(pure) Transcendence bases in φ-deformed shuffle bialgebras

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74-th Séminaire Lotharingien de Combinatoire,
March, 23-25, 2015, Ellwangen, Germany

Plan

Mais le calcul par deux, c'est-à-dire par 0 et par 1,
en récompense de sa longueur, est plus fondamental pour la science,
et donne de nouvelles découvertes, qui se trouvent utiles ensuite,
même pour la pratique des nombres, et surtout pour la géométrie ;
dont la raison est que les nombres étant réduits aux plus simples principes, comme 0 et 1,
il paraît partout un ordre merveilleux.
Gottfried Wilhelm Leibniz

INTRODUCTION
(a primer of the zoology of the shuffle products)

Schützenberger's factorization in $(A\langle X\rangle, ., 1_{X^*}, \Delta_{ш}, \epsilon_X)$

A : commutative and associative $\mathbb{Q}$-algebra with unit. The free monoid X^* is generated by $X = \{x_0, x_1\}$ and admits 1_{X^*} as neutral element.
$A\langle X\rangle$ is endowed with the concatenation and the shuffle.
$\mathcal{L}ynX$ forms (pure) transcendence basis in $(A\langle X\rangle, ш, 1_{X^*})$.

- $\{P_l\}_{l\in\mathcal{L}ynX}$: basis of $\mathcal{L}ie_A\langle X\rangle$, where P_l is defined by
  $P_l = l$ if $l \in X$ and $P_l = [P_u, P_v]$ if $l \in \mathcal{L}ynX - X$ and $\sigma(l) = (u, v)$.
- $\{P_w\}_{w\in X^*}$: PBW-L basis of $\mathcal{U}(\mathcal{L}ie_A\langle X\rangle)$ is obtained by putting
  $P_w = P_{l_1}^{i_1} \ldots P_{l_k}^{i_k}$ for $w = l_1^{i_1} \ldots l_k^{i_k}, l_1, \ldots, l_k \in \mathcal{L}ynX, l_1 > \ldots > l_k$.
- The dual basis $\{S_w\}_{w\in X^*}$ of $\{P_w\}_{w\in X^*}$, *i.e.* :

  $$\forall u, v \in X^*, \ \langle P_u | S_v\rangle = \delta_{u,v}$$

  can be obtained by putting

  $$S_l = xS_u, \qquad \text{for} \quad l = xu \in \mathcal{L}ynX,$$
  $$S_w = \frac{1}{i_1! \ldots i_k!} S_{l_1}^{ш i_1} ш \ldots ш S_{l_k}^{ш i_k}, \qquad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k.$$

Theorem (Schützenberger, 1958, Reutenauer 1988)

$$\mathcal{D}_X := \sum_{w\in X^*} w \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l).$$

Factorization in $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{⧢}, \epsilon_Y)$

The free monoid Y^* is generated $Y = \{y_k\}_{k\geq 1}$ and admits 1_{Y^*} as neutral element. $A\langle Y\rangle$ is endowed with the concatenation and the quasi-shuffle. $\mathcal{L}ynY$ forms (pure) transcendence basis in $(A\langle Y\rangle, ⧢, 1_{Y^*})$.

- Let us consider the following PBW-Lyndon basis for $\mathbb{Q}\langle Y\rangle$

$$\Pi_{y_k} = \pi_1(y_k), \quad \text{where } \forall w \in Y^+, \pi_1(w) \text{ is the primitive polynomial}$$

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k \in Y^+} \langle w | u_1 ⧢ \ldots ⧢ u_k \rangle\, u_1 \ldots u_k,$$

$$\Pi_l = [\Pi_s, \Pi_r], \quad \text{for } l \in \mathcal{L}ynY - Y \text{ and } \sigma(l) = (s, r),$$

$$\Pi_w = \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}, \quad \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY.$$

- The dual basis $\{\Sigma_w\}_{w\in Y^*}$, *i.e.* $\forall u, v \in Y^*, \langle \Sigma_v | \Pi_u \rangle = \delta_{u,v}$, can be computed as follows

$$\Sigma_w = \frac{1}{i_1! \ldots i_k!} \Sigma_{l_1}^{⧢ i_1} ⧢ \ldots ⧢ \Sigma_{l_k}^{⧢ i_k}, \quad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k.$$

Theorem (HNM, 2009)

$$\mathcal{D}_Y := \sum_{w\in Y^*} w \otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l).$$

Ce que j'avais dit de façon simpliste est faux,
mais si on le dit de façon savante, cela devient vrai.
Marcel Paul Schützenberger

