Computer Algebra for Lattice Path Combinatorics

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Inria informatics mathematics

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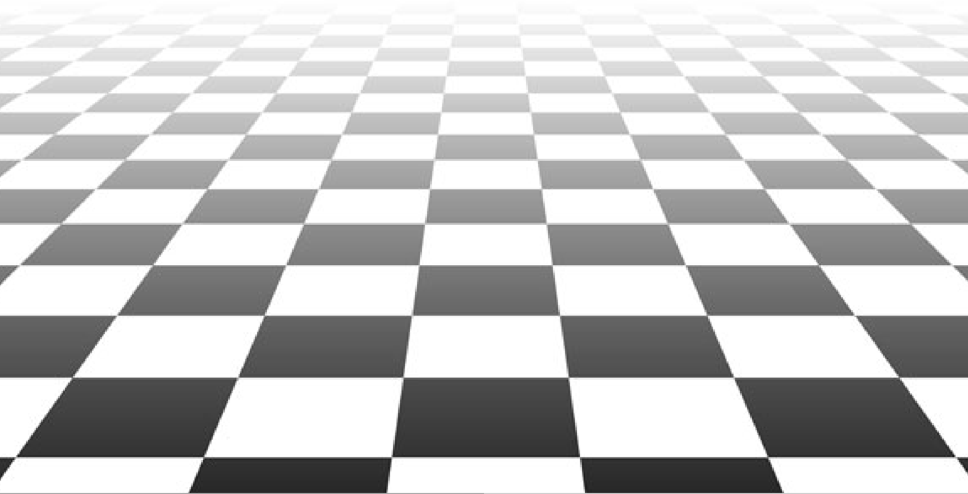
Overview

1. Monday: General presentation
2. Tuesday: Guess'n'Prove
3. Wednesday: Creative telescoping



Summary of Part I: Walks with unit steps in $\mathbb{N}^2$

- quadrant models: 79
  - $|G| < \infty$: 23
    - nonzero orbit sum: 19
      - Kernel method + CT
        - **D-finite**
    - zero orbit sum: 4
      - Guess'n'Prove
        - **algebraic**
  - $|G| = \infty$: 56
    - asymptotics + GB
      - **not D-finite**

Summary of Part I: Classification of 2D non-singular walks

The Main Theorem Let $\mathfrak{S}$ be a 2D non-singular model with small steps. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating series $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^{\alpha}$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Proof

(1) $\Rightarrow$ (2) Easy

(2) $\Rightarrow$ (3) [Denisov & Wachtel 2013] + [Chudnovsky'85, André'89, Katz'70]

(3) $\Rightarrow$ (4) [B., Raschel & Salvy 2013]

(4) $\Rightarrow$ (1) [Bousquet-Mélou & Mishna 2010] + [B. & Kauers 2010]

(5) $\Leftrightarrow$ (4) A posteriori observation

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Moreover, under (1)–(5), $F_{\mathfrak{S}}(t;x,y)$ is algebraic if and only if the model $\mathfrak{S}$ has positive covariance $\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$, and iff it has $\mathrm{OS} = 0$.

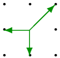
In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

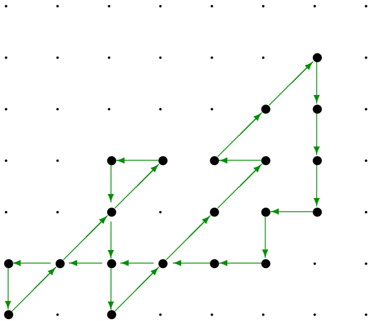
▶ Proof of the last statements: [B., Chyzak, van Hoeij, Kauers & Pech 2015]

Two important models: Kreweras and Gessel walks

$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$



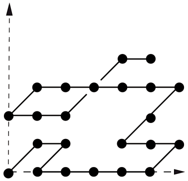


Example: A Kreweras excursion.

