Combinatorial properties in cut-and-project sets: order beyond periodicity

Christoph Richard, FAU Erlangen-Nürnberg
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motivation

every lattice $\Lambda \subset \mathbb{R}^d$ is

- uniformly discrete: $\exists r > 0 \; \forall x \in \mathbb{R}^d : |\Lambda \cap B_r(x)| \leq 1$
- relatively dense: $\exists R > 0$: $\Lambda B_R(0) = \mathbb{R}^d$
- periodic with d linearly independent periods
- "pure point diffractive"

we are interested in "ordered" point sets which generalise lattices

- I: order beyond periodicity
- II: cut-and-project sets: geometry and combinatorics
- III: cut-and-project sets: diffraction and harmonic analysis

some classes of point sets

- Here uniformly discrete point sets Λ. Then uniformly in x

$$|\Lambda \cap B_s(x)| = O(\mathrm{vol}(B_s)) \qquad (s \to \infty)$$

- Λ Delone $\iff$ Λ relatively dense, uniformly discrete
  examples: lattice, random distortion of lattice, tilings, ...
- Λ Meyer $\iff$ Λ relatively dense, $\Lambda\Lambda^{-1}$ uniformly discrete
- cut-and-project sets (certain projected subsets of a lattice)

Meyer sets are highly structured

- $\Lambda\Lambda^{-1}$ uniformly discrete: finitely many "local configurations"
- any Meyer set is a subset of a cut-and-project set (Meyer 72)
- diffraction of Delone sets: Bragg peaks of high intensity
  Meyer, if relatively dense (Lenz–Strungaru 14)

patterns in uniformly discrete point sets

consider (centered ball) patterns:

- r-pattern of Λ centered in $p \in \Lambda$

$$\Lambda \cap B_r(p), \qquad p \in \Lambda$$

- patterns equivalent if they agree up to translation

$$\Lambda \cap B_r(p) \sim \Lambda \cap B_r(q) \iff p^{-1}\Lambda \cap B_r(0) = q^{-1}\Lambda \cap B_r(0)$$

pattern counting and finite local complexity

count patterns

$$N_B^*(\Lambda) = |\{p^{-1}\Lambda \cap B \mid p \in \Lambda\}|$$

- interested in (exponential) growth of $N_B^*(\Lambda)$ with B

Definition

Λ finite local complexity (FLC) if $N_B^*(\Lambda)$ is finite for every ball B

- only finitely many "local configurations"
- examples: Meyer sets, cut-and-project sets

repetitivity

we are interested in Λ with “many equivalent patterns”

Definition

Λ is repetitive if $\forall r \, \exists R = R(r)$:
Every R-ball contains an equivalent copy of every r-pattern.

- For given r, one is interested in the smallest $R(r)$
- The above condition means: $\forall r \exists R : \forall x \in \mathbb{R}^d \, \forall p \in \Lambda \, \exists p' \in \Lambda$:

$$B_r(p') \subset B_R(x), \qquad \Lambda \cap B_r(p') \sim \Lambda \cap B_r(p)$$

- Λ repetitive $\Longrightarrow$ Λ has FLC
- FLC does not imply repetitivity: $\mathbb{Z} \setminus \{0\}$

repetitivity function $r \mapsto R(r)$

- periodic point sets are repetitive, e.g. $R(r) = r + 1$ for $\Lambda = \mathbb{Z}$.
- slow growth of $R(r)$ with r implies periodicity

Theorem (Lagarias–Pleasants 02)

Let Λ be non-empty and uniformly discrete. Assume that there exist $r > 0$ and $R(r) < \frac{4}{3} r$ such that every R-ball contains an equivalent copy of every r-pattern. Then Λ is periodic.