COMBINATORICS OF φ-DEFORMED SHUFFLE ALGEBRAS

φ-deformed shuffle products

Definition

Let $ш_\varphi$ be the product $Y^* \times Y^* \to A\langle Y\rangle$ satisfying the conditions :

i) for any $w \in Y^*$, $1_{Y^*} ш_\varphi w = w ш_\varphi 1_{Y^*} = w$,

ii) for any $a, b \in Y$ and $u, v \in Y^*$,

$$(R) \qquad au ш_\varphi bv = a(u ш_\varphi bv) + b(au ш_\varphi v) + \varphi(a,b)(u ш_\varphi v),$$

where φ is an arbitrary mapping defined by its structure constants

$$\begin{array}{rcl} \varphi : Y \times Y & \longrightarrow & AY, \\ (y_i, y_j) & \longmapsto & \displaystyle\sum_{k \in I \subset \mathbb{N}_+} \gamma_{i,j}^k \, y_k. \end{array}$$

It is said to be dualizable if there exists $\Delta_{ш_\varphi} : A\langle Y\rangle \to A\langle Y\rangle \otimes A\langle Y\rangle$ such that the dual mapping $\Big(A\langle Y\rangle \otimes A\langle Y\rangle\Big)^* \to A\langle\langle Y\rangle\rangle$ restricts to $ш_\varphi$.

Proposition

(R) and i) define a unique mapping $ш_\varphi : Y^* \times Y^* \to A\langle Y\rangle$ which is at once extended by multilinearity as a law $ш_\varphi : A\langle Y\rangle \times A\langle Y\rangle \to A\langle Y\rangle$.

Examples

Name	(recursion) Formula	φ
Shuffle	$au ш bv = a(u ш bv) + b(au ш v)$	$\varphi \equiv 0$
Quasi-shuffle	$x_i u ⊎ x_j v = x_i(u ⊎ x_j v) + x_j(x_i u ⊎ v) + x_{i+j}(u ⊎ v)$	$\varphi(x_i, x_j) = x_{i+j}$
Min-shuffle	$x_i u ⊎ x_j v = x_i(u ⊎ x_j v) + x_j(x_i u ш v) - x_{i+j}(u ⊎ v)$	$\varphi(x_i, x_j) = -x_{i+j}$
Muffle	$x_i u ш x_j v = x_i(u ш x_j v) + x_j(x_i u ш v) + x_{i\times j}(u ш v)$	$\varphi(x_i, x_j) = x_{i\times j}$
q-stuffle	$x_i u ⊎_q x_j v = x_i(u ⊎_q x_j v) + x_j(x_i u ⊎_q v) + q x_{i+j}(u ш v)$	$\varphi(x_i, x_j) = q x_{i+j}$
q-shuffle	$x_i u ш_q x_j v = x_i(u ш_q x_j v) + x_j(x_i u ш_q v) + q^{i\times j} x_{i+j}(u ш v)$	$\varphi(x_i, x_j) = q^{i\times j} x_{i+j}$
$\mathtt{LDIAG}(1, q_s)$ non-crossed, non-shifted	$au ш bv = a(u ш bv) + b(au ш v) + q_s^{\|a\|\|b\|}(a.b)(u ш v)$	$\varphi(a, b) = q_s^{\|a\|\|b\|}(a.b)$
B-shuffle	$au ш bv = a(u ш bv) + b(au ш v) + \langle a, b\rangle(u ш v)$	$\varphi(a, b) = \langle a, b\rangle = \langle b, a\rangle$
Semigroup--shuffle	$x_t u ш_\perp x_s v = x_t(u ш_\perp x_s v) + x_s(x_t u ш_\perp v) + x_{t\perp s}(u ш_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$

Properties of φ-deformed shuffle products

Lemma

Let Δ be the morphism $A\langle Y\rangle \longrightarrow A\langle\langle Y^* \otimes Y^*\rangle\rangle$ defined on the letters by

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma^s_{n,m}\, y_n \otimes y_m.$$

Then

i) $\forall u, v, w \in Y^*, \langle u \uplus_\varphi v | w\rangle = \langle u \otimes v | \Delta(w)\rangle^{\otimes 2}$.

ii) $\forall w \in Y^+, \Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) | u \otimes v\rangle u \otimes v$.

Theorem

i) *The law $\uplus_\varphi$ is associative (resp. commutative) if and only if the linear extension $\varphi : AY \otimes AY \longrightarrow AY$ is so.*

ii) *Let $\gamma^z_{x,y} := \langle \varphi(x, y)|z\rangle$ be the structure constants of φ, then $\uplus_\varphi$ is dualizable if and only if $(\gamma^z_{x,y})_{x,y,z\in Y}$ has the following property*

$$(\forall z \in Y)(\#\{(x, y) \in Y^2 | \gamma^z_{x,y} \neq 0\} < +\infty).$$

Associative commutative φ-deformed shuffle products

Theorem

Let us suppose that φ is dualizable. We still denote the dual comultiplication (of $\uplus_\varphi$) by $\Delta_{\uplus_\varphi} : A\langle Y\rangle \longrightarrow A\langle Y\rangle \otimes A\langle Y\rangle$. Then $\mathcal{B}_\varphi = (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_\varphi}, \varepsilon)$ is a bialgebra. Moreover, if φ is commutative then the following conditions are equivalent

i) $\mathcal{B}_\varphi$ *is an enveloping bialgebra.*

ii) *the algebra AY admits an increasing filtration $\Big((AY)_n\Big)_{n\in\mathbb{N}}$*

$$(AY)_0 = \{0\} \subset (AY)_1 \subset \cdots \subset (AY)_n \subset (AY)_{n+1} \subset \cdots$$

compatible with the multiplication and the comultiplication, i.e.

