# A new bijection which preserves Euler-Mahonian statistics

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1. Introduction
   - Eulerian polynomials
   - Combinatorial interpretations
   - $q$-Eulerian polynomials
2. (Simplified) construction of $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$
   - Skeleton (graph) of $\varphi(\sigma)$
   - Labelling of the graph
     - Labelling of the circles (exceedance values of $\varphi(\tau)$)
     - Labelling of the dots (non exceedance values of $\varphi(\sigma)$)
3. Thereafter
   - Extension
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# Eulerian polynomials

The sequence of Eulerian polynomials $(A_n(t))_{n\geq 1}$ can be defined by

$$\sum_{n\geq 1} A_n(t)\frac{x^n}{n!} = \frac{t-1}{t-e^{(t-1)x}}.$$

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The first values of $A_n(t)$ :

$$A_1(t) = 1,$$
$$A_2(t) = 1 + t,$$
$$A_3(t) = 1 + 4t + t^2,$$
$$A_4(t) = 1 + 11t + 11t^2 + t^3.$$

# Eulerian statistics

## Proposition (MacMahon)

*We have*

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{des(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{exc(\sigma)}$$

*where* $\mathfrak{S}_n$ *is the set of permutations on* $[n] := \{1, 2, \ldots, n\}$ *and*

$$des(\sigma) = \#\{i \in [n-1], \sigma(i) > \sigma(i+1)\},$$
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# Eulerian statistics

Example : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \in \mathfrak{S}_4$ has $\mathrm{des}(\sigma) = 2$ descents 2 and 3 and $\mathrm{exc}(\sigma) = 2$ exceedances 1 and 2.

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A statistic equidistributed with des or exc is said to be *Eulerian*.
Example : ides defined by $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$.

# Mahonian statistics

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The pairs $(i, j)$ such that $i < j$ and $\sigma(i) > \sigma(j)$ are named *inversions.*

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# 2-versions of the previous statistics

Let $\sigma \in \mathfrak{S}_n$.

- A 2-*descent* of $\sigma$ is an integer $i \in [n-1]$ such that $\sigma(i) \geq \sigma(i+1) + 2$.

$$\mathrm{des}_2(\sigma) := \text{number of 2-descents of } \sigma.$$

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- A 2-*inversion* of $\sigma$ is a pair $(i,j) \in [n]^2$ such that $i < j$ and $0 \leq \sigma(i) - \sigma(j) < 2$ (*i.e.* $\sigma(i) = \sigma(j) + 1$).

$$\mathrm{inv}_2(\sigma) := \text{number of 2-inversions of } \sigma.$$

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Consider the pairs of statistics $(\mathrm{maj}_2, \mathrm{inv}_2)$ and $(\mathrm{maj} - \mathrm{exc}, \mathrm{exc})$ where, for all $\sigma \in \mathfrak{S}_n$.

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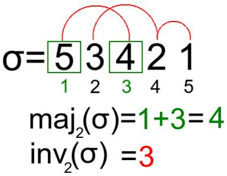

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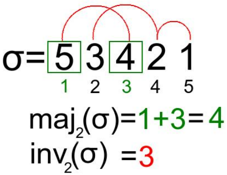


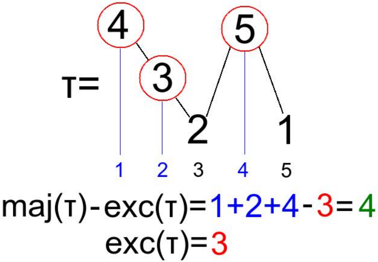

# $q$-Eulerian polynomials

Let $A_n(q,t)$ and $A_n^{(2)}(q,t)$ be the $q$-Eulerian polynomials

$$A_n(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)},$$

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*For all $n \geq 1$, we have*

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The proof relies on quasisymmetric function techniques.

