Optimal embeddings of trees in the line

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- $G = (N, E)$ simple **connected** graph, $N = \{1, \ldots, n\}$, $E \subseteq \binom{N}{2}$

Embedding problem

$$\text{minimise } \sum_{i=1}^{n} \|v_i\|^2 \quad \text{s.t.} \begin{cases} \|v_i - v_j\|^2 \geq 1 \ (ij \in E) \\ v_i \in \mathbb{R}^n \ (i \in N) \end{cases}$$

Where does it come from?

- *Weighted Laplacian of G*: For edge weights $w \in \mathbb{R}^E$, $w_{ij} \geq 0$, $L_w(G) \in \mathbb{R}^{N \times N}$ is defined as

$$L_w(G)_{ij} = L_w(G)_{ji} = \begin{cases} -w_{ij} & \text{if } ij \in E, \\ \sum_{ik \in E} w_{ik} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

$$\left(\Leftrightarrow L_w(G) = \sum_{ij \in E} w_{ij}(e_i - e_j)(e_i - e_j)^T \right)$$

Embedding problem

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is (almost) the Lagrangian dual of

Minimise largest eigenvalue of weighted Laplacian $L_w(G)$ (Fiedler '90)

$$\text{maximise } \frac{1}{\lambda_n} \quad \text{s.t.} \begin{cases} \lambda_n I - L_w(G) \succeq 0, \\ \sum_{ij \in E} w_{ij} = 1, \\ w_{ij} \geq 0 \ (ij \in E), \ \lambda_n \in \mathbb{R} \end{cases}$$

- Strong duality: Both programs attain their optimal values which coincide! (Göring, Helmberg, Reiß)
- This work provides a combinatorial algorithm to compute the largest eigenvalue of $L_w(G)$ for a bipartite graph.

Embedding problem

$$\text{minimise} \sum_{i=1}^{n} \|v_i\|^2 \quad \text{s.t.} \begin{cases} \|v_i - v_j\|^2 \geq 1 \ (ij \in E) \\ v_i \in \mathbb{R}^n \ (i \in N) \end{cases}$$

- Solvable by a compactness argument.

Existence of 1d optimal embeddings of bipartite graphs

Every optimal embedding $V^* = (v_1^*, \ldots, v_n^*)$ of a bipartite $G = (N = W \dot{\cup} B, E \subseteq W \times B)$ comes with an associated *optimal one-dimensional* embedding

$$v_i = \begin{cases} -\|v_i^*\|, \text{ if } i \in W, \\ \|v_i^*\|, \text{ if } i \in B. \end{cases}$$

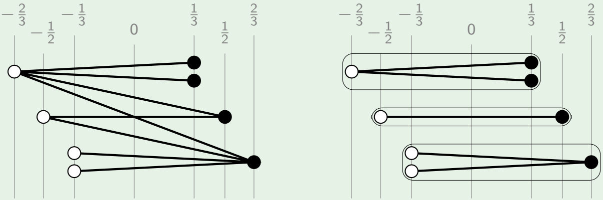
(opt. obj. and feasible: $1 \leq \|v_i^* - v_j^*\|^2 \leq |\|v_i^*\| + \|v_j^*\||^2 \leq |v_i - v_j|^2$)

- Every opt. embedding lives within unit ball.

Definition

Let $G = (N = W \dot\cup B, E \subseteq W \times B)$ connected bipartite graph and $V = [v_1, \ldots, v_n] \in \mathbb{R}^{1\times n}$ opt. sol. to $\left(\text{minimise } \sum_{i=1}^n \|v_i\|^2 \quad \text{s.t.} \begin{cases} \|v_i - v_j\|^2 \geq 1 \ (ij \in E) \\ v_i \in \mathbb{R}^n \ (i \in W \cup B) \end{cases}\right)$.
Call $G_V = (N, \{ij \in E \colon v_j - v_i = 1\})$ the *active subgraph* of G w.r.t. V.