$$\begin{aligned} (AY)_p(AY)_q &\subset (AY)_{p+q} \\ \Delta_{\uplus_\varphi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q. \end{aligned}$$

iii) $\mathcal{B}_\varphi$ *is isomorphic to $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\text{Ш}}, \epsilon)$ as a bialgebra.*

In the previous equivalent cases, φ will be called of moderate growth.

φ-extended Ree theorem

Lemma (φ-extended Friedrichs criterion)

Let $S \in A\langle\langle Y \rangle\rangle$. Then

1. *If $\langle S | 1_{Y^*} \rangle = 0$ then S is primitive, if and only if, for any u and $v \in Y^+$, one has $\langle S | u \,\text{⧢}_{\varphi} v \rangle = 0$.*
2. *If $\langle S | 1_{Y^*} \rangle = 1$, then S is group-like, if and only if, for any u and $v \in Y^*$, one has $\langle S | u \,\text{⧢}_{\varphi} v \rangle = \langle S | u \rangle \langle S | v \rangle$.*

Theorem (φ-extended Ree theorem)

Let $A_+\langle Y \rangle := \{P \in A\langle Y \rangle \mid \langle P | 1_{Y^*} \rangle = 0\}$. Then

$$A_+\langle Y \rangle = \mathcal{P}_Y \overset{\perp}{\oplus} \mathcal{I}_Y,$$

where

$$\begin{aligned} \mathcal{P}_Y &:= \{P \in A\langle Y \rangle \mid \Delta_{\text{⧢}_{\varphi}}(P) = P \otimes 1_{Y^*} + 1_{Y^*} \otimes P\}, \\ \mathcal{I}_Y &:= span_A\{u \,\text{⧢}_{\varphi} v\}_{u,v \in Y^+}. \end{aligned}$$

φ-extended eulerian projectors

Let π_1^φ be the linear endomorphism of $A\langle\langle Y\rangle\rangle$ given by the $\log_{\uplus_\varphi} \mathcal{D}_Y$.
In particular, for the letters, one has

$$\forall y \in Y, \qquad \pi_1^\varphi(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l\in Y} \gamma^y_{x_1,\ldots,x_l}\, x_1\ldots x_l,$$

where, with the Einstein notation,

$$\gamma^y_{x_1,\ldots,x_l} = \sum_{t_1,\ldots,t_{l-2}\in Y} \gamma^y_{x_1,t_1}\gamma^{t_1}_{x_2,t_2}\ldots\gamma^{t_{l-2}}_{x_{l-1},x_l}.$$

Proposition

Let $\mathcal{Y}_1$ be a copy of Y.
Let $\Phi_{\pi_1^\varphi}$ be the conc*-morphism of algebras defined as follows :*

$$\begin{aligned} \Phi_{\pi_1^\varphi} : A\langle\mathcal{Y}_1\rangle &\longrightarrow A\langle Y\rangle, \\ y_{y_k} &\longmapsto \Phi_{\pi_1^\varphi}(y_{y_k}) = \pi_1^\varphi(y_k). \end{aligned}$$

Then $\Phi_{\pi_1^\varphi}$ is an isomorphism of bialgebras from $(A\langle\mathcal{Y}_1\rangle, \mathrm{conc}, \Delta_{Ш}, \epsilon_{\mathcal{Y}_1})$ to $(A\langle Y\rangle, \mathrm{conc}, \Delta_{\uplus_\varphi}, \epsilon_Y)$.

PBW-Lyndon basis in $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi}, \epsilon_Y)$

$$\begin{aligned}
\Pi_{y_k} &= \pi_1^{\varphi}(y_k), && \text{for } k \geq 1,\\
\Pi_l &= [\Pi_s, \Pi_r], && \text{for } l \in \mathcal{L}ynY - Y \text{ and } \sigma(l) = (s, r),\\
\Pi_w &= \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}, && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY.
\end{aligned}$$

Proposition

$\{\Pi_w\}_{w\in Y^*}$ *is upper triangular :*

$$\forall w \in Y^*, \qquad \Pi_w = w + \sum_{|v|>|w|} c_v v,$$

In particular,

$$\forall l \in \mathcal{L}ynY, \qquad \Pi_l = l + \sum_{|v|>|l|} c_v v.$$

Theorem

The free associative algebra $A\langle Y\rangle$ is isomorphic to $\mathcal{U}(\mathcal{P}_Y)$.
$\mathcal{P}_Y$ as a A-vector space (A is here supposed to be a field of zero characteristic) is freely generated by $\{\Pi_l\}_{l\in\mathcal{L}ynY}$.

