 \cdots m_k(m_k + 1)$ is well defined. We have $\sigma(k) = m_k$, hence condition (2) is satisfied. Moreover, the first m_k letters of $1 \cdots n$, after removing $\sigma(n), \ldots, \sigma(k)$, are $1, 2, \cdots, m_k - 1, m_k + 1$. Thus $\sigma(k-1) \in \{1, \ldots, m_k - 1, m_k + 1\}$ as $m_{k-1} \leqslant m_k$, which proves condition (1).

Conversely, let us suppose that σ satisfies conditions (1) and (2). We will show that the encoding $(m_1, \ldots, m_n)$ of the corresponding tree-like tableau satisfies the conditions of Observation 3.5. As $(m_1 - 1, \ldots, m_n - 1)$ is the non-inversion table of σ, for $j \in [\![1, n]\!]$, $m_j - 1$ corresponds to the number of $\sigma(i)$ such that $i < j$ and $\sigma(i) < \sigma(j)$. First, we notice that $\sigma(k) + 1$ is at the left of $\sigma(k)$ by condition (2). For $j > k$, in the word σ, at the left of $\sigma(j)$ there are $\sigma(k) + 1$, $\sigma(k)$, and all the integers that are at the left of $\sigma(k)$ and smaller than $\sigma(k)$. Hence $m_j - 1 \geqslant 2 + m_k - 1$, which is equivalent to $m_j > m_k + 1$. Condition (2) implies that, if for $j < k - 1$ we have $\sigma(j) < \sigma(k-1)$, this implies $\sigma(j) < \sigma(k)$. Thus, $m_{k-1} \leqslant m_k$. $\qquad\square$

With the help of this last interpretation, we obtain a result refining Theorem 3.2.

Proposition 3.7. *For* $k \in [\![2, n]\!]$*, the number of tree-like tableaux* T *of size* n *having* p_k *in a corner is*

$$\frac{n!}{(n-k+2)(n-k+1)} + \begin{cases} (n-1)!, & \text{if } k = n; \\ 0 & , \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. In order to show this, we enumerate the permutations for which k satisfies the conditions of Proposition 3.6. First, if $k = n$, only condition (1) matters. Thus, if $\sigma(n) = n$, we can choose the other value of σ as we want, hence we have $(n-1)!$ possibilities to choose σ. Otherwise, if we denote by i the integer $\sigma(n) \leqslant n-1$, we have i possibilities for $\sigma(n-1)$, namely $1, \ldots, i-1, i+1$, and there is no other condition on σ, hence we have $i(n-2)!$ such permutations. By summing all these numbers, we obtain

$$\sum_{i=1}^{n-1} i(n-2)! + (n-1)! = \frac{n!}{2} + (n-1)!,$$

which is equal to Equation (3.2) when $k = n$. Suppose now $k \neq n$, and let $\sigma(k) = i$. Condition (2) tells us that the integers $\sigma(k+1), \ldots, \sigma(n)$ are inside $[\![i+2, n]\!]$, hence $n - k \leqslant n - i - 1$, which is equivalent to $i \leqslant k - 1$. There are $\binom{n-i-1}{n-k}$ choices for those integers and $(n-k)!$ ways of ordering them. We have i choices for $\sigma(k-1)$, namely

$1, \ldots, i-1$ and $i+1$. Finally, to define $\sigma(1), \ldots, \sigma(k-2)$, we have $(k-2)!$ possibilities. By summing all these numbers, we obtain

$$(n-k)! \, (k-2)! \sum_{i=1}^{k-1} i \binom{n-i-1}{n-k}.$$

The sum is easy to evaluate:

$$\begin{aligned}
\sum_{i=1}^{k-1} i \binom{n-i-1}{n-k} &= \sum_{i=1}^{k-1} \left(n \binom{n-i-1}{n-k} - (n-k+1) \binom{n-i}{n-k+1} \right) \\
&= \left(n \binom{n-1}{n-k+1} - (n-k+1) \binom{n}{n-k+2} \right) \\
&= \binom{n}{n-k+2}.
\end{aligned}$$

We get the desired result since

$$(n-k)! \, (k-2)! \binom{n}{n-k+2} = \frac{n!}{(n-k+2)(n-k+1)}. \qquad \square$$

3.4. **Extension to symmetric tree-like tableaux.** Symmetric tree-like tableaux were introduced in [ABN13, Section 2.2]. They are the tree-like tableaux which are invariant under a reflection with respect to their main diagonal. They are in bijection with symmetric alternative tableaux from [Nad11, Section 3.5] and type B permutation tableaux from [LW08, Section 10.2.1]. The root point is the only point of the main diagonal of a symmetric tree-like tableau. Indeed, if there was another one, it would have a point to its left and above it, or neither of them, which would contradict condition (2) of Definition 2.1. Hence such a tree-like tableau has odd size. We denote the set of symmetric tree-like tableaux of size $2n+1$ by $\mathcal{T}_{2n+1}^{Sym}$. There are $2^n n!$ of them (see [ABN13, Section 2.2]).

