Computer Algebra for Lattice Path Combinatorics

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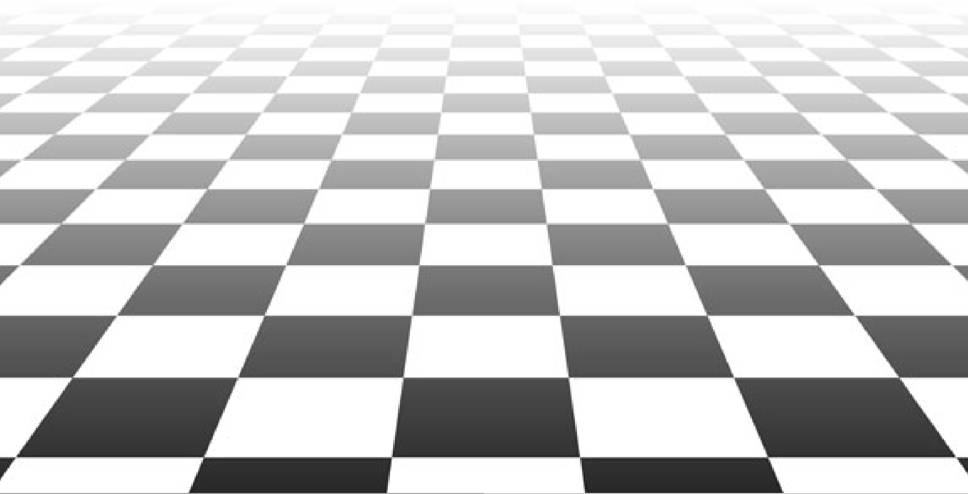
informatics mathematics Inria

The 74th Séminaire Lotharingien de Combinatoire
Ellwangen, March 23–25, 2015

Overview

1. Monday: General presentation
2. Tuesday: Guess'n'Prove
3. Wednesday: Creative telescoping

Part III: Creative telescoping



DIAGONALS

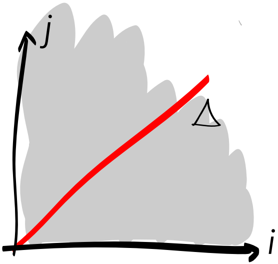
Definition

If F is a formal power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

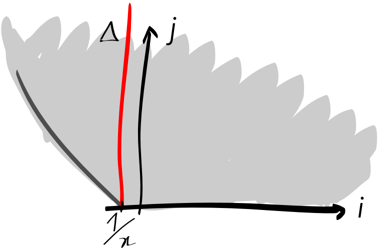
its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.



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Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

Proof:

$$\mathrm{Diag}\,(F) = [x^{-1}]\frac{1}{x}F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right)\frac{dx}{x}$$

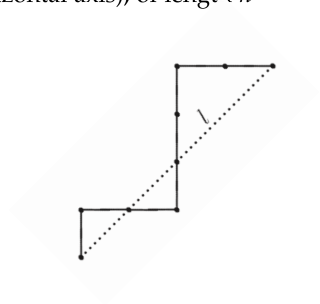
and evaluating the integral by residues concludes (residues are algebraic)

Example: Dyck walks

$$\mathfrak{S} = \{(1,1),(1,-1)\}$$

Let B_n be the number of Dyck bridges (i.e. $\mathfrak{S}$-walks in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis), of length n

Rotating a Dyck bridge



counterclockwise by $\pi/4$

B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n)

$$\Longrightarrow \qquad B(t) = \sum_{n \geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x - x^2 - t} = \left.\frac{1}{1-2x}\right|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

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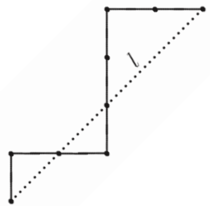


counterclockwise by $\pi/4$

B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) $= \binom{2n}{n}$

$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{\sqrt{1-4t}} = \sum_{n\geq 0}\binom{2n}{n} t^n$$

Let $A, B \in \mathbb{K}[x]$ with $\deg(A) < \deg(B)$ and squarefree monic denominator B. The rational function $F = A/B$ has simple poles only.

If $F = \sum_i \frac{\gamma_i}{x - \beta_i}$, then the residue γ_i of F at the pole β_i equals $\gamma_i = \frac{A(\beta_i)}{B'(\beta_i)}$.