Dual basis in $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus_\varphi}, \epsilon_Y)$

Here, φ is supposed dualizable and the elements $\{\Sigma_w\}_{w\in Y^*}$ (computed below) are supposed to be polynomials.

$\{\Sigma_w\}_{w\in Y^*}$ = dual basis of $\{\Pi_w\}_{w\in Y^*}$: $\forall u,v\in Y^*, \langle\Sigma_v|\Pi_u\rangle=\delta_{u,v}$.

For any $w=l_1^{i_1}\dots l_k^{i_k}$, with $l_1,\dots,l_k\in\mathcal{L}ynY$ and $l_1>\dots>l_k$,

$$\Sigma_w=\frac{1}{i_1!\dots i_k!}\Sigma_{l_1}^{\uplus_\varphi i_1}\uplus_\varphi\dots\uplus_\varphi\Sigma_{l_k}^{\uplus_\varphi i_k}.$$

Proposition

$\{\Sigma_w\}_{w\in Y^*}$ is lower triangular :

$$\forall w\in Y^*,\qquad \Sigma_w=w+\sum_{|v|<|w|}d_v v,$$

And, in particular,

$$\forall l\in\mathcal{L}ynY,\qquad \Sigma_l=l+\sum_{|v|<|l|}c_v v.$$

Theorem

Let $\varphi: AY\otimes AY\longrightarrow AY$ be an associative and commutative law. Then $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ and $\{\Sigma_w\}_{w\in Y^*}$ are respectively (pure) transcendence and linear bases of $(A\langle Y\rangle, \uplus_\varphi, 1_{Y^*})$.

φ-extended Schützenberger's factorization

Theorem

Let $\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w \in A\langle Y\rangle \hat{\otimes} A\langle Y\rangle$.

Then $\mathcal{D}_Y = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l)$.

Corollary

1. *The families* $\{\Sigma_{l_2}^{\uplus_\varphi i_2} \uplus_\varphi \ldots \uplus_\varphi \Sigma_{l_k}^{\uplus_\varphi i_k}\}_{l_2,\ldots,l_k \in \mathcal{L}ynY}^{i_2,\ldots,i_k \geq 1}$ *and* $\{l_2^{\uplus_\varphi i_2} \uplus_\varphi \ldots \uplus_\varphi l_k^{\uplus_\varphi i_k}\}_{l_2,\ldots,l_k \in \mathcal{L}ynY}^{i_2,\ldots,i_k \geq 1}$ *form bases for* $\mathcal{I}_Y$.
2. *The family* $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ *is a transcendence basis of* $A\langle Y\rangle$.
3. *The family* $\{\Pi_{l_1}^{\uplus_\varphi i_1} \uplus_\varphi \ldots \uplus_\varphi \Pi_{l_k}^{\uplus_\varphi i_k}\}_{l_1,\ldots,l_k \in \mathcal{L}ynY}^{i_1,\ldots,i_k \geq 1}$ *is a linear basis of* $A\langle Y\rangle$.
4. $\mathcal{I}_Y = \bigoplus_{k \geq 2} \mathcal{P}_Y^{\uplus_\varphi k}$.

Lorsque le bon point de vue est poursuivi, la raison est inondée de lumière,
et les conséquences qu'elle en tire ont alors le caractère de l'évidence.
Néanmoins elle n'a pas la force d'atteindre d'un seul coup ce point de vue.
Hermann Weyl

BACK TO POLYLOGARITHMS AND TO HARMONIC SUMS

Polylogarithms and iterated path integrals

The iterated integral, associated to $x_{i_1}\cdots x_{i_k}\in X^*$, along the path $z_0 \rightsquigarrow z$ and over the differential forms $\omega_0(z)=\dfrac{dz}{z}$ and $\omega_1(z)=\dfrac{dz}{1-z}$, is defined, on any appropiate simply connected domain, as follows

$$\alpha_{z_0}^z(1_{X^*})=1 \quad \text{and} \quad \alpha_{z_0}^z(x_{i_1}\ldots x_{i_k})=\int_{z_0}^z \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).$$

For any $u,v\in X^*$, one has $\alpha_{z_0}^z(u ш v)=\alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$.
The polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is well defined and its Taylor expansion is

$$\mathrm{Li}_{s_1,\ldots,s_r}(z):=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}=\alpha_0^z(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1), \quad \text{for} \quad |z|<1.$$

Putting $\mathrm{Li}_{x_0^k}(z)=\log^k z/k!$ and using the correspondence

$$\mathbf{s}=(s_1,\ldots,s_r)\leftrightarrow u=y_{s_1}\ldots y_{s_r}\underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} v=x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1,$$

one has $\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}\cong\mathbb{Q}\langle X\rangle$ and

$$(\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*},\times,1)\cong(\mathbb{Q}\langle X\rangle, ш, 1_{X^*})\cong(\mathbb{Q}[\mathcal{L}ynX], ш, 1_{X^*}).$$