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(n;i,j)$ = number of walks from $(0,0)$ to (i,j) with n steps in $\mathfrak{S}$

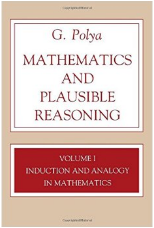
Question: Find the nature of the generating function

$$G(t;x,y) = \sum_{i,j,n=0}^{\infty} g(n;i,j)\, x^i y^j t^n \in \mathbb{Q}[[x,y,t]]$$



Theorem (B.-Kauers 2010) $G(t;x,y)$ is an algebraic function.†

$\rightarrow$ Effective, computer-driven discovery and proof



Guessing and Proving

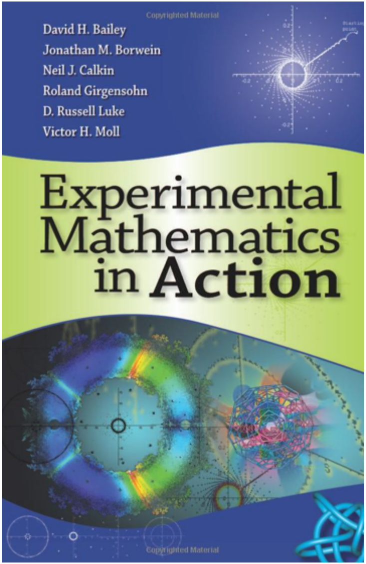
George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.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Experimental Mathematics in Action
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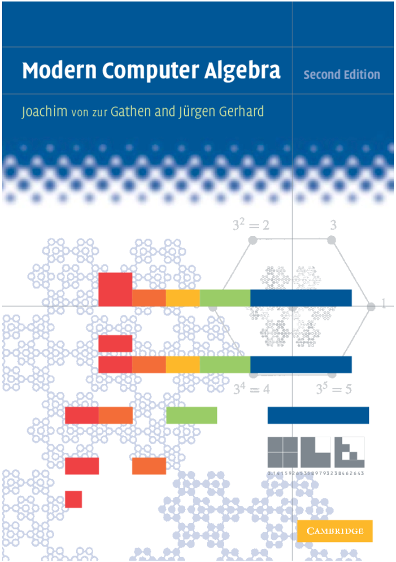
Modern Computer Algebra
Second Edition
Joachim von zur Gathen and Jürgen Gerhard
$3^2 = 2$
3
1
$3^4 = 4$
$3^5 = 5$
CAMBRIDGE

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data

(S2) Conjecture

(S3) Prove

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rigorously certify the minimal polynomials, using (exact) polynomial computations.

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+ Efficient Computer Algebra

Step (S1): high order series expansions

$f_{\mathfrak{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1;i,j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ initial conditions $f_{\mathfrak{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n;-1,j) = f_{\mathfrak{S}}(n;i,-1) = 0$.

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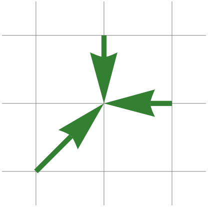
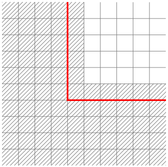
Example: for the Kreweras walks,

$$\begin{aligned} k(n+1;i,j) &= k(n;i+1,j) \\ &\quad + k(n;i,j+1) \\ &\quad + k(n;i-1,j-1) \end{aligned}$$





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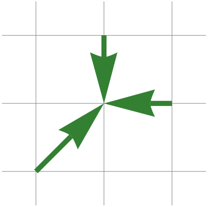

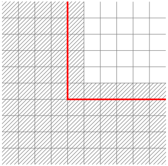

▷ Recurrence is used to compute $F_{\mathfrak{S}}(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &\quad + (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &\quad + (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,y)$, a first idea

In terms of generating series, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&\big(xy-(x+y+x^2y^2)t\big)K(t;x,y)\\ &\qquad = xy - xt\,K(t;x,0) - yt\,K(t;0,y)\end{aligned}} \tag{KerEq}$$

- A similar kernel equation holds for $F_{\mathfrak{S}}(t;x,y)$, for any $\mathfrak{S}$-walk.

Corollary. $F_{\mathfrak{S}}(t;x,y)$ is algebraic (resp. D-finite) if and only if $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$ are both algebraic (resp. D-finite).

- Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx$30Gb)

Task 1: Given the first N terms of $S = F_{\mathfrak{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \mod t^N.$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,0)$ & $F_{\mathfrak{S}}(t;0,y)$

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- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

Step (S2): guessing equations for $K(t;x,0)$

Using $N = 80$ terms of $K(t;x,0)$, one can guess

- a linear differential equation of order 4, degrees $(14,11)$ in (t,x), such that

$$\begin{aligned} &t^3 \cdot (3t-1) \cdot (9t^2+3t+1) \cdot (3t^2+24t^2x^3-3xt-2x^2) \cdot \\ &\cdot (16t^2x^5+4x^4-72t^4x^3-18x^3t+5t^2x^2+18xt^3-9t^4) \cdot \\ &\qquad \cdot (4t^2x^3-t^2+2xt-x^2) \cdot \frac{\partial^4 K(t;x,0)}{\partial t} + \cdots \\ &\qquad\qquad = 0 \bmod t^{100} \end{aligned}$$

- a polynomial of tridegree $(6,10,6)$ in (T,t,x)

$$\begin{aligned} \mathcal{P}_{x,0} &= x^6t^{10}T^6 - 3x^4t^8(x-2t)T^5 + \\ &+ x^2t^6\left(12t^2+3t^2x^3-12xt+\frac{7}{2}x^2\right)T^4 + \cdots \end{aligned}$$

such that $\mathcal{P}_{x,0}(K(t;x,0),t,x) = 0 \bmod t^{100}$.

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x,t,T]$ of degree $(32,43,24)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y,t,T]$ of degree $(40,44,24)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(x,t,G(t;x,0)) = \mathcal{P}_{0,y}(y,t,G(t;0,y)) = 0 \mod t^{1200}.$$

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▶ We actually first guessed differential equations†, then computed their p-curvatures to empirically certify them. This led us suspect the algebraicity of $G(t; x, 0)$ and $G(t; 0, y)$, using a conjecture of Grothendieck (on differential equations modulo p) as an oracle.

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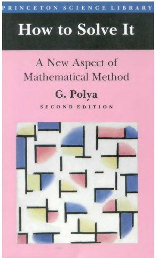
▶ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!



Guessing and Proving

George Pólya

Guessing is good, proving is better.

Step (S3): warm-up – Gessel excursions are algebraic

Theorem. $g(t) := G(\sqrt{t};0,0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ is algebraic.

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n):

$$(n+2)(3n+5) r_{n+1} - 4(6n+5)(2n+1) r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy - (x+y+x^2y^2)t)}_{\overset{!}{=}\,0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;0,y)$$

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shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x\cdot y_0 - x\cdot t\cdot U(t,x) - y_0\cdot t\cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

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2. $U = K(t;x,0)$ is the unique solution in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

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$$\underbrace{(xy - (x+y+x^2y^2)t)}_{\stackrel{!}{=}\,0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the unique solution in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. Resultant computations + verification of initial terms
$\Longrightarrow \quad U = H(t,x)$ also satisfies (RKerEq).

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\stackrel{!}{=}\, 0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the unique solution in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. Resultant computations + verification of initial terms
$\implies \quad U = H(t,x)$ also satisfies (RKerEq).

5. Uniqueness: $H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

Algebraicity of Kreweras walks: our Maple proof in a nutshell

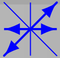
```
[bostan@inria ~]$ maple
    |\^/|     Maple 19 (APPLE UNIVERSAL OSX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2014
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                   3.62
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                     1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                 7, 617
```

Step (S3): rigorous proof for Gessel walks 

Same strategy, but several complications:

- stepset diagonal symmetry is lost: $G(t;x,y) \neq G(t;y,x)$;
- $G(t;0,0)$ occurs in (KerEq) (because of the step $\swarrow$);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace equation (RKerEq) by a system of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B., Flajolet, Salvy & Schost 2006] for computations with algebraic series).

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www.elsevier.com/locate/jsc

Fast computation of special resultants

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INSIDE THE BOX

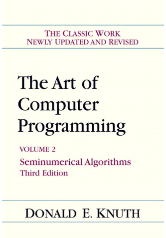
–Computer algebra–

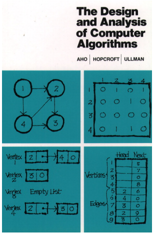
General framework

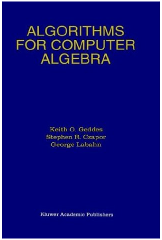
Computer algebra = effective mathematics and algebraic complexity

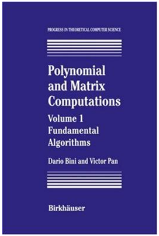
- Effective mathematics: what can we compute?
- algebraic complexity: how fast?