Theorem 3.8. *The number of occupied corners in the set $\mathcal{T}_{2n+1}^{Sym}$ is given by*

$$oc(\mathcal{T}_{2n+1}^{Sym}) = 2^n n!.$$

Proof. Similarly to Subsection 3.2, we will put the set of occupied corners of $\mathcal{T}_{2n+1}^{Sym}$ in bijection with the set of triples consisting of a symmetric tree-like tableau of size $2n-1$, an integer of $[\![1, n]\!]$, and an element ρ of $\{a, b\}$. Let OC be an occupied corner in $\mathcal{T}_{2n+1}^{Sym}$, and let T be the tree-like tableau corresponding to OC. If OC is below the main diagonal of T, we set ρ to b, and we denote the index of the horizontal border edge of OC by i. Otherwise, we consider the occupied corner which is the mirror image of OC, and we denote the index of its horizontal border edge by i. Moreover, we set ρ to a. We write T' for the symmetric tableau obtained after removing the nearly empty row or column in which we find OC and its mirror image. We associate to OC the triple (T', i, ρ). From the pair (T', i), we reconstruct the tree-like tableau T by inserting a row or a column at $e_i(T')$ and putting a point in the right-most or the bottom cell, and doing also the mirror images of these operations. In this way, we constructed two new

occupied corners, OC corresponds to the one below the main diagonal if $\rho = b$, and to the one above the main diagonal if $\rho = a$. Finally, we have

$$oc(\mathcal{T}_{2n+1}^{Sym}) = 2^{n-1}(n-1)! \times n \times 2 = 2^n n!. \quad \square$$

Let $Q_n = \sum_{T \in \mathcal{T}_{2n+1}^{Sym}} x^{oc(T)} = \sum_{k \geqslant 0} b_{n,k} x^{2k}$. In the case of symmetric tree-like tableaux, we obtain the following recurrence relation. For $n \geqslant 2$ and $k \geqslant 1$, we have

$$2k \cdot b_{n,k} = 2 \left[2k \cdot b_{n-1,k} + (n - 2(k-1))b_{n-1,k-1} \right].$$

This can be translated into a recurrence relation for the polynomials Q_n, namely

$$xQ_n = 2\left(xQ'_{n-1} + nx^2 Q_{n-1} - x^3 Q'_{n-1} \right),$$

which is equivalent to

$$Q'_n = 2nx \cdot Q_{n-1} + 2(1 - x^2)Q'_{n-1}. \tag{3.3}$$

If we set $Q_0 = 1$, Equation (3.3) is also true for $n = 1$. The sequence of polynomials $(Q_n)_{n \geqslant 0}$ is uniquely defined by

$$\begin{cases} \text{for all } n \geqslant 1, \ Q'_n = 2nx \cdot Q_{n-1} + 2(1 - x^2)Q'_{n-1} \text{ and } Q_n(1) = 2^n n!; \\ Q_0 = 1. \end{cases}$$

To give an illustration, we present the first terms of the sequence $(Q_n)_{n \geqslant 0}$:

$$Q_0 = 1,$$
$$Q_1 = x^2 + 1,$$
$$Q_2 = 4x^2 + 4,$$
$$Q_3 = 2x^4 + 20x^2 + 26,$$
$$\vdots$$
$$Q_9 = 384x^{10} + 55680x^8 + 1386240x^6 + 13566720x^4 + 61380480x^2$$
$$+ 109405056.$$

As we did for occupied corners in $\mathcal{T}_n$, we can compute the variance. It is equal to $\frac{n-1}{n}$, hence we can draw the same conclusion, namely that it tends to 1 as n approaches infinity.

3.5. **A refinement of the main result.** A *non-ambiguous class* is a set of tree-like tableaux which have their points arranged in the same way (see Figure 7). These classes are linked with the non-ambiguous trees introduced in [ABBS14]. Indeed, they can be described as the set of tree-like tableaux which have the same underlying non-ambiguous tree.