Harmonic sums and quasi-symmetric monomial functions

For any $w \in Y^*$ associated to $\mathbf{s}$, the quasi-symmetric monomial function, on the commutative alphabet $\mathbf{t} = \{t_i\}_{i\geq 1}$, is defined as follows

$$M_w(\mathbf{t}) = M_{\mathbf{s}}(\mathbf{t}) = \sum_{n_1>\ldots>n_k>0} t_{n_1}^{s_1}\ldots t_{n_k}^{s_k} \quad \text{and} \quad M_{1_{Y^*}}(\mathbf{t}) = M_{\emptyset}(\mathbf{t}) = 1.$$

One has, for any $u, v \in Y^*$, $M_u(\mathbf{t})M_v(\mathbf{t}) = M_{u \uplus v}(\mathbf{t})$.
The multiple harmonic sum

$$\mathrm{H}_{\mathbf{s}}(N) = \mathrm{H}_w(N) := \sum_{N\geq n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}$$

is obtained by specializing the indeterminates $\mathbf{t} = \{t_i\}_{i\geq 1}$ in the quasi-symmetric monomial function $M_{\mathbf{s}}(\mathbf{t}) = M_w(\mathbf{t})$ as follows :

$$t_i = i^{-1} \quad \text{and} \quad \forall i > N, t_i = 0.$$

Then one has $\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*} \cong \mathbb{Q}\langle Y\rangle$ and

$$(\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1) \cong (\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}) \cong (\mathbb{Q}[\mathcal{L}ynY], \uplus, 1_{Y^*}).$$

Let $N, r > 0$. Let w associated to $(s_1, \ldots, s_r)$. If $s_1 > 1$ then

$$\lim_{z\to 1} \mathrm{Li}_w(z) = \lim_{N\to\infty} \mathrm{H}_w(N) = \zeta(w) = \sum_{n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}.$$

Abel like theorem for noncommutative generating series

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w \; w = (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l}, \qquad \Delta_{ш}\mathrm{L} = \mathrm{L}\hat{\otimes}\mathrm{L}.$$

$$Z_{ш} := \prod_{l\in\mathcal{L}ynX - X}^{\searrow} e^{\zeta(S_l)P_l}, \qquad \Delta_{ш} Z_{ш} = Z_{ш}\hat{\otimes} Z_{ш}.$$

$$\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w \; w = (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}, \qquad \Delta_{\uplus}\mathrm{H} = \mathrm{H}\hat{\otimes}\mathrm{H}.$$

$$Z_{\uplus} := \prod_{l\in\mathcal{L}ynY - \{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}, \qquad \Delta_{\uplus} Z_{\uplus} = Z_{\uplus}\hat{\otimes} Z_{\uplus}.$$

Theorem (HNM, 2005)

$$\lim_{z\to 1} \exp\left[-y_1 \log\frac{1}{1-z}\right]\pi_Y \mathrm{L}(z) = \lim_{N\to\infty} \exp\left[\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\mathrm{H}(N) = \pi_Y Z_{ш}.$$

Or equivalently,

$$\mathrm{L}(z) \underset{z\to 1}{\widetilde{\ \ }} \exp\left[x_1 \log\frac{1}{1-z}\right] Z_{ш} \text{ and } \mathrm{H}(N) \underset{N\to\infty}{\widetilde{\ \ }} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\pi_Y Z_{ш}.$$

Finite parts of divergent polyzetas

For any $w \in Y^*$ and $k \in \mathbb{N}_+$, there exists $\gamma_w, \alpha_i, \beta_{i,j} \in \mathcal{Z}[\gamma]$, such that

$$\mathrm{H}_w(N) = \sum_{i=1}^{|w|} \alpha_i \log^i(N) + \gamma_w + \sum_{j=1}^{k} \sum_{i=0}^{|w|-1} \beta_{i,j} \frac{1}{N^j} \log^i(N) + \mathrm{O}\left(\frac{1}{N^k}\right).$$

Let us define then

$$Z_\gamma := \sum_{w \in Y^*} \gamma_w \, w.$$

$\gamma_\bullet$ is a character and Z_γ is then group-like, for $\Delta_{\uplus}$.
The monoidal factorization yields

$$Z_\gamma = e^{\gamma y_1} Z_{\uplus} \quad \text{and} \quad Z_\gamma = \Gamma(y_1 + 1) \pi_Y Z_{\text{ш}}.$$

Hence, by cancellation, we get finally

$$Z_{\uplus} = B'(y_1) \pi_Y Z_{\text{ш}}, \quad \text{where} \quad B'(y_1) = 1 + \sum_{m \geq 2} B^{(m)} y_1^m$$

and for any $m \geq 2$,

$$B^{(m)} = \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{\substack{k_1, \ldots, k_i \geq 2 \\ k_1 + \ldots + k_i = m}} (-1)^{m-i} \frac{\zeta(k_1) \ldots \zeta(k_i)}{k_1 \ldots k_i}.$$