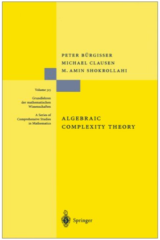
Computer algebra books

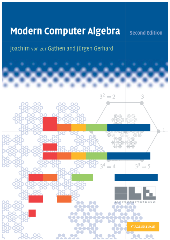


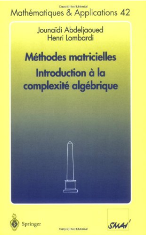












Complexity yardsticks

Important features:

- addition is easy: naive algorithm already optimal
- multiplication is the most basic (non-trivial) problem
- almost all problems can be reduced to multiplication

Are there quasi-optimal algorithms for:

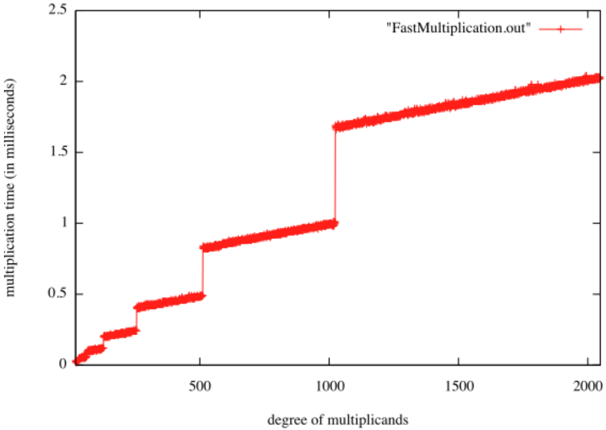
- integer/polynomial/power series multiplication? Yes!
- matrix multiplication? Big open problem!

$$
\begin{aligned}
\mathsf{M}(n) &= \text{complexity of multiplication in } \mathbb{K}[x]_{<n}, \text{ and of } n\text{-bit integers}\\
&= O(n^2) \ \text{ by the naive algorithm}\\
&= O\left(n^{1.58}\right) \ \text{ by Karatsuba's algorithm}\\
&= O\left(n^{\log_\alpha(2\alpha-1)}\right) \ \text{ by the Toom-Cook algorithm } (\alpha \geq 3)\\
&= O\big(n \log n \operatorname{loglog} n\big) \ \text{ by the Schönhage-Strassen algorithm}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{MM}(r) &= \text{complexity of matrix product in } \mathcal{M}_r(\mathbb{K})\\
&= O(r^3) \ \text{ by the naive algorithm}\\
&= O(r^{2.81}) \ \text{ by Strassen's algorithm}\\
&= O(r^{2.38}) \ \text{ by the Coppersmith-Winograd algorithm}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{MM}(r,n) &= \text{complexity of polynomial matrix product in } \mathcal{M}_r(\mathbb{K}[x]_{<n})\\
&= O(r^3\,\mathsf{M}(n)) \ \text{ by the naive algorithm}\\
&= O(\mathsf{MM}(r)\, n \log(n) + r^2 n \log n \operatorname{loglog} n) \ \text{ by the Cantor-Kaltofen algo}\\
&= O(\mathsf{MM}(r)\, n + r^2\,\mathsf{M}(n)) \ \text{ by the B-Schost algorithm}
\end{aligned}
$$

Fast polynomial multiplication in practice



Practical complexity of Magma's multiplication in $\mathbb{F}_p[x]$, for $p = 29 \times 2^{57} + 1$.

What can be computed in 1 minute with a CA system

on a PC, (Intel Xeon X5160, 3GHz processor, with 8GB RAM), running Magma V2.16-7

- polynomial product† in degree 14,000,000 (>1 year with schoolbook)
- product of two integers with 500,000,000 binary digits
- factorial of $N = 20,000,000$ (output of 140,000,000 digits)
- gcd of two polynomials of degree 600,000
- resultant of two polynomials of degree 40,000
- factorization of a univariate polynomial of degree 4,000
- factorization of a bivariate polynomial of total degree 500
- resultant of two bivariate polynomials of total degree 100 (output 10,000)
- product/sum of two algebraic numbers of degree 450 (output 200,000)
- determinant (char. polynomial) of a matrix with 4,500 (2,000) rows
- determinant of an integer matrix with 32-bit entries and 700 rows

INSIDE THE BOX

–Hermite-Padé approximants–

Definition of Hermite-Padé approximants

Definition: Given a column vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and an n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, a Hermite-Padé approximant of type $\mathbf{d}$ for $\mathbf{F}$ is a row vector $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n$, ($\mathbf{P} \neq 0$), such that:

(1) $\mathbf{P} \cdot \mathbf{F} = P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,

(2) $\deg(P_i) \leq d_i$ for all i.