Optimally embedded graph and active subgraph



- No isolated vertices in G_V.
- Let $C \subseteq G_V$ connected component with vertex set $W(C) \dot\cup B(C)$. Then $W(C) \ni i \mapsto y$ and $B(C) \ni i \mapsto y + 1$.
- C contributes $|W(C)| \cdot y^2 + |B(C)| \cdot (1 + y)^2$ to objective.
- Minimal at $y_c = \frac{-|B(C)|}{|B(C)|+|W(C)|} \Rightarrow y = y_C$.

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Definition

For a bipartite graph $C = (W(C)\dot{\cup}B(C), E(C))$ the embedding

$$W(C) \ni i \mapsto \frac{-|B(C)|}{|B(C)|+|W(C)|} =: y_C$$
$$B(C) \ni i \mapsto \frac{|W(C)|}{|B(C)|+|W(C)|} = 1 + y_C$$

is called the *two point embedding* (2PE) of C.

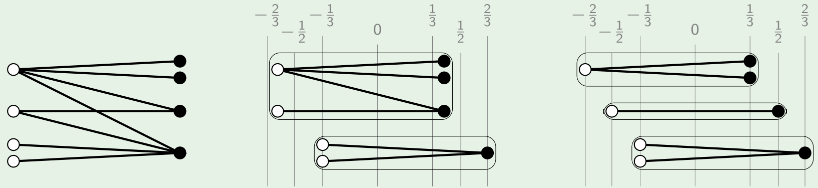
Lemma

Let $G = (W\dot{\cup}B, E \subseteq W \times B)$ be a connected bipartite graph and V be an optimal 1d embedding. Then V decomposes into optimal 2PEs, i.e. there are partitions $(W_i)_{i\in\mathcal{J}}$ and $(B_i)_{i\in\mathcal{J}}$ of W resp. B such that $C_i = G[W_i \cup B_i] \to \{y_{C_i}, 1 + y_{C_i}\}$ and

1. $\forall\, wb \in E\colon\ w \in W_i,\ b \in B_j \Rightarrow y_{C_i} \le y_{C_j}$ *(feasibility),*
2. $\forall\, i$ *the 2PE of C_i is an optimal embedding of C_i.*

Conversely, any such decomposition yields an optimal embedding.

Suboptimal and optimal decomposition into 2PEs



Optimality criterion for two point embeddings

The 2PE of a bipartite graph $G = (W \dot\cup B, E)$ is optimal
$\Leftrightarrow \forall Q \subseteq W \colon \frac{|Q|}{|N_G(Q)|} \leq \frac{|W|}{|B|}$ $\left(\Leftrightarrow \forall P \subseteq B \colon \frac{|N_G(P)|}{|P|} \geq \frac{|W|}{|B|}\right)$ where
$N_G(Q) = \{j \in W \cup B \colon \exists\, i \in Q \colon ij \in E\}$.

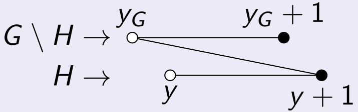
Proof.

$\Rightarrow$: Observe:

$y_C < y_{C'} \Leftrightarrow \frac{-|B(C)|}{|B(C)|+|W(C)|} < \frac{-|B(C')|}{|B(C')|+|W(C')|} \Leftrightarrow \frac{|W(C)|}{|B(C)|} < \frac{|W(C')|}{|B(C')|}$

Consider $Q \subseteq W$ with $\frac{|Q|}{|N_G(Q)|} > \frac{|W|}{|B|}$, $H = G\left[N_G(Q) \cup Q\right]$.

$\forall y > y_G$ this embedding is also feasible:



H's contribution $|Q| \cdot y^2 + |N_G(Q)| \cdot (1+y)^2$ is min. at $y = y_H > y_G$
$\Rightarrow$ the 2PE of G not optimal.

□

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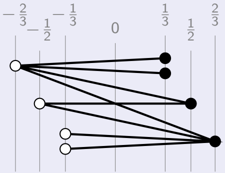
$N_G(Q) = \{j \in W \cup B \colon \exists\, i \in Q \colon ij \in E\}$.