# Main result

**Theorem 2 (B.,2015)**

*There exists a bijection* $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ *such that*

$$(maj_2(\sigma), inv_2(\sigma)) = (maj(\tau) - exc(\tau), exc(\tau))$$

*for all* $\sigma \in \mathfrak{S}_n$ *and* $\tau = \varphi(\sigma)$.

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This proves combinatorially Theorem 1.

# First step

Let $\sigma \in \mathfrak{S}_n$. We compute a sequence $(c_{\mathrm{des}_2(\sigma)}, \ldots, c_1, c_0)$ such that $\sum c_i = \mathrm{inv}_2(\sigma)$.

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**Principle** : for $k$ from $\mathrm{des}_2(\sigma)$ to 0,

1. we consider the maximal sequence of 2-inversions $(i_1, j_1), \ldots, (i_p, j_p)$ such that $d_2^k \leq i_1 < \ldots < i_p$ and $\sigma(i_1) < \ldots < \sigma(i_p)$;

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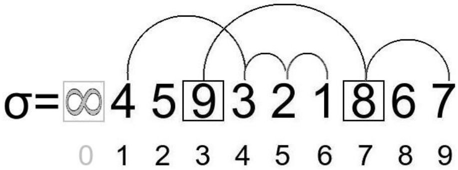
2. we define $c_k \geq p$ as $\sum_{q=1}^{p} n_q$ where $n_q$ is the number of consecutive 2-inversions $(i_q, j_q = i_q^1), (i_q^1, j_q^1 = i_q^2), \dots$, and we erase every of those 2-inversions.

# First step

Example : $\sigma = 459321867 \in \mathfrak{S}_9$.



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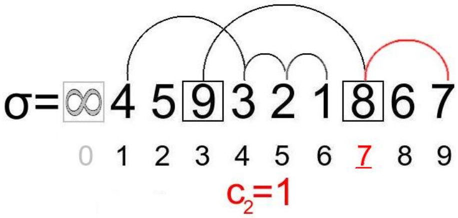

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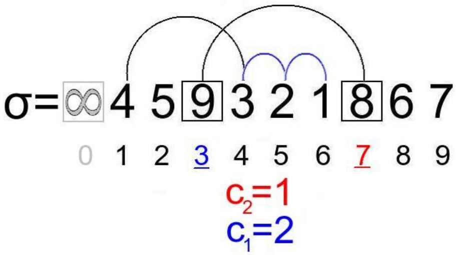

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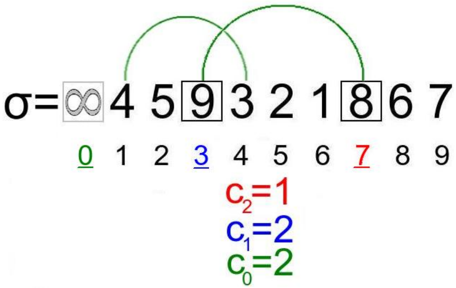

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**(Main) principle** :

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2. we draw ascending slopes of dots at the remaining abscissas.

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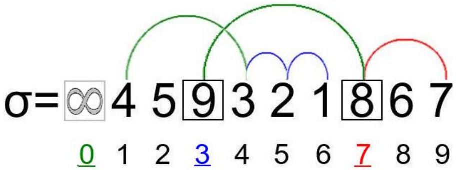
$(c_0, c_1, c_2) = (2, 2, 1)$



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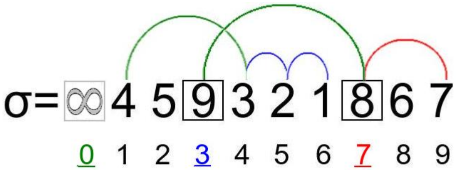


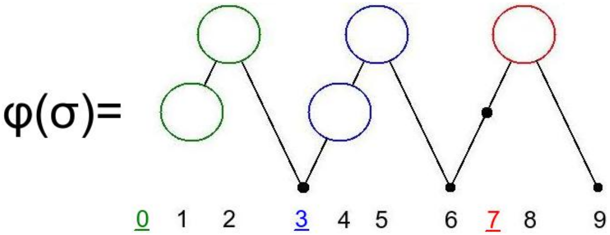

# Labelling of the circles (exceedances of $\varphi(\tau)$)

In order to label the circles (which will be the exceedance values of $\varphi(\tau)$) of the skeleton of $\varphi(\sigma)$, we compute a word $\omega(\sigma)$ whose letters will be the labels of the circles.