σ is called the order of the approximant $\mathbf{P}$.

- Very useful concept in number theory (irrationality/transcendence):
  - [Hermite 1873]: e is transcendent.
  - [Lindemann 1882]: π is transcendent; so does e^α for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$.
  - [Apéry 1978, Beukers 1981]: $\zeta(3) = \sum_n \frac{1}{n^3}$ is irrational.
  - [Rivoal 2000]: there exist infinite values of k such that $\zeta(2k+1) \notin \mathbb{Q}$.

Let us compute a Hermite-Padé approximant of type $(1,1,1)$ for $(1, C, C^2)$, where $C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + O(x^6)$.
This boils down to finding $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$ such that

$$\alpha_0 + \alpha_1 x + (\beta_0 + \beta_1 x)(1 + x + 2x^2 + 5x^3 + 14x^4) + (\gamma_0 + \gamma_1 x)(1 + 2x + 5x^2 + 14x^3 + 42x^4) = O(x^5).$$

Identifying coefficients, this is equivalent to a homogeneous linear system:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 5 & 2 \\ 0 & 0 & 5 & 2 & 14 & 5 \\ 0 & 0 & 14 & 5 & 42 & 14 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \\ \gamma_1 \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 5 & 2 & 14 \\ 0 & 0 & 14 & 5 & 42 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \end{bmatrix} = -\gamma_1 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 14 \end{bmatrix}.$$

By homogeneity, one can choose $\gamma_1 = 1$.
Then, the violet minor shows that one can take $(\beta_0, \beta_1, \gamma_0) = (-1, 0, 0)$.
The other values are $\alpha_0 = 1,\ \alpha_1 = 0$.

Thus the approximant is $(1, -1, x)$, which corresponds to $P = 1 - y + xy^2$ such that $P(x, C(x)) = 0 \bmod x^5$.

Algebraic and differential approximation = guessing

- Hermite-Padé approximants of $n = 2$ power series are related to Padé approximants, i.e. to approximation of series by rational functions
- algebraic approximants = Hermite-Padé approximants for $f_\ell = A^{\ell-1}$, where $A \in \mathbb{K}[[x]]$ seriestoalgeq, listtoalgeq
- differential approximants = Hermite-Padé approximants for $f_\ell = A^{(\ell-1)}$, where $A \in \mathbb{K}[[x]]$ seriestodiffeq, listtodiffeq

```
> listtoalgeq([1,1,2,5,14,42,132,429],y(x));
                                                  2
                           [1 - y(x) + x y(x) , ogf]

> listtodiffeq([1,1,2,5,14,42,132,429],y(x));
                                         / 2        \
                     /d       \          |d         |
[{-2 y(x) + (2 - 4 x) |-- y(x)| + x |--- y(x)|, y(0) = 1, D(y)(0) = 1}, egf]
                     \dx      /          |  2       |
                                         \dx        /
```

Existence and naive computation

Theorem For any vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and for any n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exists a Hermite-Padé approx. of type $\mathbf{d}$ for $\mathbf{F}$.

Proof: The undetermined coefficients of $P_i = \sum_{j=0}^{d_i} p_{i,j} x^j$ satisfy a linear homogeneous system with $\sigma = \sum_i (d_i + 1) - 1$ eqs and $\sigma + 1$ unknowns.

Corollary Computation in $O(\mathsf{MM}(\sigma)) = O(\sigma^\theta)$, for $2 \leq \theta \leq 3$.

- There are better algorithms:
  - The linear system is structured (Sylvester-like / quasi-Toeplitz)
  - Derksen's algorithm (Gaussian-like elimination) $O(\sigma^2)$
  - Beckermann-Labahn's algorithm (DAC) $\tilde{O}(\sigma) = O(\sigma \log^2 \sigma)$

Quasi-optimal computation

Theorem [Beckermann-Labahn, 1994] One can compute a Hermite-Padé approximant of type $(d, \ldots, d)$ for $\mathbf{F} = (f_1, \ldots, f_n)$ in $O(\mathsf{MM}(n, d) \log(nd))$.