Proof.

$\Leftarrow$: If not opt. consider an opt. embedding of G.

The subgraph embedded in the rightmost pair $(y, 1+y)$ of coordinates is of the form $Q \cup N_G(Q)$, $Q \subseteq W$ and $\frac{|Q|}{|N_G(Q)|} > \frac{|W|}{|B|} > \frac{|W \backslash Q|}{|B \backslash N_G(Q)|}$.



□

Trees

Optimality criterion for two point embeddings

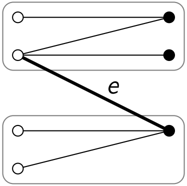
The 2PE of a bipartite graph $G = (W \dot{\cup} B, E)$ is optimal
$\Leftrightarrow \forall Q \subseteq W\colon \frac{|Q|}{|N_G(Q)|} \leq \frac{|W|}{|B|} \left(\Leftrightarrow \forall P \subseteq B\colon \frac{|N_G(P)|}{|P|} \geq \frac{|W|}{|B|} \right)$ where
$N_G(Q) = \{j \in W \cup B\colon \exists\, i \in Q\colon ij \in E\}$.

That criterion is in general hard to check but simple for trees!

Let $T = (W \dot{\cup} B, E \subseteq W \times B)$ be a tree. The removal of any edge $e = wb \in E$ $(w \in W, b \in B)$ decomposes T into two subtrees, namely

the *white subtree* $T^w(e) \ni w$

the *black subtree* $T^b(e) \ni b$



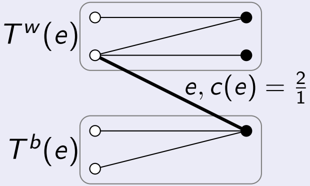
- $T^b(e)$ is of the form $Q \cup N_T(Q)$ with $Q \subseteq W$.

Definition

Let $T = (W \dot{\cup} B, E \subseteq W \times B)$ be a tree.

For every edge e define its label $c(e)$ as

$$c(e) = \frac{|W(T^b(e))|}{|B(T^b(e))|}.$$



T is called *balanced* if $\forall\, e \in E\colon c(e) \leq \frac{|W|}{|B|}$.

Optimality criterion for two point embeddings of trees

The two point embedding of a tree $T = (W \dot{\cup} B, E)$

$$W \ni i \mapsto \frac{-|B|}{|B| + |W|} =: y_T$$
$$B \ni i \mapsto \frac{|W|}{|B| + |W|} = 1 + y_T$$

is optimal if and only if T is *balanced*.

Decompose T into balanced (=opt. 2PE) trees!

Algorithm

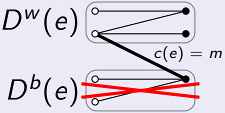
INPUT: $T = (W \dot\cup B, E)$ tree. OUTPUT: feasible decomposition into balanced trees. INITIALISATION: $\mathcal{OUT} = \emptyset$, $\mathcal{QUEUE} = \{T\}$.

For $D \in \mathcal{QUEUE}$ **do:**

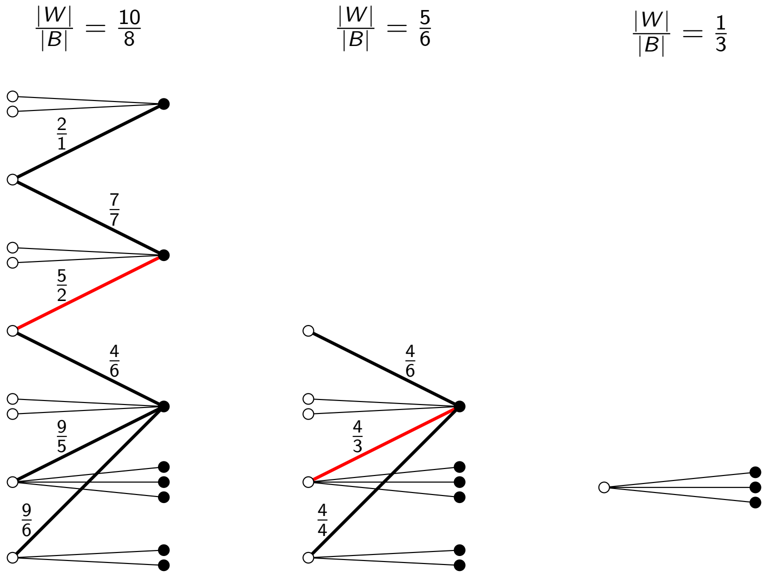
1. $\mathcal{QUEUE} \leftarrow \mathcal{QUEUE} \setminus \{D\}$, $D' \leftarrow D$
2. Compute $c(e)$ for every $e \in E(D)$.
3. **while** $m = \max\{c(e)\colon e \in E(D)\} > \frac{|W(D)|}{|B(D)|}$ **do**
    1. Choose an edge e with $c(e) = m$.
    2. $D \leftarrow D^w(e)$.
    3. Update edge labels in D.