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**Definition 1**

Let $(i_1, j_1), \ldots, (i_{\mathrm{inv}_2(\sigma)}, j_{\mathrm{inv}_2(\sigma)})$ (with $i_1 < \ldots < i_{\mathrm{inv}_2(\sigma)}$) be the 2-inversions of $\sigma$.
We define $\omega(\sigma)$ as the word $j_1 j_2 \ldots j_{\mathrm{inv}_2(\sigma)}$.

# Computation of $\omega(\sigma)$

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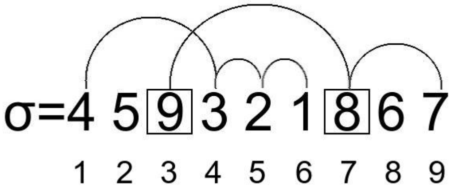

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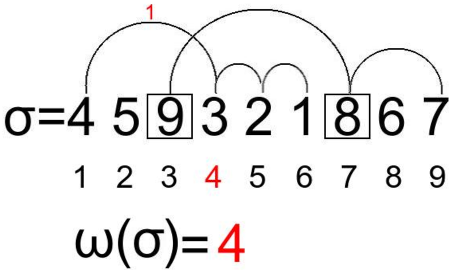

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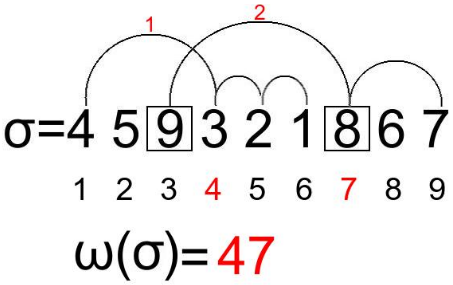

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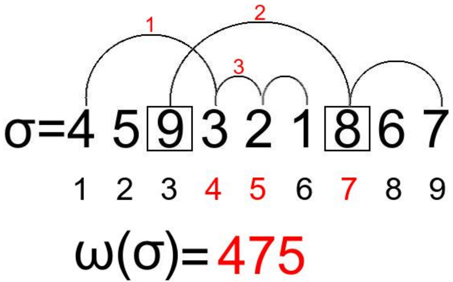

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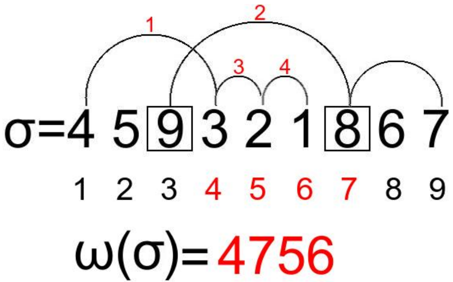

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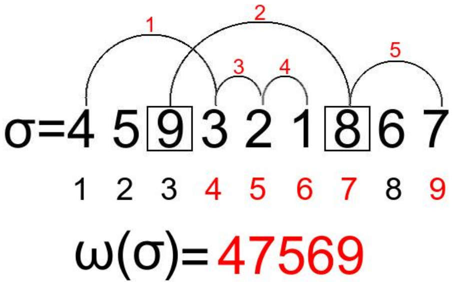

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3. if $l_1$ and $l_2$ are two labels that can be "exchanged" (*i.e.*, the circles labelled by $l_1$ and $l_2$ respectively can be labelled by $l_2$ and $l_1$ while respecting the above two rules), then the order of appearance of $l_1$ and $l_2$ in the graph is the same as in $\omega(\sigma)$.