Ideas:

- Compute a whole matrix of approximants
- Exploit divide-and-conquer

Algorithm:

1. If $\sigma = n(d+1) - 1 \leq$ threshold, call the naive algorithm
2. Else:
   1. recursively compute $\mathbf{P}_1 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_1 \cdot \mathbf{F} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_1) \approx \frac{d}{2}$
   2. compute "residue" $\mathbf{R}$ such that $\mathbf{P}_1 \cdot \mathbf{F} = x^{\sigma/2} \cdot \left(\mathbf{R} + O(x^{\sigma/2})\right)$
   3. recursively compute $\mathbf{P}_2 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_2 \cdot \mathbf{R} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_2) \approx \frac{d}{2}$
   4. return $\mathbf{P} := \mathbf{P}_2 \cdot \mathbf{P}_1$

- ▶ The precise choices of degrees is a delicate issue
- ▶ Corollary: Gcd, extended gcd, Padé approximants in $O(\mathsf{M}(n) \log n)$

INSIDE THE BOX

–Linear differential operators–

Linear differential operators

Definition: If $\mathbb{K}$ is a field, $\mathbb{K}\langle x, \partial; \partial x = x\partial + 1\rangle$, or simply $\mathbb{K}(x)\langle\partial\rangle$, denotes the associative algebra of linear differential operators with coefficients in $\mathbb{K}(x)$.

$\mathbb{K}(x)\langle\partial\rangle$ is called the (rational) Weyl algebra. It is the algebraic formalization of the notion of linear differential equation with rational function coefficients:

$$a_r(x)y^{(r)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0$$

$$\Longleftrightarrow$$

$$L(y) = 0, \quad \text{where} \quad L = a_r(x)\partial^r + \cdots + a_1(x)\partial + a_0(x)$$

The commutation rule $\partial x = x\partial + 1$ formalizes Leibniz's rule $(fg)' = f'g + fg'$.

- Implementation in the `DEtools` package: `diffop2de, de2diffop, mult`

```
DEtools[mult](Dx,x,[Dx,x]);
                           x Dx + 1
```

Weyl algebra is Euclidean

Theorem [Libri 1833, Brassinne 1864, Wedderburn 1932, Ore 1932]
$\mathbb{K}(x)\langle\partial\rangle$ is a non-commutative (left and right) Euclidean domain: for any $A, B \in \mathbb{K}(x)\langle\partial\rangle$, there exist unique operators $Q, R \in \mathbb{K}(x)\langle\partial\rangle$ such that

$$A = QB + R, \qquad \text{and} \qquad \deg_\partial(R) < \deg_\partial(B).$$

This is called the Euclidean right division of A by B.

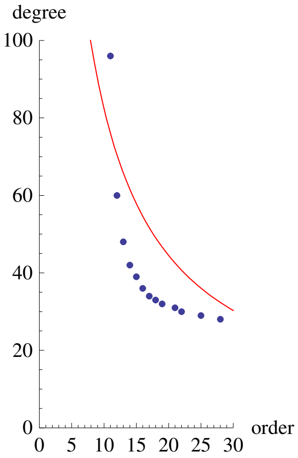
Moreover, any $A, B \in \mathbb{K}(x)\langle\partial\rangle$ admit a greatest common right divisor (GCRD) and a least common left multiple (LCLM). They can be computed by a non-commutative version of the extended Euclidean algorithm.

- `rightdivision`, `GCRD`, `LCLM` from the `DEtools` package

```
> rightdivision(Dx^10,Dx^2-x,[Dx,x])[2];
                              3                  2    5
                        (20 x  + 80) Dx + 100 x  + x
```

proves that $\mathrm{Ai}^{(10)}(x) = (20x^3 + 80)\mathrm{Ai}'(x) + (100x^2 + x^5)\mathrm{Ai}(x)$

Application to differential guessing



1000 terms of a series are enough to guess candidate differential equations below the red curve. GCRD of candidates could jump above the red curve.

Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite.

Proof: Let $f(x) \in \mathbb{K}[[x]]$ such that $P(x, f(x)) = 0$, with $P \in \mathbb{K}[x, y]$ irreducible.

Differentiate w.r.t. x:

$$P_x(x, f(x)) + f'(x) P_y(x, f(x)) = 0 \qquad \Longrightarrow \qquad f' = -\frac{P_x}{P_y}(x, f).$$

Bézout relation: $\gcd(P, P_y) = 1 \quad \Longrightarrow \quad UP + VP_y = 1,\;$ for $U, V \in \mathbb{K}(x)[y]$

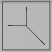
$$\Longrightarrow \quad f' = -\Big(P_x V \bmod P\Big)(x, f) \;\in\; \mathsf{Vect}_{\mathbb{K}(x)}\left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right).$$

By induction, $f^{(\ell)} \in \mathsf{Vect}_{\mathbb{K}(x)}\left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right)$, for all ℓ. $\square$

- Implemented in gfun: algeqtodiffeq
- Generalization: g D-finite, f algebraic $\rightarrow g \circ f$ D-finite algebraicsubs

BACK TO THE EXERCISE

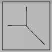
–A hint–

An exercise involving the model 

Let $\mathfrak{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$.