Theorem. The residues γ_i of F are roots of the Rothstein-Trager resultant

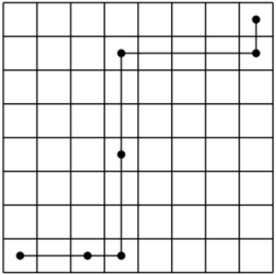
$$R(t) = \mathrm{Res}_x\big(B(x),\, A(x) - t \cdot B'(x)\big).$$

Proof. Poisson's formula: $R(t) = \prod_i \Big(A(\beta_i) - t \cdot B'(\beta_i)\Big)$.

- This resultant is useful for symbolic integration of rational functions.
- [Bronstein 1992] generalized this result to multiple poles.

Example: diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves right or up at each step.



$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$

Example: diagonal Rook paths

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \dots$$

is

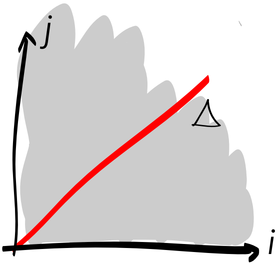
$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2i\pi} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By residue theorem, $\mathsf{Diag}(F)$ is a sum of roots $y(t)$ of the Rothstein-Trager resultant

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
```

$$t^2(1-t)(2y-1)(36ty^2 - 4y^2 + 1 - t)$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right)$.



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$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.[x]

[x] The converse is also true [Furstenberg, 1967]

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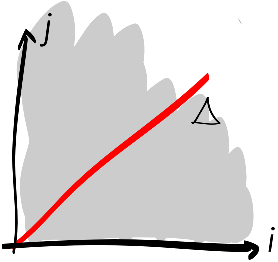

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

- This is false for more than 2 variables. E.g.

$$\mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \binom{3n}{n,n,n} t^n = {}_2F_1\left(\left.\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\right| 27t\right) \quad \text{is transcendental}$$

Pólya's theorem

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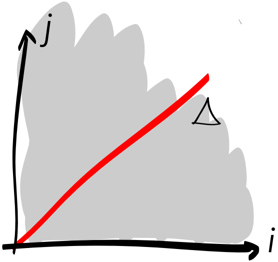

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

- Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
- Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]

Definition

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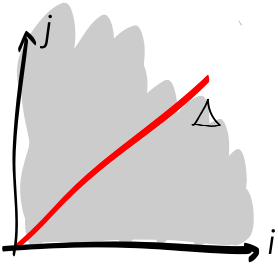

Theorem (Lipshitz, 1988)

Diagonals of rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]
How many ways can a Rook move from $(0,0,0)$ to (N,N,N), where each step is a positive integer multiple of $(1,0,0),(0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, ...

Answer [B.-Chyzak-Hoeij-Pech 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3 \\ & 2\end{matrix} \,\middle|\, \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\, dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\mathrm{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

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Argument: if one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y \text{ })$$

that annihilates $G = \frac{1}{xy}\cdot F\left(x,\frac{y}{x},\frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

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Proof:

1. $\mathrm{Diag}(F) = [x^0y^0]\, F\left(x, \frac{y}{x}, \frac{t}{y}\right)$
2. $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
3. $0 = [x^{-1}y^{-1}]L(G) = [x^{-1}y^{-1}]P(G) = P([x^{-1}y^{-1}]G) = P(\mathrm{Diag}(F))$

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▶ Remaining task: Show that such an L does exist.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i + j + k + \ell \le N$$

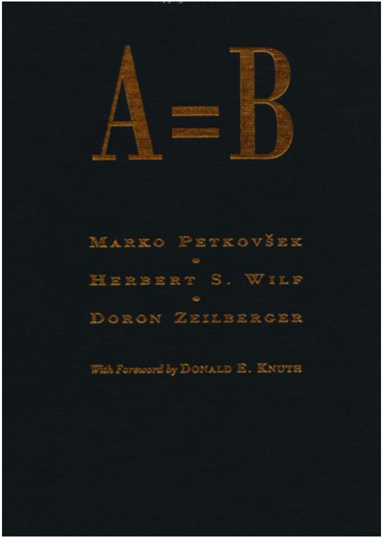
are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

- If $\binom{N+4}{4} > 18(N+1)^3$, then there exists $L(t, \partial_t, \partial_x, \partial_y)$ (resp. $P(t, \partial_t)$) of total degree at most N, such that $LG = 0$ (resp. $P(\text{Diag}(F)) = 0$).
- $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$
- Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations!
- A better solution is provided by creative telescoping.

Creative Telescoping

General framework in computer algebra –initiated by Zeilberger in the ’90s– for computing multiple integrals and sums with parameters.



Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1891]

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{n}{k}^3$ [Strehl 1992]

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$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(*The specific problem was mentioned to Don Zagier, who solved it with irritating speed* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{n}{k}^3$ [Strehl 1992]

Examples II: Integrals and Diagonals

- $\displaystyle\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\,du = \frac{\pi}{2}J_0(z);$
- $\displaystyle\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$ [Glasser-Montaldi 1994];
- $\displaystyle\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2\rfloor!}$ [Doetsch 1930];
- $\displaystyle\operatorname{Diag}\frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n\geq 0} A_n t^n$ [Straub 2014].

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

$$F_n = \sum_k u_{n,k} = ?$$

IF one knows $P(n, S_n)$ and $R(n, k, S_n, S_k)$ s.t.

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$), then the sum “telescopes”, leading to

$$P(n, S_n) \cdot F_n = 0.$$

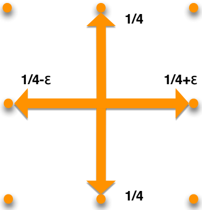
Zeilberger's Algorithm [1990]

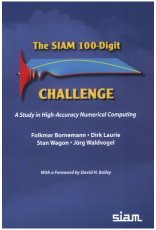
Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions in n and k;

Output:

- a linear recurrence (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), s.t. checking the result is easy from $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Example: SIAM flea





$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}},$$

$$p_n = \sum_{k=0}^{n} U_{n,k} \quad = \text{probability of return to } (0,0) \text{ at step } 2n.$$