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Example : $\sigma = 459321867 \in \mathfrak{S}_9$.

$$\omega(\sigma) = 47569$$

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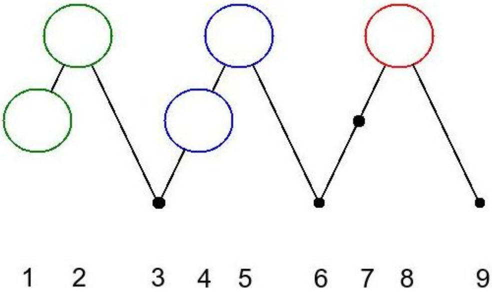

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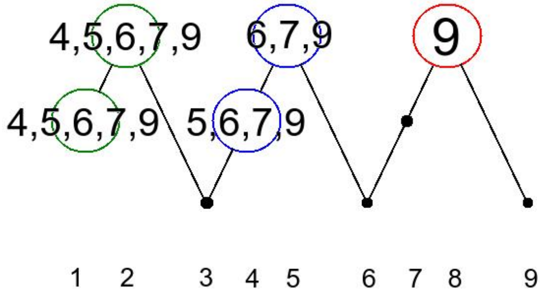
$\varphi(\sigma) =$ 

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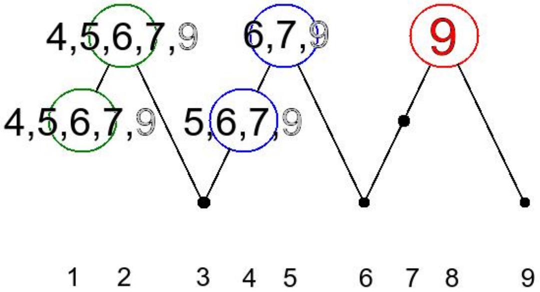

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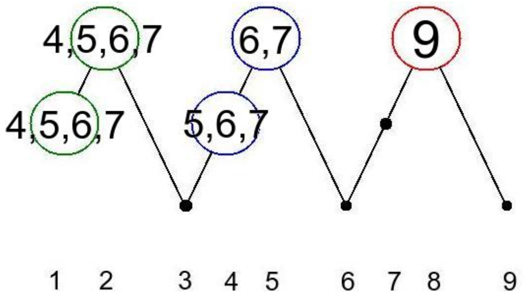

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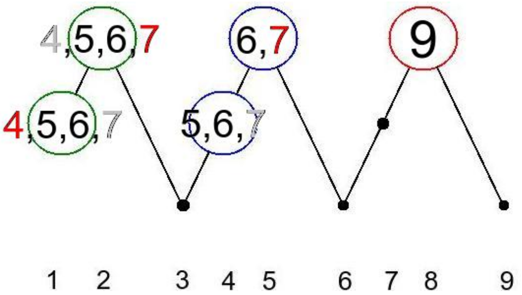

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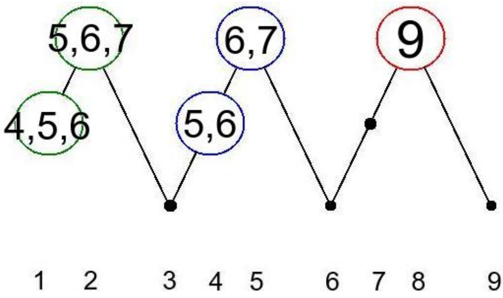

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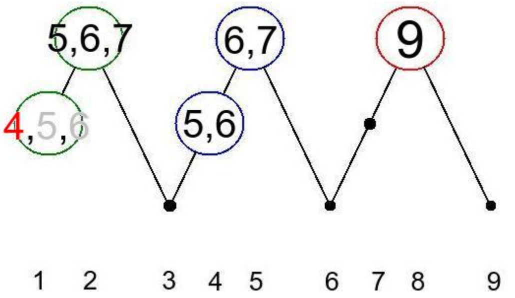