Theorem 3.9. *In a non-ambiguous class, on average there is one occupied corner per tree-like tableau.*

Figure 7 gives an example of a non-ambiguous class of size 5 with 5 occupied corners in total. The above theorem is a refinement of Theorem 3.2 since we can partition $\mathcal{T}_n$ in non-ambiguous classes. We first show that the result of Theorem 3.9 can be restated in terms of paths. For each non-ambiguous class, we choose a canonical representative: the
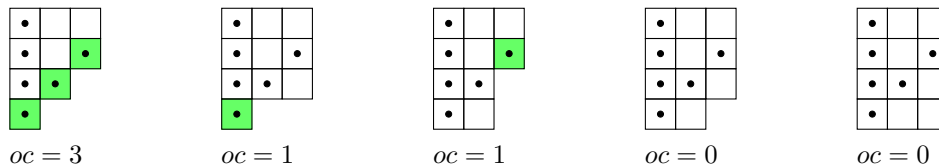
FIGURE 7. A non-ambiguous class in $\mathcal{T}_5$, and the enumeration of occupied corners.

only tree-like tableau whose corners are all occupied. The other tree-like tableaux of the class are obtained by choosing a lattice path, with steps East and North, weakly below the southeast border of the canonical representative. Figure 8 shows an example of this correspondence. Only the points which are in the corners of the canonical representative are drawn. In this study, the only thing that matters is the subpath of the southeast

A canonical representative of a non-ambiguous class.

Another element of a non-ambiguous class.
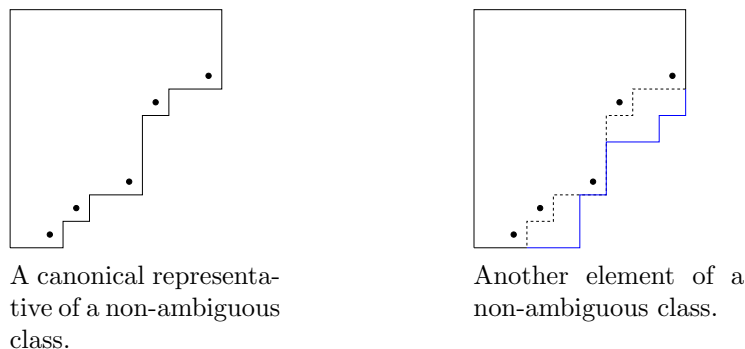
FIGURE 8. Correspondence between non-ambiguous classes and paths.

border of the canonical representative which is between the southwest-most corner and the northeast-most corner. In the rest of the section, we will only consider lattice paths with North (N) and East (E) steps. Let P be a lattice path starting with a *corner*, i.e., an E step followed by a N step, and ending with a corner. We denote the set of lattices weakly below P by $\mathcal{P}(P)$. For $P' \in \mathcal{P}(P)$, we denote the number of corners P' and P have in common by $cc(P')$. Theorem 3.9 can be restated as follows.

**Proposition 3.10.** *Let P be a lattice path starting and ending with a corner. Then*

$$|\mathcal{P}(P)| = \sum_{P' \in \mathcal{P}(P)} cc(P').$$

Proof. Let P be such a path. We denote the set of corners of $\mathcal{P}(P)$ common with P by $CC(P)$. Let us consider $cc \in CC(P)$ and P' its corresponding lattice path. We will bijectively construct an element P'' of $\mathcal{P}(P)$. If we denote the E step corresponding to cc by $\dot{E}$, P' is of the form $u\dot{E}Nv$, where u and v are (possibly empty) words in $\{E, N\}$. We define P'' as the path $u\dot{E}vN$. P'' is weakly below P', so, as desired, P'' belongs to $\mathcal{P}(P)$. In particular, since P' is weakly below P, $\dot{E}$ is the last E step that P and P'' have in common. Conversely, let P'' be a path weakly below P. If we denote the last