Assume w.l.o.g. $0 \in \Lambda$ and define, with the above r,

$$P_r = \{p \in \Lambda \,|\, \Lambda \cap B_r(p) \sim \Lambda \cap B_r(0)\}$$

- $\big(B_{r/3}(p)\big)_{p \in P_r}$ covers $\mathbb{R}^d$:
  repetitivity: $\forall x \in \mathbb{R}^d \, \exists p' \in P_r$ such that $B_r(p') \subset B_R(x)$.
  Hence $d(x, p') \leq R - r < r/3$.
- in particular $P_r \cap B_{r/3}(x) \neq \varnothing$ for all $x \in \mathbb{R}^d$
- $P_r \cap B_{2r/3}(0)$ contains d linearly independent vectors:
  Let $x_1, \ldots, x_k \in P_r \cap B_{2r/3}(0)$ be linearly independent. Every $r/3$-ball which intersects $\langle x_1, \ldots, x_k \rangle$ only in 0 contains some linearly independent $x_{k+1} \in P_r$.

Lemma

$x \in P_r \cap B_{2r/3}(0)$ is a period, i.e., $px \in \Lambda$ for every $p \in \Lambda$.

Proof.

For $q \in P_r$ we have

i) $p \in \Lambda \cap B_r(0) \Rightarrow pq \in \Lambda$ by definition of P_r

ii) $p \in \Lambda \cap B_{r/3}(q) \Rightarrow px \in \Lambda$:

- $q^{-1}p \in \Lambda \cap B_{r/3}(0)$ by definition of P_r
- hence $q^{-1}px \in \Lambda \cap B_r(0)$ by *i)*
- hence $px \in \Lambda$ by definition of P_r

The lemma follows since $\left(B_{r/3}(q)\right)_{q \in P_r}$ covers Λ. □

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Beatty sequences (Morse–Hedlund 38)

For irrational $\alpha \in (0,1)$ define

$$b_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor \in \{0,1\}$$

repetitivity properties of $\Lambda_\alpha = \{n \in \mathbb{Z} \mid b_n = 1\}$

- for every r there exists finite $R(r)$
- Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be any continuous non-decreasing function. Then there is α such that for infinitely many $r \in \mathbb{N}$ the sequence Λ_α cannot satisfy

$$R(r) \leq g(r)$$

- Any Λ_α arises naturally from a cut-and-project construction.

Fibonacci substitution

- letters: a, b, substitution rule: $a \to ab$, $b \to a$
- (right-infinite) Fibonacci chain: start with a!

$$a, ab, aba, abaab, abaababa, abaababaabaab, \ldots$$

- 01-sequence by $a \mapsto 1$, $b \mapsto 0$.
- substitution matrix:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- eigenvalues of M:

$$\tau = \frac{1+\sqrt{5}}{2} = 1.618034 \qquad \tau' = \frac{1-\sqrt{5}}{2} = -0.618034\ldots$$

letter frequencies

- n-fold substitution: no letters $M^n e_1$

$$M^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

- Fibonacci numbers: $(f_n)_{n\in\mathbb{N}_0} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 44, \ldots$

$$f_n = f_{n-1} + f_{n-2} = \frac{\tau^n - \tau'^n}{\tau - \tau'} = \mathrm{round}\left(\frac{\tau^n}{\tau - \tau'}\right)$$

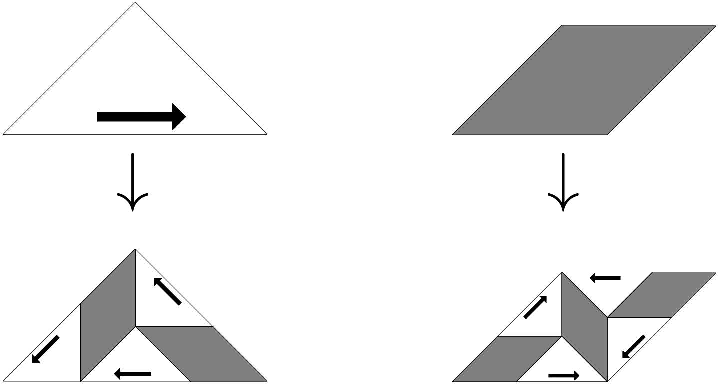
- relative frequency $(h_n(a))_{n\in\mathbb{N}}$ of a converges:

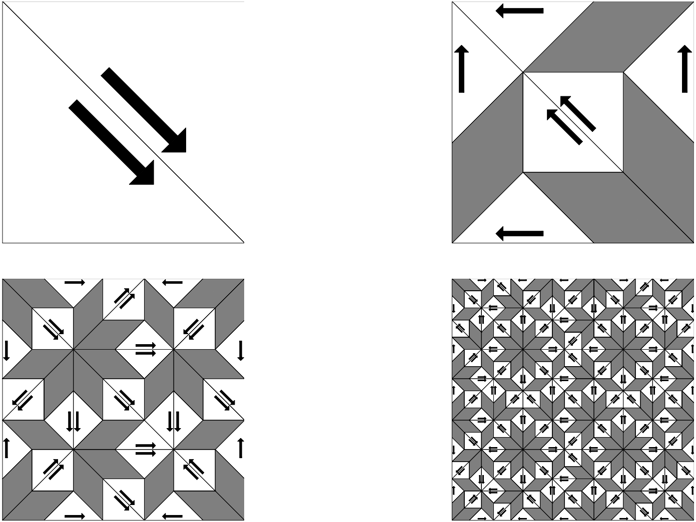
$$h_n(a) = \frac{f_{n+1}}{f_{n+1} + f_n} = \frac{f_{n+1}}{f_{n+2}} \to \frac{1}{\tau} \qquad (n \to \infty)$$

- irrational limit, hence Fibonacci chain not periodic!

Ammann–Beenker substitution

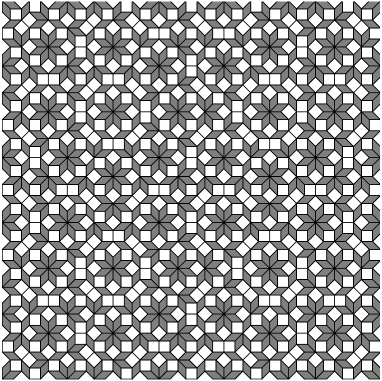
isosceles triangle (side lengths 1 and $\sqrt{2}$)
45°-rhombus (side lengths 1)





inflation has fix point!

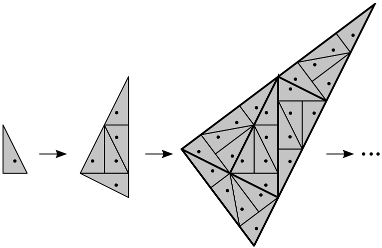
Ammann–Beenker tiling



non-periodic tiling of the plane, non-periodic vertex point set

pinwheel tiling

triangle of side lengths 1, 2, $\sqrt{5}$



- discovered by Conway and Radin 94
- triangle orientations dense in $\mathbb{S}^1$
- hence not FLC w.r.t. translations

some larger patch ...

Federation Square Melbourne, Australia (Paul Bourke)

linearly repetitive examples

- study lineary repetitive point sets: $R(r) = O(r)$ as $r \to \infty$
- there is an abundance of such point sets:

Theorem (Solomyak 97)

Let $\mathcal{T}$ be a substitution tiling with primitive substitution matrix. If $\mathcal{T}$ has finite local complexity, then $\mathcal{T}$ is linearly repetitive.

Here FLC and repetitivity are defined on "tile configurations".

pattern frequencies

box decompositions (Lagarias–Pleasants 03)