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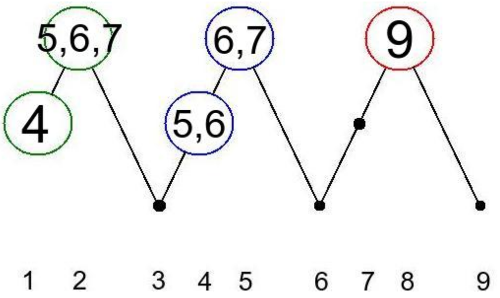
$$\omega(\sigma) = 47569$$

$$\varphi(\sigma) =$$



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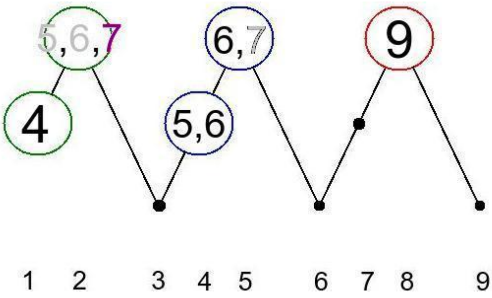

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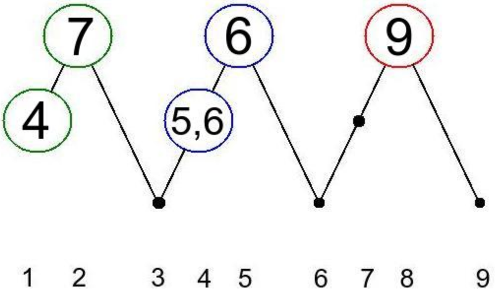

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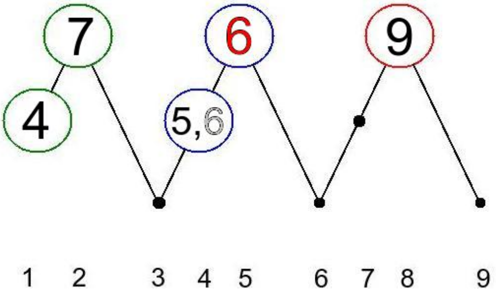

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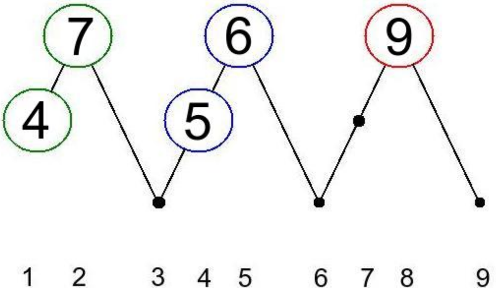

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In order to label the dots (which will be the non exceedance values of $\varphi(\tau)$) of the skeleton of $\varphi(\sigma)$, we compute a subpermutation $\pi(\sigma)$.

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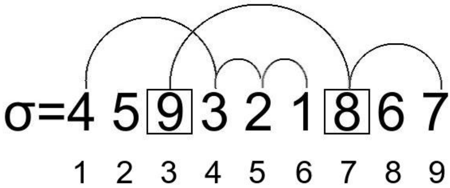
**Definition 2**

Let $(i_1, j_1), \ldots, (i_{\mathrm{inv}_2(\sigma)}, j_{\mathrm{inv}_2(\sigma)})$ (with $i_1 < \ldots < i_{\mathrm{inv}_2(\sigma)}$) be the 2-inversions of $\sigma$.
We define $\pi(\sigma)$ as the subpermutation of $\sigma$ whose input set $I_\sigma$ is $[n] \backslash \{i_1, i_2, \ldots, i_{\mathrm{inv}_2(\sigma)}\}$.

# Computation of $\pi(\sigma)$

Example : $\sigma = 459321867 \in \mathfrak{S}_9$.