```
> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$[\left(4n^2+16n+16\right)Sn^2+\left(-4n^2+32c^2n^2+96c^2n-12n+72c^2-9\right)Sn$$
$$+128c^4n+64c^4n^2+48c^4, \text{ ...(BIG certificate)...}]$$

$$I(t) = \oint_\gamma H(t,x)\,dx = ?$$

IF one knows $P(t,\partial_t)$ and $R(t,x,\partial_t,\partial_x)$ s.t.

$$\big(P(t,\partial_t) + \partial_x R(t,x,\partial_t,\partial_x)\big) \cdot H(t,x) = 0,$$

then the integral "telescopes", leading to

$$P(t,\partial_t) \cdot I(t) = 0.$$

Example: diagonal Rook paths again

Generating function of the sequence

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is

$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2i\pi} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By the creative telescoping, $\mathsf{Diag}(F)$ satisfies the differential equation

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> Zeilberger(G, t, x, Dt)[1];
```

$$(9t^2 - 10t + 1)\partial_t^2 + (18t - 14)\partial_t$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right)$.

CT for Multiple rational integrals

Problem:

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \ldots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ H(t, \mathbf{x}) & \text{— rational function} \\ \gamma & \text{— } n\text{-cycle in } \mathbb{C}^n \end{array}\right\} \oint_\gamma H(t, \mathbf{x}) \mathrm{d}\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \implies \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma H \mathrm{d}\mathbf{x} = 0$$

Task:

1. find the $c_k(t)$ which satisfy a telescopic relation,
2. ideally, without computing the certificate (A_i).

Perimeter of an ellipse with eccentricity e and semi-major axis 1 [Euler, 1733]

$$p(e) = \int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\mathrm{d}x = \oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}},$$

CT finds the telescopic relation:

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right) \cdot \left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x \left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\, \partial_y \left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0$.
(Note the size of the certificate.)

Example: 3D rook paths [B.-Chyzak-Hoeij-Pech 2011]

Task: Given G in $\mathbb{Q}(t,x,y)$, construct a linear differential operator $P(t,\partial_t)$, and two rational functions R and S in $\mathbb{Q}(t,x,y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$

Solution: Creative telescoping!