E step P and P'' have in common by $\dot{E}$, P'' is of the form $u\dot{E}vN$, where u and v are (possibly empty) words in $\{E, N\}$. $\dot{E}$ exists, since they both start by an E step. We define cc as the corner $\dot{E}N$ of the path $P' = u\dot{E}Nv$. The definition of $\dot{E}$ implies that P' belongs to $\mathcal{P}(P)$, hence cc belongs to $CC(P)$. An example of the bijection is given in Figure 9. $\qquad\square$
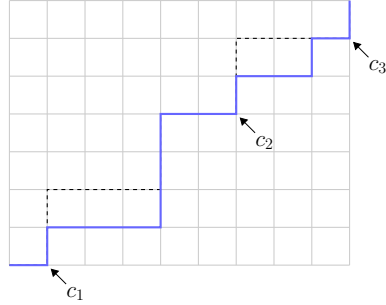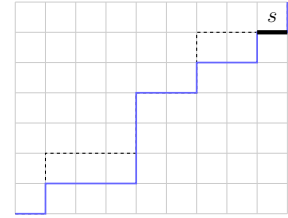

A path P (dashed) and a path P' (blue) weakly below P, with $cc(P') = 3$.
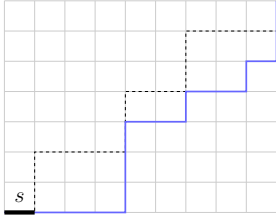
Path associated to c_1.
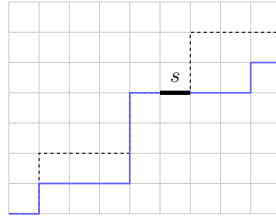
Path associated to c_2.

Path associated to c_3.

FIGURE 9. Bijective proof of Proposition 3.10.

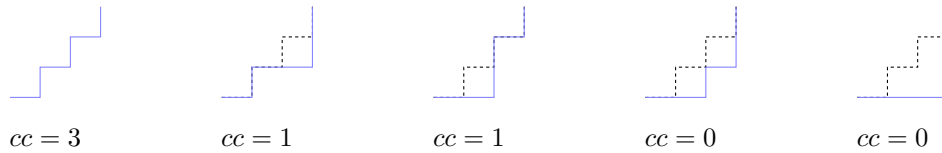Figure 10 shows the example corresponding to Figure 7 in terms of paths.

$cc = 3$ $\qquad$ $cc = 1$ $\qquad$ $cc = 1$ $\qquad$ $cc = 0$ $\qquad$ $cc = 0$

FIGURE 10. A path P (dashed) and the set of paths weakly below P (blue) with the enumeration of common corners.

4. The link between corners and the PASEP

As explained in the introduction, the stationary state of the PASEP can be described by tree-like tableaux. We define the weight $w(T)$ of a tree-like tableau as follows: $w(T) = q^{cr(T)}\alpha^{-tp(T)}\beta^{-lp(T)}$, where $cr(T)$, $tp(T)$, and $lp(T)$ correspond to the number of crossings, top points, and left points of T, respectively (see [ABN13] for the definitions of those statistics). Let Z_n be the partition function $\sum_{T \in \mathcal{T}_{n+1}} w(T)$. Furthermore, let s

be a word of size n corresponding to a state in the PASEP. We denote by $\mathcal{T}_{n+1}^{\lambda(s)}$ the set of tree-like tableaux of size $n+1$ that project to s. As shown in [CW07b], the stationary probability of s is equal to

$$\frac{\sum_{T \in \mathcal{T}_{n+1}^{\lambda(s)}} w(T)}{Z_n}.$$

Let $X(s)$ be the random variable counting the number of locations of a state s, where a particle may jump to the right or to the left. Let T be a tree-like tableau that projects to s. An *inner corner* of T is a succession of a vertical border edge by a horizontal border edge. We denote the number of inner corners of T by $ic(T)$. We have the immediate relation $ic(T) = c(T) - 1$, where $c(T)$ represents the number of corners of T. With these notations, $X(s) = c(T) + ic(T) = 2c(T) - 1$. We can now compute the theoretical value of the expected value of X in terms of tree-like tableaux:

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{s \in \{\circ, \bullet\}^n} \mathbb{P}(s) \cdot X(s) \\
&= \frac{1}{Z_n} \sum_{s \in \{\circ, \bullet\}^n} \sum_{T \in \mathcal{T}_{n+1}^{\lambda(s)}} w(T)(2c(T) - 1) \\
&= \frac{1}{Z_n} \sum_{T \in \mathcal{T}_{n+1}} w(T)(2c(T) - 1).
\end{aligned}$$

By computer exploration using `Sage` [S+15], we found the following conjecture.