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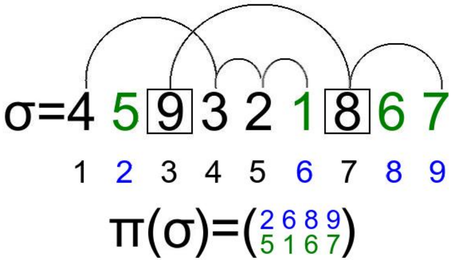

# Labelling of the dots

The idea is to label the dots with the elements of $[n]\backslash\{j_1, j_2, \ldots, j_{\mathrm{inv}_2(\sigma)}\}$ so that :

1. if $l$ is the label of the $k$-th dot (whose abscissa is $i$), then $l \leq \min(i, i^k)$ (where $i^k$ is the $k$-th element of $I_\sigma$);

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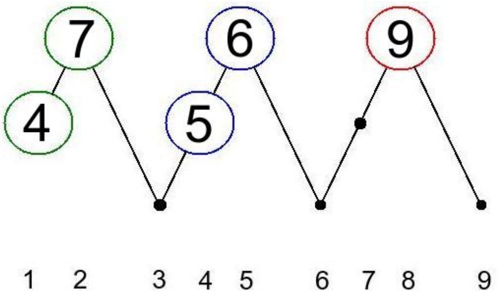
3. if $l_1$ and $l_2$ are two labels that can be "exchanged" (*i.e.*, the dots labelled by $l_1$ and $l_2$ respectively can be labelled by $l_2$ and $l_1$ while respecting the above two rules), then if the dots labelled by $l_1$ and $l_2$ are respectively the $k_1$-th dot and the $k_2$-th dot of the graph, we have $l_1 < l_2 \Leftrightarrow \sigma(i^{k_1}) < \sigma(i^{k_2})$.

# Labelling of the dots

Example : $\sigma = 459321867 \in \mathfrak{S}_9$.

$$\pi(\sigma) = \begin{pmatrix} 2 & 6 & 8 & 9 \\ 5 & 1 & 6 & 7 \end{pmatrix}$$

$\varphi(\sigma) =$



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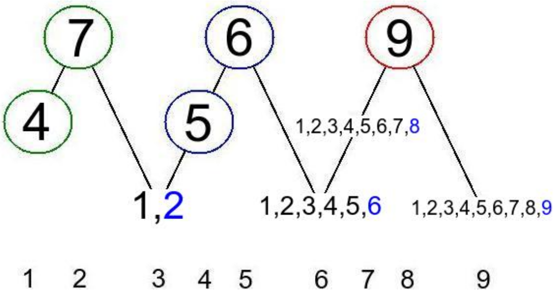

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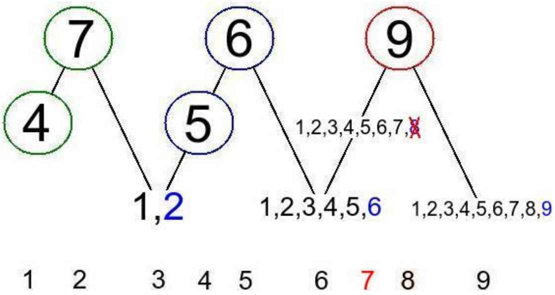

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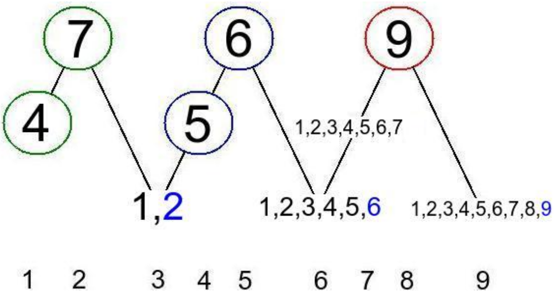