(ii) the number of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$;

An exercise involving the model 

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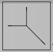
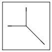
(ii) the number of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$;

For instance, for $n = 3$, this common value is 3:

(i) $(0,0) \mapsto (-1,0) \mapsto (-1,1) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (-1,1) \mapsto (0,0)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0)$, i.e., W–N–SE, N–W–SE, N–SE–W

(ii) $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (0,2) \mapsto (1,1)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1)$, i.e., N–SE–W, N–N–SE, N–SE–N

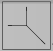
A recurrence relation for -walks in $\mathbb{Z} \times \mathbb{N}$

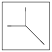
$h(n;i,j)$ = # walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j), with $\mathfrak{S} =$ 
The numbers $h(n;i,j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
  end:

> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

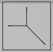
A recurrence relation for -walks in $\mathbb{N}^2$

$q(n;i,j)$ = # walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j), with $\mathfrak{S} =$ 
The numbers $q(n;i,j)$ satisfy

$$q(n;i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j') \in \mathfrak{S}} q(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> q:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else q(n-1,i,j-1)+q(n-1,i+1,j)+q(n-1,i-1,j+1) fi
  end:

> B:=series(add(add(q(n,k,k),k=0..n)*t^n,n=0..12),t,12);
```

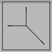
$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

Guessing the answer for -excursions in $\mathbb{Z} \times \mathbb{N}$

```
> seriestorec(series(add(h(n,0,0)*t^n,n=0..30),t,30), u(n))[1];
           2                       2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                        u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n));

{    (n/3)                                           1/2
{ 27      GAMMA(n/3 + 2/3) GAMMA(n/3 + 1/3) 3    (n/3 + 1)
{ -----------------------------------------------------       irem(n, 3) = 0
{                                           2
{                  2 Pi  GAMMA(n/3 + 2)
{
{                               0                             irem(n-1, 3) = 0
{
{                               0                             irem(n-2, 3) = 0

> A:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                         3
              A := hypergeom([1/3, 2/3], [2], 27 t )
```

- Thus, differential guessing predicts

$$A(t) = {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

Guessing the answer for diagonal -walks in $\mathbb{N}^2$

```
> series(add(add(q(n,k,k),k=0..n)*t^n,n=0..30),t,30), u(n))[1];
        2                        2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                              u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n));

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{ -----------------------------------------------------     irem(n, 3) = 0
{                                         2
{                  2 Pi GAMMA(n/3 + 2)
{
{                              0                           irem(n-1, 3) = 0
{
{                              0                           irem(n-2, 3) = 0

> B:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                        3
            B := hypergeom([1/3, 2/3], [2], 27 t )
```
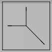

- Thus, differential guessing predicts

$$A(t) = B(t) = {}_2F_1\left(\left.\begin{matrix}1/3 \;\; 2/3\\ 2\end{matrix}\,\right|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

Guessing the answer for diagonal -walks in $\mathbb{N}^2$

```
> series(add(add(q(n,k,k),k=0..n)*t^n,n=0..30),t,30), u(n))[1];
        2                       2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                        u(0) = 1, u(1) = 0, u(2) = 0}

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{
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{
{                          0                                irem(n-2, 3) = 0

> B:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                   3
             B := hypergeom([1/3, 2/3], [2], 27 t )
```

- Tomorrow, we will prove this using creative telescoping

Summary

- ☺ Guess'n'Prove is a powerful method, especially when combined with efficient computer algebra
- ☺ It is robust: can be used to uniformly prove
  - D-finiteness in all the cases with finite group
  - algebraicity in all the cases with finite group and zero orbit sum
- ☺ In the D-finite cases, failure of algebraic guessing proves transcendence: $\exists N$ (depending only on the differential equation) such that if algebraic guessing mod t^N only produces the trivial equation, then there is no non-trivial equation [B., Bousquet-Mélou, Kauers, Melczer 2015]
- ☺ Brute-force and/or use of naive algorithms = hopeless. E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

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Thanks for your attention!