Conjecture 4.1. *The number of corners in $\mathcal{T}_n$ is $n! \cdot \frac{n+4}{6}$.*

In the case $\alpha = \beta = q = 1$ and $\delta = \gamma = 0$, we have $w(T) = 1$, hence this conjecture would imply that

$$\mathbb{E}(X) = 2 \frac{(n+1)+4}{6} - 1 = \frac{n+2}{3}.$$

We were not able to adapt the proofs of Theorem 3.2 to prove this conjecture. To adapt the first proof, we should construct in average $\frac{n(n+4)}{6}$ corners in $\mathcal{T}_n$ from a tree-like tableau of size $n-1$. For an adaptation of the second proof, we should control the behaviour of corners during an insertion, and translate this behaviour in terms of the code.

We also give the version of Conjecture 4.1 for symmetric tree-like tableaux.

Conjecture 4.2. *The number of corners in $\mathcal{T}_{2n+1}^{Sym}$ is $2^n \times n! \times \frac{4n+13}{12}$.*

Remark 4.3. Since the submission of this article, Conjectures 4.1 and 4.2 were proved by Gao, Gao, L.-Z. and Sun in [GGLZS16a] and [GGLZS16b]. Independently, Hitczenko and Lohss [HL15] also proved both conjectures.

References

[ABBS14] Jean-Christophe Aval, Adrien Boussicault, Mathilde Bouvel, and Matteo Silimbani. Combinatorics of non-ambiguous trees. *Adv. Appl. Math.*, 56:78–108, 2014.

[ABN13]   Jean-Christophe Aval, Adrien Boussicault, and Philippe Nadeau. Tree-like tableaux. *Electron. J. Combin.*, 20(4):Paper 34, 24 pp., 2013.

[Bur07]   Alexander Burstein. On some properties of permutation tableaux. *Ann. Comb.*, 11(3-4):355–368, 2007.

[CN09]   Sylvie Corteel and Philippe Nadeau. Bijections for permutation tableaux. *European J. Combin.*, 30(1):295–310, 2009.

[CW07a]   Sylvie Corteel and Lauren K. Williams. A Markov chain on permutations which projects to the PASEP. *Int. Math. Res. Not. IMRN*, (17):Art. ID rnm055, 27 pp., 2007.

[CW07b]   Sylvie Corteel and Lauren K. Williams. Tableaux combinatorics for the asymmetric exclusion process. *Adv. Appl. Math.*, 39(3):293–310, 2007.

[CW10]   Sylvie Corteel and Lauren K. Williams. Staircase tableaux, the asymmetric exclusion process, and Askey-Wilson polynomials. *Proc. Natl. Acad. Sci. USA*, 107(15):6726–6730, 2010.

[GGLZS16a] Alice L. L. Gao, Emily X. L. Gao, Patxi Laborde-Zubieta and Brian Y. Sun. Enumeration of corners in tree-like tableaux and a conjectural (a, b)-analogue, 2015; arχiv:1511.05434.

[GGLZS16b] Alice L. L. Gao, Emily X. L. Gao, Patxi Laborde-Zubieta and Brian Y. Sun. Enumeration of corners in symmetric tree-like tableaux, 2015; arχiv:1511.05456.

[HL15]   Pawel Hitczenko and Amanda Lohss. Corners in tree-like tableaux, 2015; arχiv:1511.04989.

[LW08]   Thomas Lam and Lauren K. Williams. Total positivity for cominuscule Grassmannians. *New York J. Math.*, 14:53–99, 2008.

[LZ15]   Patxi Laborde Zubieta. Alternative proof of the average number of occupied corners. Available at `http://www.labri.fr/~plaborde/unpublished_notes`, May 2015.

[Nad11]   Philippe Nadeau. The structure of alternative tableaux. *J. Combin. Theory, Ser. A*, 118(5):1638–1660, 2011.

[S+15]   W. A. Stein et al. *Sage Mathematics Software (Version 6.2.beta7)*. The Sage Development Team, 2015. `http://www.sagemath.org`.

[SCc08]   The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. `http://combinat.sagemath.org`.

[Slo]   N. J. A. Sloane. The on-line encyclopedia of integer sequences. `http://oeis.org`.

[SW07]   Einar Steingrímsson and Lauren K. Williams. Permutation tableaux and permutation patterns. *J. Combin. Theory, Ser. A*, 114(2):211–234, 2007.

[Vie08]   Xavier Viennot. Alternative tableaux, permutations and partially asymmetric exclusion process. Slides of a talk at the Isaac Newton Institute in Cambridge, 2008.

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