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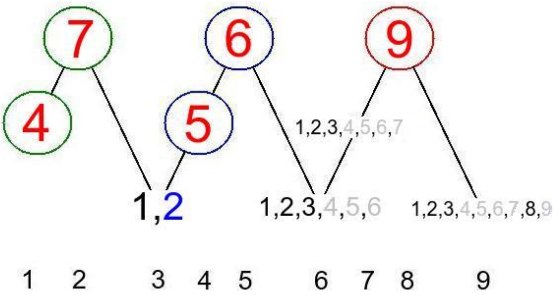

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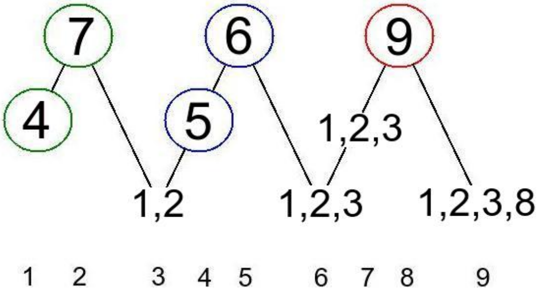

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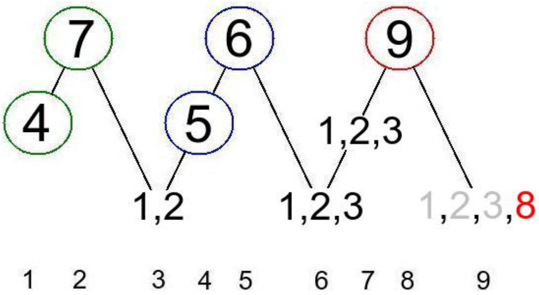

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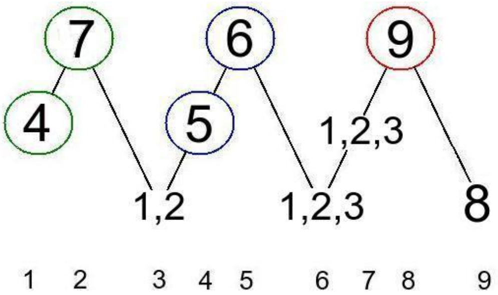

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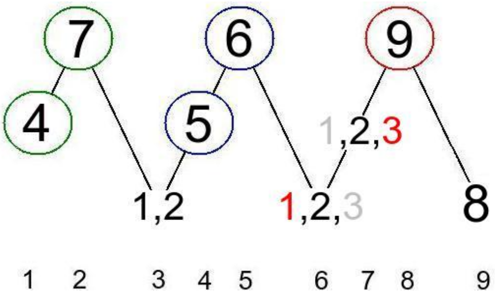

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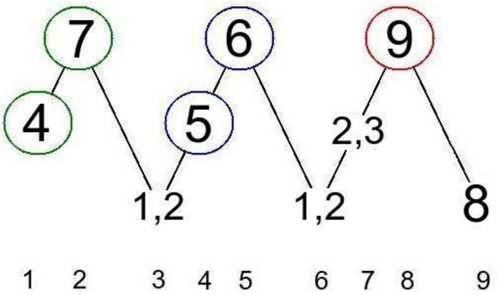

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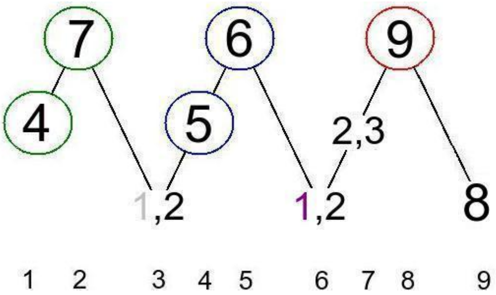

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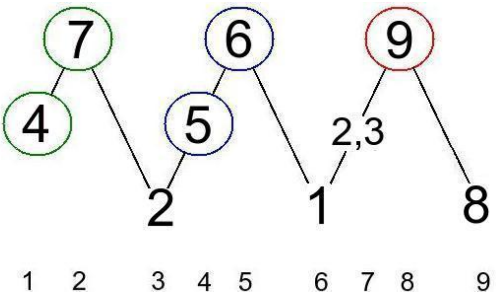