Homogenous algebraic relations among local coordinates

	Relations among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Relations among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

Homogenous rewriting systems among local coordinates

	Rewriting among $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY - \{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX - X}$
3	$\zeta(\Sigma_{y_2 y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0 x_1^2}) \to \zeta(S_{x_0^2 x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3 x_1}) \to \frac{2}{5}\zeta(S_{x_0 x_1})^2$
	$\zeta(\Sigma_{y_3 y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2 x_1^2}) \to \frac{1}{10}\zeta(S_{x_0 x_1})^2$
	$\zeta(\Sigma_{y_2 y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0 x_1^3}) \to \frac{2}{5}\zeta(S_{x_0 x_1})^2$
5	$\zeta(\Sigma_{y_3 y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3 x_1^2}) \to -\zeta(S_{x_0^2 x_1})\zeta(S_{x_0 x_1}) + 2\zeta(S_{x_0^4 x_1})$
	$\zeta(\Sigma_{y_4 y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2 x_1 x_0 x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4 x_1}) + \zeta(S_{x_0^2 x_1})\zeta(S_{x_0 x_1})$
	$\zeta(\Sigma_{y_2^2 y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2 x_1^3}) \to -\zeta(S_{x_0^2 x_1})\zeta(S_{x_0 x_1}) + 2\zeta(S_{x_0^4 x_1})$
	$\zeta(\Sigma_{y_3 y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0 x_1 x_0 x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4 x_1})$
	$\zeta(\Sigma_{y_2 y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0 x_1^4}) \to \zeta(S_{x_0^4 x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5 x_1}) \to \frac{8}{35}\zeta(S_{x_0 x_1})^3$
	$\zeta(\Sigma_{y_4 y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4 x_1^2}) \to \frac{6}{35}\zeta(S_{x_0 x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2 x_1})^2$
	$\zeta(\Sigma_{y_5 y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3 x_1 x_0 x_1}) \to \frac{4}{105}\zeta(S_{x_0 x_1})^3$
	$\zeta(\Sigma_{y_3 y_1 y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3 x_1^3}) \to \frac{23}{70}\zeta(S_{x_0 x_1})^3 - \zeta(S_{x_0^2 x_1})^2$
	$\zeta(\Sigma_{y_3 y_2 y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2 x_1 x_0 x_1^2}) \to \frac{2}{105}\zeta(S_{x_0 x_1})^3$
	$\zeta(\Sigma_{y_4 y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2 x_1^2 x_0 x_1}) \to -\frac{89}{210}\zeta(S_{x_0 x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2 x_1})^2$
	$\zeta(\Sigma_{y_2^2 y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2 x_1^4}) \to \frac{6}{35}\zeta(S_{x_0 x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2 x_1})^2$
	$\zeta(\Sigma_{y_3 y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0 x_1 x_0 x_1^3}) \to \frac{8}{21}\zeta(S_{x_0 x_1})^3 - \zeta(S_{x_0^2 x_1})^2$
	$\zeta(\Sigma_{y_2 y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0 x_1^5}) \to \frac{8}{35}\zeta(S_{x_0 x_1})^3$

Polyzetas and Zagier's dimension conjecture

The length and the weight of $y_{s_1}\ldots y_{s_r} \in Y^*$ are respectivement the numbers $|w| = r$ and $(w) = s_1 + \ldots + s_r$.

Theorem

$$\{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3 y_1^5}), \zeta(\Sigma_{y_9}),$$
$$\zeta(\Sigma_{y_3 y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2 y_1^9}), \zeta(\Sigma_{y_3 y_1^9}), \zeta(\Sigma_{y_2^2 y_1^8})\}$$

(*resp.* $\{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}),$
$\zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}$

are algebraicly independant if and only if the Zagier's dimension conjecture holds up to weight 12.

Conjecture (Zagier, 1992)

Let $d_n = \dim$ *of* $span_{\mathbb{Q}}\{\zeta(s_1, \ldots, s_r) | s_1 + \ldots + s_r = n\}$.
Then $d_1 = 0,\ d_2 = d_3 = 1$ *and* $\forall n \geq 4, d_n = d_{n-2} + d_{n-3}$.

More about polylogarithms and harmonic sums

Let $\mathrm{Li}_0(z) := \dfrac{z}{1-z}$ and, for any $|z| < 1, N \in \mathbb{N}_+$, let

$$\mathrm{Li}^-_w(z) := \sum_{n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}\ z^{n_1} \quad \text{and} \quad \mathrm{H}^-_w(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}.$$

$$\mathrm{L}^- := \sum_{w \in Y^*} \mathrm{Li}^-_w\ w \quad \text{and} \quad \mathrm{H}^- := \sum_{w \in Y^*} \mathrm{H}^-_w\ w.$$

One has

$$\frac{\mathrm{Li}^-_w(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}^-_w(N)\ z^N,$$

and

$$\mathrm{H}^-(N) = \Big(1 + \sum_{k \geq 1} y_k N^k\Big)\mathrm{H}^-(N-1) = \prod_{n=0}^{N}\Big(1 + \sum_{k \geq 1} y_k n^k\Big) = \sum_{\substack{w \in Y^* \\ |w| \leq N}} \mathrm{H}^-_w(N)\ w.$$

Hence, for any $w \in Y^*$, $\mathrm{H}^-_w(N) = 0$, for $|w| > N$, and $\omega(\mathrm{Li}^-_w) = |w|$.