- box $B = \bigtimes_{i=1}^{d} [a_i, b_i)$, $\mathrm{vol}(B) > 0$
- box decomposition $(B_i)_i$ of B

$$B = \bigcup_i B_i, \qquad B_i \cap B_j = \varnothing \quad (i \neq j)$$

- $\mathcal{B}(U)$ set of squarish U-boxes, i.e., $b_i - a_i \in [U, 2U]$
- Every box in $\mathcal{B}(W)$ admits decomposition in boxes from $\mathcal{B}(U)$, if $W \geq U$
- $\mathcal{B} = \bigcup_{U>0} \mathcal{B}(U)$ set of squarish boxes

pattern counting function

$$w_\Lambda(B) = |\Lambda \cap B|$$

- boundedness: $\exists C \forall B \in \mathcal{B}$
$$w_\Lambda(B) \leq C \mathrm{vol}(B)$$
- additivity: for every box decomposition $(B_i)_i$ of B
$$w_\Lambda(B) = \sum_i w_\Lambda(B_i)$$
- covariance: $\forall x \in \mathbb{R}^d \forall B \in \mathcal{B}$
$$w_{x\Lambda}(xB) = w_\Lambda(B)$$
- invariance: $\forall B, B' \in \mathcal{B}$
$$\Lambda \cap B = \Lambda \cap B' \Longrightarrow w_\Lambda(B) = w_\Lambda(B')$$

pattern frequencies

- upper and lower frequencies on squarish U-boxes

$$f^+(U) = \sup\left\{\frac{w_\Lambda(B)}{\mathrm{vol}(B)} \mid B \in \mathcal{B}(U)\right\}$$
$$f^-(U) = \inf\left\{\frac{w_\Lambda(B)}{\mathrm{vol}(B)} \mid B \in \mathcal{B}(U)\right\}$$

  finite due to boundedness

- behaviour for $U \to \infty$?

Theorem (Lagarias–Pleasants 03)

i) *$f^+(U)$ decreases to a finite limit f as $U \to \infty$.*

ii) *If Λ is linearly repetitive, then*

$$\lim_{U \to \infty} f^-(U) = \lim_{U \to \infty} f^+(U) = f$$

iii) *If Λ is linearly repetitive, then for every box sequence $(B_n)_{n \in \mathbb{N}}$ in $\mathcal{B}$ of diverging inradius*

$$\lim_{n \to \infty} \frac{w_\Lambda(B_n)}{\mathrm{vol}(B_n)} = f,$$

and this convergence is uniform in the center of the boxes.

note: same result for arbitrary pattern frequencies

proof of i)

- Fix $W \geq U$, choose $B \in \mathcal{B}(W)$ and a box decomposition $(B_i)_i$ of B with boxes from $\mathcal{B}(U)$.
- Then by additivity

$$\frac{w_\Lambda(B)}{\mathrm{vol}(B)} = \sum_i \frac{w_\Lambda(B_i)}{\mathrm{vol}(B_i)} \cdot \frac{\mathrm{vol}(B_i)}{\mathrm{vol}(B)} \leq f^+(U)$$

- As $B \in \mathcal{B}(W)$ was arbitrary, we conclude

$$0 \leq f^+(W) \leq f^+(U),$$

and the claim follows by monotonicity.

proof of ii) a la Damanik–Lenz 01

- indirect proof: assume

$$\liminf_{U} f^{-}(U) < \lim_{U} f^{+}(U) = f$$

- Then there are many big boxes with small frequencies: There is $\varepsilon > 0$ and $B_{U_k} \in \mathcal{B}(U_k)$ such that $U_k \to \infty$ and

$$\frac{w_\Lambda(B_{U_k})}{\mathrm{vol}(B_{U_k})} \leq f - \varepsilon$$

Due to linear repetitivity, such boxes will reduce the limiting value of the upper frequency f!

proof of ii)

- Choose constant K of linear repetitivity and take arbitrary $B \in \mathcal{B}(3KU_k)$.
- By partitioning each side of B into 3 parts of equal length, B can be decomposed into 3^d equivalent smaller boxes, each belonging to $\mathcal{B}(KU_k)$. Denote by $B^{(i)} \in \mathcal{B}(KU_k)$ the box which does not touch the boundary of B.
- By linear repetitivity, there exists $x \in \mathbb{R}^d$ such that $B_0 = xB_{U_k} \subset B^{(i)}$ and $x(\Lambda \cap B_{U_k}) = \Lambda \cap B_0$.
- Using $B \in \mathcal{B}(3KU_k)$ and $B_0 \in \mathcal{B}(U_k)$, we may estimate

$$\frac{\mathrm{vol}(B_0)}{\mathrm{vol}(B)} \geq \frac{U_k^d}{(2 \cdot 3KU_k)^d} = \frac{1}{(6K)^d}$$

proof of ii)