```
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

$$\begin{aligned} P = \quad & t(t-1)(64t-1)(3t-2)(6t+1)\partial_t^3 \\ & +(4608t^4-6372t^3+813t^2+514t-4)\partial_t^2 \\ & +4(576t^3-801t^2-108t+74)\partial_t \end{aligned}$$

- The whole computation takes < 10 seconds on a personal laptop.
- Proves a recurrence conjectured by [Erickson 2010]

Brief review on CT algorithms

Brief and incomplete

General-purpose creative telescoping algorithms:

- using linear algebra [Lipshitz, 1988];
- using non-commutative Gröbner bases:
  - and elimination [Takayama, 1990];
  - and rational resolution of differential equations [Chyzak, 2000];
  - and heuristics [Koutschan, 2010].

▶ Drawbacks: Bad or unknown complexity; unsatisfactory performance on medium-sized problems; all compute certificates.

Rational case:

- univariate integrals [B., Chen, Chyzak, Li, 2010];
- double integrals [Chen, Kauers, Singer, 2012].

Univariate case: CT by Hermite reduction

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

CT Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

▶ $L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r \eta_k g_k$ the corresponding certificate).

Multiple case: Polynomial time computation

$H = \frac{P}{Q}$ — a rational function in t and $\mathbf{x} = x_1, \dots, x_n$

$d_{\mathbf{x}}$ — the degree of Q w.r.t. $\mathbf{x}$

d_t — $\max(\deg_t P, \deg_t Q)$

Theorem (B., Lairez, Salvy, 2013)

A telescoper for H can be computed using $\widetilde{\mathcal{O}}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$ operations.
The minimal telescoper has order $\leq d_{\mathbf{x}}^n$ and degree $\mathcal{O}(e^n d_{\mathbf{x}}^{3n} d_t)$.
These size bounds are generically reached.

- First polynomial time algorithm for rational creative telescoping.
- It avoids the costly computation of certificates.
- Generically, certificates have size $\Omega(d_{\mathbf{x}}^{n^2/2})$.
- General-purpose algorithms have double-exponential complexity.
- Applies to diagonals: $\mathrm{Diag}(F)(t) = \frac{1}{(2\pi i)^n} \oint F\left(\frac{t}{x_1 \dots x_n}, x_1, \dots, x_n\right)$.

Main ingredients of the integration algorithm

Griffiths–Dwork method for the generic case

Linear reduction classical in algebraic geometry;
Generalization of Hermite's reduction.

Fast linear algebra on polynomial matrices

Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case

Input perturbation using a new free variable.

► Recent, highly non-trivial, extension by [Lairez, 2015] tremendously improves the efficiency of the algorithm.

WALKS IN THE QUARTER PLANE

The 19 D-finite cases with nonzero orbit sum

Task: Prove Cases 1–19 in the tables [B. & Kauers 2009] for $F(t;1,1)$

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

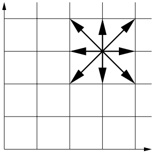
Equation sizes = {order, degree}@(algeq, diffeq)

Task: Prove Cases 1–19 in the tables [B. & Kauers 2009] for $F(t;1,1)$

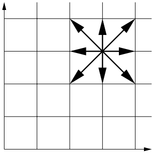
	OEIS	$\mathfrak{S}$	alg?	asympt		OEIS	$\mathfrak{S}$	alg?	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}\frac{3^n}{n^{1/2}}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}\frac{3^n}{n^{3/2}}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}\frac{5^n}{n^{1/2}}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}\frac{6^n}{n^{3/2}}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}\frac{5^n}{n^{1/2}}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}\frac{7^n}{n^{1/2}}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

The group of a model



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$



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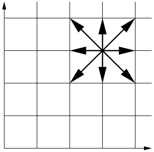
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$



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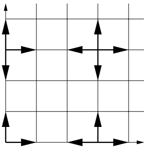
and thus under any element of the group

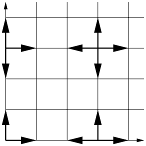
$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$



$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$

D-Finiteness via the finite group [Bousquet-Mélou & Mishna 2010]



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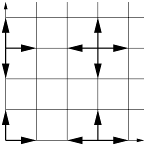
"Kernel equation":

$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$



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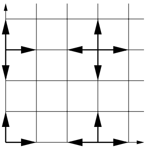

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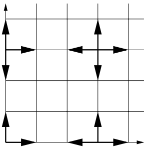

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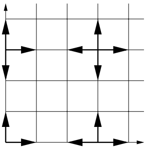
\end{aligned}$$



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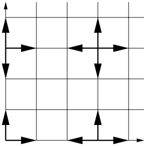
Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,F(t;x,y)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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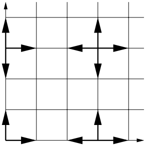
\end{aligned}$$