Extended Eulerian polynomials

Proposition

1. *Let* $\{A^-_w(z)\}_{w\in Y^*}$ *be the polynomials defined recursively as follows*
$$A^-_w(z) = \begin{cases} \sum_{k=0}^{n-1} A_{n,k} z^k & \text{if} \quad w = y_k \in Y, \\ \sum_{i=0}^{s_1} \binom{s_1}{i} A_{y_i} A^-_{y_{(s_1+s_2-i)} y_{s_3} \ldots y_{s_r}} & \text{if} \quad w = y_k u \in YY^*, \end{cases}$$
where $A_{n,k}$ *are Eulerian numbers. Then*
$$\mathrm{Li}^-_w(z) = \left(\frac{z}{1-z}\right)^{|w|} \frac{A^-_w(z)}{(1-z)^{(w)}}.$$

2. *For any* $w \in Y^*$, *let us define* $\{G^-_w(n)\}_{n\in\mathbb{N}}$ *by the following generating series*
$$\sum_{n\geq |w|} \frac{(n+1)!}{(n-|w|)!} G^-_w(n) z^n = \frac{\mathrm{Li}^-_w(z)}{1-z}.$$
Then $\mathrm{H}^-_w(N) = (N+1)N(N-1)\ldots(N-|w|+1)G^-_w(N)$.

3. $\mathrm{Li}^-_w(z) \in \mathbb{Q}[(1-z)^{-1}]$ *and* $\mathrm{H}^-_w(N) \in \mathbb{Q}[N]$ *of degree* $|w| + (w)$ *and of valuation* 1.

Extended Faulhaber's formula

Denoting $B_{y_k}(z)$ is the k-th Bernoulli polynomial, we define the polynomials $\{B_w(z)\}_{w\in Y^+}$ recursively as follows

$$B_w(z+1) = B_w(z) + kz^{k-1}B_u(z) \quad \text{if} \quad w = y_k u \in YY^*.$$

Thus, for any $w = y_k u \in YY^*$ such that $k > 1$, one has $B_w(0) = B_w(1)$.

Proposition

For any $w \in Y^*$, let $b_w := B_w(0)$ and $\beta_w(z) := B_w(z) - b_w$. Then

$$\forall y_{s_1}\ldots y_{s_r} \in Y^*, \qquad \beta_{y_{s_1}\ldots y_{s_r}}(N) = \sum_{k=1}^{r}(\prod_{i=1}^{k} s_i) b_{y_{s_{i+1}}\ldots y_{s_r}} \mathrm{H}^-_{y_{s_1}-1\ldots y_{s_i}-1}(N).$$

Conversely, one also has

$$\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}}(N) = \frac{\beta_{y_{s_1+1}\ldots y_{s_r+1}}(N+1) - \sum_{i=1}^{r-1} b_{y_{s_{i+1}+1}\ldots y_{s_r+1}} \beta_{y_{s_1+1}\ldots y_{s_i+1}}(N+1)}{\prod_{k=1}^{r}(s_k+1)}.$$

Examples by computer (1/2)

Example $(r = 1)$

$$\mathrm{Li}_1^-(z) = \frac{z}{(1-z)^2} = -\frac{1}{1-z} + \frac{1}{(1-z)^2}.$$

$$\mathrm{Li}_2^-(z) = \frac{z(z+1)}{(1-z)^3} = \frac{1}{1-z} - \frac{3}{(1-z)^2} + \frac{2}{(1-z)^3}.$$

$$\mathrm{Li}_3^-(z) = \frac{z(z^2+4z+1)}{(1-z)^4} = -\frac{1}{1-z} + \frac{7}{(1-z)^2} - \frac{12}{(1-z)^3} + \frac{20}{(1-z)^4}.$$

Example $(r = 2)$

$$\begin{aligned}
\mathrm{Li}_{1,1}^-(z) &= \mathrm{Li}_0(z)\,\mathrm{Li}_2^-(z) + (\mathrm{Li}_{-1}^-(z))^2 \\
&= -\frac{1}{1-z} + \frac{3}{(1-z)^2} + \frac{3}{(1-z)^3} - \frac{1}{(1-z)^4} \\
\mathrm{Li}_{2,1}^-(z) &= \mathrm{Li}_0(z)\,\mathrm{Li}_3^-(z) + 3\,\mathrm{Li}_1^-(z)\,\mathrm{Li}_2^-(z) \\
&= \frac{1}{1-z} - \frac{9}{(1-z)^2} + \frac{17}{(1-z)^3} - \frac{23}{(1-z)^4} - \frac{14}{(1-z)^5}, \\
\mathrm{Li}_{1,2}^-(z) &= \mathrm{Li}_0(z)\,\mathrm{Li}_3^-(z) + \mathrm{Li}_1^-(z)\,\mathrm{Li}_2^-(z) \\
&= \frac{1}{1-z} - \frac{7}{(1-z)^2} + \frac{9}{(1-z)^3} - \frac{13}{(1-z)^4} - \frac{18}{(1-z)^5}.
\end{aligned}$$