Choose a box decomposition $(B_i)_{i=0}^n$ of B, with $B_i \in \mathcal{B}(U_k)$ for $i \in \{1, \ldots, n\}$ and estimate

$$\begin{aligned}
\frac{w_\Lambda(B)}{\mathrm{vol}(B)} &= \sum_{i=1}^n \frac{w_\Lambda(B_i)}{\mathrm{vol}(B)} + \frac{w_\Lambda(B_0)}{\mathrm{vol}(B)} = \sum_{i=1}^n \frac{w_\Lambda(B_i)}{\mathrm{vol}(B)} + \frac{w_\Lambda(B_{U_k})}{\mathrm{vol}(B)} \\
&\le \sum_{i=1}^n f^+(U_k)\frac{\mathrm{vol}(B_i)}{\mathrm{vol}(B)} + (f-\varepsilon)\,\frac{\mathrm{vol}(B_0)}{\mathrm{vol}(B)} \\
&\le f^+(U_k) + \left(f - f^+(U_k)\right)\frac{\mathrm{vol}(B_0)}{\mathrm{vol}(B)} - \frac{\varepsilon}{(6K)^d} \\
&\le f^+(U_k) - \frac{\varepsilon}{(6K)^d}
\end{aligned}$$

Since $B \in \mathcal{B}(3KU_k)$ was arbitrary, we have

$$f^+(3KU_k) \le f^+(U_k) - \varepsilon/(6K)^d,$$

a contradiction for $k \to \infty$.

proof of iii)

Let $(B_n)_{n\in\mathbb{N}}$ any box sequence in $\mathcal{B}$ of diverging inradius. Since $B_n \in \mathcal{B}(U_n)$ for some U_n, we have the estimate

$$f^-(U_n) \le \frac{w_\Lambda(B_n)}{\mathrm{vol}(B_n)} \le f^+(U_n).$$

Since $U_n \to \infty$ as $n \to \infty$, this yields the claimed uniform convergence.

□

from boxes to balls

The above result also holds for sequences of balls $(D_n)_{n\in\mathbb{N}}$ of diverging radius.

- tile n–ball D_n by $\sqrt{n}$–boxes $B^i_{\sqrt{n}}$, some of which may protude the boundary of D_n
- with additivity we get

$$\frac{w_\Lambda(D_n)}{\mathrm{vol}(D_n)} = \sum_i \frac{w_\Lambda(B^i_{\sqrt{n}})}{\mathrm{vol}(B^i_{\sqrt{n}})} \cdot \frac{\mathrm{vol}(B^i_{\sqrt{n}})}{\mathrm{vol}(D_n)}$$

- result follows from uniform convergence on boxes
- boundary boxes are asymptotic irrelevant since

$$\frac{\mathrm{vol}(D_{\sqrt{n}}\partial D_n)}{\mathrm{vol}(D_n)} \to 0 \qquad (n \to \infty)$$

outlook: point set dynamical systems

dynamical system naturally associated to point set

- identify Λ with Dirac measure $\sum_{p\in\Lambda}\delta_p$
- vague topology on collection of uniformly discrete point sets
- compact hull $\mathbb{X}_\Lambda = \overline{\{x\Lambda \mid x \in \mathbb{R}^d\}}$
- topological dynamical system, continuous translation action

Proposition

Let Λ be FLC Delone. Then

- *Λ repetitive $\iff$ $\mathbb{X}_\Lambda$ minimal.*
- *Λ has uniform pattern frequencies $\iff$ $\mathbb{X}_\Lambda$ uniquely ergodic.*

Analogous result for Delone sets of infinite local complexity!

references

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