Taking positive parts yields:

$$[x^>][y^>]\sum_{\theta\in\mathcal{G}}(-1)^\theta\theta\big(xy\,F(t;x,y)\big) = [x^>][y^>]\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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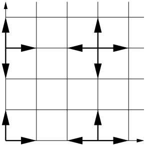

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\end{aligned}$$

$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right)$$

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\, F(t;x,y) = [x^>][y^>]\, \frac{\sum_{\theta \in \mathcal{G}} (-1)^\theta \theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

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Proof: Use [Lipshitz'88] for positive parts of rational functions:
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▶ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$.

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► Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$; in fact, any such ODE is probably
TOO LARGE TO BE MERELY WRITTEN!

Explicit Expressions for the Cases 1–19

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Explicit Expressions for the Cases 1–19

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Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;1,1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Explicit Expressions for the Cases 1–19

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

- Proof uses Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum_\theta \frac{(-1)^\theta\,\theta(xy)}{J(t;x,y)}$, then $F = [u^{-1}v^{-1}]H$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.

2. If $P \in \mathbf{Q}(x,y)[t]\langle\partial_t\rangle$ and $U,V \in \mathbf{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L.

3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

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   Use creative telescoping for finding L.
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

For $F(t;x,y)$, run whole process for $F(t;0,0)$, $F(t;x,0)$, and $F(t;0,y)$, then substitute into Kernel equation!

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{3}{2}\\ 2\end{matrix}\middle\vert\, w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4}\ \frac{11}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{3}{2}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{5}{2}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2}\ \frac{3}{2}\\ 2\end{matrix}\middle\vert\, w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2}\ \frac{3}{2}\\ 2\end{matrix}\middle\vert\, w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4}\ \frac{11}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{3}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4}\ \frac{11}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4}\ \frac{9}{4}\\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{matrix}-\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 2\end{matrix}\middle\vert\, w\right)$	$16t^2$

Theorem

- In cases 1–19, both $F(t;x,y)$ and $F(t;0,0)$ are transcendental.
- In cases 1–16 and 19, $F(t;1,1)$ is transcendental.
- Specific simplifications prove algebraicity of $F(t;1,1)$ in cases 17–18.

Proof: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- F is algebraic $\Longrightarrow$ G is algebraic.
- Computing a few coefficients of G shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to L_2 decides whether L_2 has nonzero algebraic solutions.

Coefficient Asymptotics $\kappa n^\alpha \rho^n$

Local theory of D-finite functions $\longrightarrow$
Systematic method for coefficient asymptotics
(Flajolet and Odlyzko's singularity analysis)

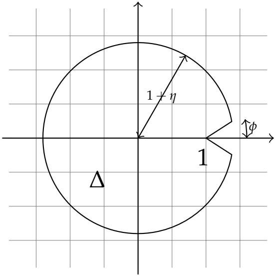
$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \longrightarrow \qquad f_n \sim \dots$$

- Determine dominant singularities of the complex-analytic function f.
- Find asymptotic expansion

$$f(z) =_{z \to s} \sum_{\alpha,\gamma} c_{\alpha,\gamma} (s-z)^\alpha \left(\ln \frac{1}{s-z} \right)^\gamma \qquad (1)$$

- Syntactic transfer into an asymptotic expansion for f_n

Transfer Theorems [Flajolet & Odlyzko 1990]



For $f(z) = \sum_{n=0}^{\infty} f_n z^n$ analytic in $\Delta \setminus \{1\}$:

$f(z)$	f_n	assumptions
$O\big((1-z)^\alpha)\big)$	$O\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$o\big((1-z)^\alpha)\big)$	$o\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$\sim C(1-z)^\alpha$	$\sim \frac{Cn^{-(\alpha+1)}}{\Gamma(-\alpha)}$	$\alpha \in \mathbb{R} \setminus \mathbb{N}$
$\sum_{j=0}^{m-1} c_j(1-z)^{\alpha_j} + O\big((1-z)^A\big)$	$\sum_{j=0}^{m-1} \frac{c_j n^{-(\alpha_j+1)}}{\Gamma(-\alpha_j)} + O\big(n^{-(A+1)}\big)$	$\alpha_1 \leq \cdots \leq \alpha_{m-1} < A$
$O\big((1-z)^\alpha(\ln(1-z)^{-1})^\gamma\big)$	$O\big(n^{-(\alpha+1)}(\ln n)^\gamma\big)$	$\alpha, \gamma \in \mathbb{R}$
$o\big((1-z)^\alpha(\ln(1-z)^{-1})^\gamma\big)$	$o\big(n^{-(\alpha+1)}(\ln n)^\gamma\big)$	$\alpha, \gamma \in \mathbb{R}$
$\sim C(1-z)^\alpha(\ln(1-z)^{-1})^\gamma$	$\sim \frac{Cn^{-(\alpha+1)}(\ln n)^\gamma}{\Gamma(-\alpha)}$	$\alpha, \gamma \in \mathbb{R} \setminus \mathbb{N}$
$\vdots$	$\vdots$	

One Example:  at $(1,1)$

$$Q = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}\,{}_2F_1\left(\left.\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\right| w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)}$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t \to \frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \quad \longrightarrow \quad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t \to -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \quad \longrightarrow \quad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

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$$f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}6^n$$

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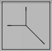
Summary – the 19 cases with finite group

Creative telescoping helps the uniform treatment of several questions:

- compute differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients.

BACK TO THE EXERCISE

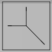
–Solution–

An exercise involving the model 

Let $\mathfrak{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$.

(ii) the number of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$;

An exercise involving the model 

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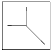
For instance, for $n = 3$, this common value is 3:

(i) $(0,0) \mapsto (-1,0) \mapsto (-1,1) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (-1,1) \mapsto (0,0)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0)$, i.e., W–N–SE, N–W–SE, N–SE–W

(ii) $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (0,2) \mapsto (1,1)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1)$, i.e., N–SE–W, N–N–SE, N–SE–N

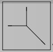
$h(n;i,j)$ = # walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j), with $\mathfrak{S}$ = 
The numbers $h(n;i,j)$ satisfy

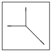
$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
  end:
```

```
> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A recurrence relation for -walks in $\mathbb{N}^2$

$q(n;i,j)$ = # walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j), with $\mathfrak{S} =$ 
The numbers $q(n;i,j)$ satisfy

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  end:

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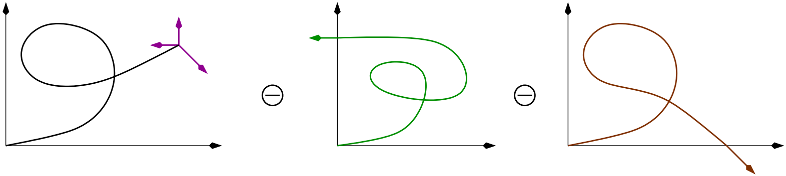
```

$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

Generating function: $Q(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} q(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

Kernel equation ($\bar{x} = 1/x,\ \bar{y} = 1/y$):

$$Q(t;x,y) \equiv Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$



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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x,y) = 1 - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0),$$

... and the corresponding functional equation for $\mathbb{N}^2$

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$$Q(t;x,y) \equiv Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x,y) = 1 - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0),$$

or

$$\boxed{(1 - t(y + \bar{x} + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)}$$

Task (Q): Find $B(t) = [x^0]\, Q(x,\bar{x})$

... and the corresponding functional equation for $\mathbb{Z} \times \mathbb{N}$

Generating function: $H(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=-n}^{n} \sum_{j=0}^{\infty} h(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,\bar{x},y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$H(t;x,y) \equiv H(x,y) = 1 + t(y + \bar{x} + x\bar{y})H(x,y) - tx\bar{y}H(x,0)$$

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or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big) y H(x,y) = y - tx H(x,0)}$$

Task (H): Find $A(t) = [x^0]\, H(x,0)$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$

- Let

$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} \qquad = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

be the (unique) root in $\mathbb{Q}[x,\bar{x}][[t]]$ of $K(x,y_0) = 0$.

- Then

$$0 = K(x,y_0)yH(x,y_0) = y_0 - txH(x,0),$$

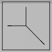
thus

$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad H(x,y) = \frac{y - y_0}{tx}.$$

- In conclusion: the GF of excursions in the half-plane is

$$A(t) = \left[x^0\right] \frac{y_0}{tx}.$$

The group of the model



Step set $\mathfrak{S} = \{(-1,0),(0,1),(1,-1)\}$, with characteristic polynomial

$$\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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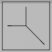

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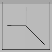

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Φ and Ψ are involutions, and generate a finite dihedral group $\mathfrak{G}$ of order 6:

$$(x,y) \xrightarrow{\Phi} (\bar{x}y, y) \xrightarrow{\Psi} (\bar{x}y, \bar{x}) \xrightarrow{\Phi} (\bar{y}, \bar{x})$$

$$(x,y) \xrightarrow{\Psi} (x, x\bar{y}) \xrightarrow{\Phi} (\bar{y}, x\bar{y}) \xrightarrow{\Psi} (\bar{y}, \bar{x})$$

The kernel method for $\mathbb{N}^2$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)xyQ(x,y) = xy - tx^2Q(x,0) - tyQ(0,y)$$

- The orbit of (x,y) under $\mathfrak{G}$ is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y,y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y,\bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y},\bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y},x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x,x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

- All transformations of $\mathfrak{G}$ leave $K(x,y)$ invariant. Hence

$$\begin{array}{rcccccc} K(x,y)\, xyQ(x,y) & = & xy & - & tx^2Q(x,0) & - & tyQ(0,y) \\ -K(x,y)\, \bar{x}y^2Q(\bar{x}y,y) & = & -\bar{x}y^2 & + & t\bar{x}^2y^2Q(\bar{x}y,0) & + & tyQ(0,y) \\ K(x,y)\, \bar{x}^2yQ(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & t\bar{x}Q(0,\bar{x}). \end{array}$$

- Summing up yields the (half) orbit equation

$$K(x,y)\left(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x})\right)$$
$$= xy - \bar{x}y^2 + \bar{x}^2y - tx^2Q(x,0) - t\bar{x}Q(0,\bar{x}).$$

Conclusion

- We finally solve the (half) orbit equation

$$K(x,y)\left(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x})\right) = xy - \bar{x}y^2 + \bar{x}^2y - tx^2Q(x,0) - t\bar{x}Q(0,\bar{x}).$$

- Substitute y in this equation by $\bar{x}$, resp. by y_0, yields

$$\begin{aligned} K(x,\bar{x})Q(x,\bar{x}) &= 1 \quad -tx^2Q(x,0) - t\bar{x}Q(0,\bar{x}) \\ 0 &= xy_0 - \bar{x}y_0^2 + \bar{x}^2y_0 \quad -tx^2Q(x,0) - t\bar{x}Q(0,\bar{x}) \end{aligned}$$

- By subtraction:

$$Q(x,\bar{x}) = \frac{1-(xy_0 - \bar{x}y_0^2 + \bar{x}^2y_0)}{1-t(2\bar{x}+x^2)} = \frac{y_0}{tx}$$

- In conclusion: the GF of diagonal walks in the quarter-plane is

$$B(t) = \left[x^0\right] Q(x,\bar{x}) = \left[x^0\right] \frac{y_0}{tx},$$

thus equal to $A(t)$, the GF of excursions in the half-plane. QED

Bonus: explicit expression

We have proved that both $A(t)$ and $B(t)$ are equal to

$$\left[x^0\right] \frac{-\sqrt{(t-x)^2 - 4t^2x^3}}{2t^2x^2}$$

Creative telescoping gives a differential equation for $A(t)$ and $B(t)$:

```
> G:=sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2);
> deq:=diffop2de(Zeilberger(G/x, t, x, Dt)[1], [Dt,t], y(t));
```

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$

Its solution is

$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 \quad 2/3\\ 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Bonus 2: Solving directly the kernel equation for $\mathbb{N}^2$

- Orbit equation:

$$
\begin{aligned}
xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) \\
- \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \\
\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\end{aligned}
$$

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- [Bousquet-Mélou & Mishna, 2010]

$$xyQ(x,y) = [x^{>0}y^{>0}]\, \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\, \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Multivariate Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

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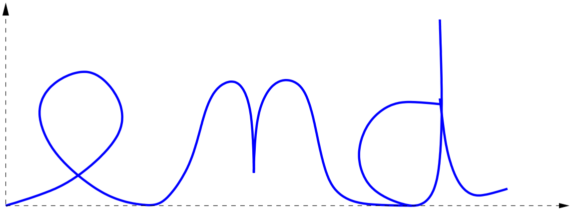

Thanks for your attention!