Examples by computer (2/2)

Example ($r = 1$)

$$
\begin{aligned}
\mathrm{H}_1^-(N) &= \frac{(N+1)^2}{2} - \frac{N+1}{2} &&= \frac{1}{2}N(N+1),\\
\mathrm{H}_2^-(N) &= \frac{1}{3}(N+1)^3 - \frac{1}{2}(N+1)^2 + \frac{1}{6}(N+1) &&= \frac{1}{6}N(2N+1)(N+1),\\
\mathrm{H}_3^-(N) &= \frac{1}{4}(N+1)^4 - \frac{1}{2}(N+1)^3 + \frac{1}{4}(N+1)^2 &&= \left(\frac{1}{2}N(N+1)\right)^2.
\end{aligned}
$$

Example ($r = 2$)

$$
\begin{aligned}
\mathrm{H}_{2,1}^-(N) &= \frac{1}{120}N(N^2-1)(10N^2+15N+2),\\
\mathrm{H}_{2,2}^-(N) &= \frac{1}{180}N(10N^5+12N^4-10N^3-35N^2+5N+3),\\
\mathrm{H}_{2,3}^-(N) &= \frac{1}{8}N^2(N-1)(2N^2+N-2)(N+1)^2,\\
\mathrm{H}_{2,4}^-(N) &= \frac{1}{60}N(N+1)(12N^6+6N^5-35N^4-10N^3+30N^2+N-5),\\
\mathrm{H}_{2,5}^-(N) &= \frac{1}{120}N(N-1)(N+1)(20N^6+30N^5-50N^4-75N^3+31N^2\\
&\qquad +40N-4),\\
\mathrm{H}_{3,3}^-(N) &= \frac{1}{24}N(N-1)(N+1)(6N^5+12N^4-2N^3-12N^2-N+2),\\
\mathrm{H}_{5,6}^-(N) &= \frac{1}{1764}N(N+1)(252N^{11}+504N^{10}-1638N^9-3213N^8+4998N^7\\
&\qquad +8232N^6-9360N^5-9897N^4+10170N^3+4383N^2-4411N+1).
\end{aligned}
$$

More on structures

Theorem

1. $\{\mathrm{H}^-_{y_k}\}_{k\geq 1}$ *(resp.* $\{\mathrm{Li}^-_{y_k}\}_{k\geq 1}$*) are linearly independent.*
2. *There is a law of algebra* $\top$ *in* $\mathbb{Q}\langle Y\rangle$*, such that following maps are morphisms of algebras*

$$\begin{aligned}
\mathrm{H}^-_\bullet : (\mathbb{Q}\langle Y\rangle, \,\text{⧢}\,) &\longrightarrow (\mathbb{Q}\{\mathrm{H}^-_w\}_{w\in Y^*}, .), \qquad w \longmapsto \mathrm{H}^-_w,\\
\mathrm{L}^-_\bullet : (\mathbb{Q}\langle Y\rangle, \top) &\longrightarrow (\mathbb{Q}\{\mathrm{L}^-_w\}_{w\in Y^*}, .), \qquad w \longmapsto \mathrm{Li}^-_w .
\end{aligned}$$

   and, moreover, $\ker \mathrm{H}^-_\bullet = \ker \mathrm{Li}^-_\bullet = \mathbb{Q}\{w - w\top 1^*_Y | w \in Y^*\}$.
3. H^- *is group-like, for* $\Delta_{\,\text{⧢}}$.
4. *For any* $w \in Y^*$*, let* $C^-_w = \prod_{w=uv, v\neq 1_{Y^*}} \frac{1}{(v) + |v|} \in \mathbb{Q}$*. Then*

$$\begin{aligned}
\mathrm{L}^-(z) &\underset{z\to 1}{\widetilde{\quad}} \sum_{w\in Y^*} \frac{(|w| + (w))!}{(1-z)^{|w|+(w)}} C^-_w \; w,\\
\mathrm{H}^-(N) &\underset{N\to+\infty}{\widetilde{\quad}} \sum_{w\in Y^*} N^{|w|+(w)} C^-_w \; w.
\end{aligned}$$

Les choses l'univers ne se lassent jamais de parler d'elles-même
et de se révéler, à celui qui se soucie d'entendre.
Alexandre Grothendieck

THANK YOU FOR YOUR ATTENTION