   end while
4. $\mathcal{OUT} \leftarrow \mathcal{OUT} \cup \{D\}$.
5. $\mathcal{QUEUE} \leftarrow \mathcal{QUEUE} \cup (D' \setminus D)$.

end for



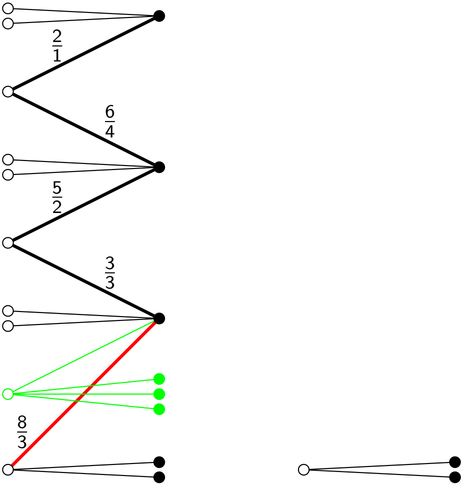
Example: 1st round of for loop



Example: 2nd iteration of for loop

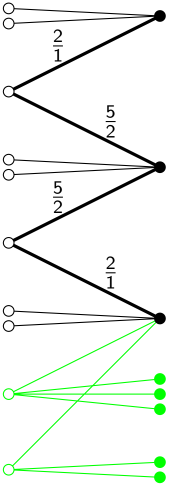
$$\frac{|W|}{|B|} = \frac{9}{5}$$

$$\frac{|W|}{|B|} = \frac{1}{2}$$

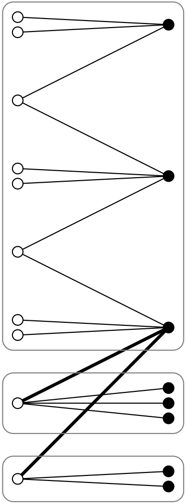


3rd iteration of for loop

$$\frac{|W|}{|B|} = \frac{8}{3}$$



Opt. decomposition



What's the time?

Worst case running time

For a tree $T = (W \dot{\cup} B, E)$ the algorithm makes at most $O\left(\min\left(|W|, |B|\right)^2 \cdot (|W| + |B|)\right)$ updates of edge labels.

And general bipartite graphs?

A subroutine

For a bipartite graph $G = (W \dot\cup B, E)$ without isolated vertices the set

$$M = \text{Argmin}\left(\frac{|N_G(X)|}{|X|} : X \subseteq B, X \neq \emptyset \right)$$

contains a unique element $S(G) \subseteq B$ of maximal cardinality which can be computed in at most $\min(|W|, |B|)^2 \cdot \max(|W|, |B|)$ calls of the tree algorithm.

- $G[S(G) \cup N_G(S(G))]$ is balanced (i.e. its 2PE optimal).
- Construct decomposition into balanced subgraphs: Set $G_0 = G$ and inductively $G_i = G_{i-1} \setminus (S(G_{i-1}) \cup N_{G_{i-1}}(S(G_{i-1})))$