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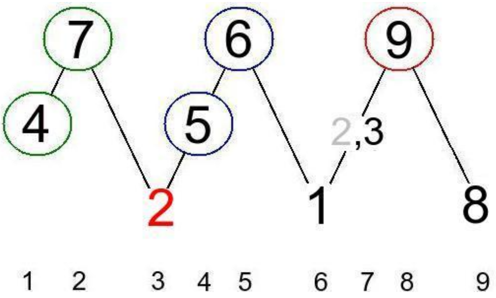

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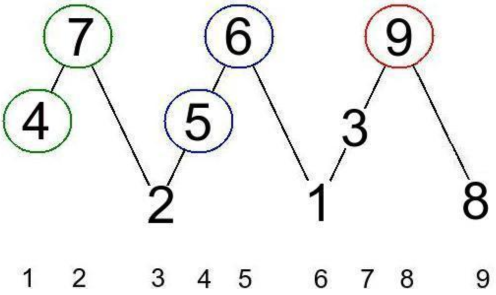

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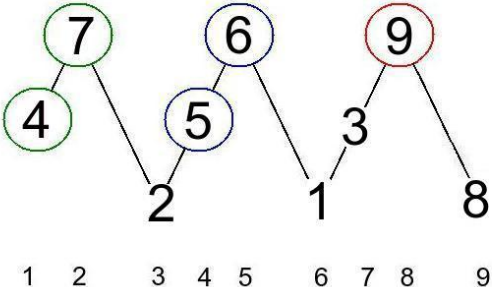
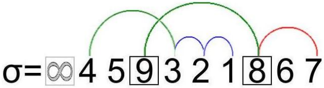
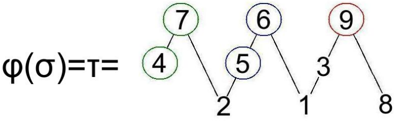



$= 472561398 \in \mathfrak{S}_9.$




0 1 2 3 4 5 6 7 8 9

$(\mathrm{maj}_2(\sigma),\mathrm{inv}_2(\sigma))=(0+3+7,2+2+1)=(10,5)$

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$(\mathrm{maj}(\tau)-\mathrm{exc}(\tau),\mathrm{exc}(\tau))=(\mathbf{2}-2+\mathbf{5}-2+\mathbf{8}-1,2+2+1)=(10,5)$

# Extension

By using the same quasisymmetric function methode as Shareshian and Wachs, Hance and Li proved that

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{amaj}_2(\sigma)} y^{\mathrm{ides}(\sigma)} z^{\widetilde{\mathrm{asc}_2}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{des}(\sigma)}$$

where

$$\mathrm{asc}_2(\sigma) = \#\{i \in [n-1], \sigma(i) < \sigma(i+1)+1\} \text{ (number of 2-}\textit{ascents}\text{)},$$

$$\widetilde{\mathrm{asc}_2}(\sigma) = \begin{cases} \mathrm{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\ \mathrm{asc}_2(\sigma)+1 & \text{if } \sigma(1) \neq 1, \end{cases}$$

$$\mathrm{amaj}(\sigma) = \sum_{\sigma(i)<\sigma(i+1)+1} i.$$

# Extension

The bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ provides the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} z^{\widetilde{\mathrm{des}_2}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{des}(\sigma)}$$

where

$$\widetilde{\mathrm{des}_2}(\sigma) = \begin{cases} \mathrm{des}_2(\sigma) & \text{under certain conditions,} \\ \mathrm{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

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By composing with a simple bijection which maps $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ to $(\mathrm{amaj}_2, \widetilde{\mathrm{asc}_2}, \mathrm{ides})$, we obtain the equality of Hance and Li.

# Open problem

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} z^{?} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{inv}(\sigma)}$$

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Thank you for your attention.