A family of posets defined from acyclic digraphs.

François Viard

ICJ, Lyon

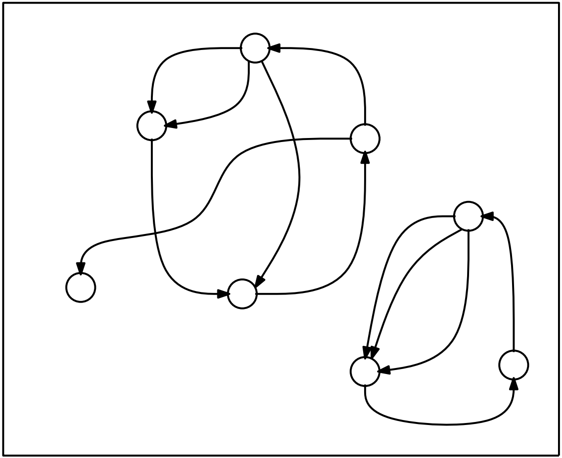
Mars 2015, SLC

Plan

1. Some definitions about digraphs and posets
2. Definition of the posets $\mathcal{P}(\mathcal{G})$
3. Applications (with emphasis on the weak order on the symmetric group)

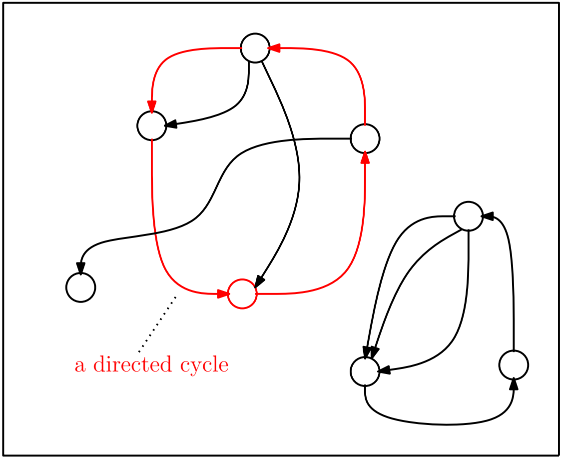
Simple acyclic digraphs: definition

$G = (V, E)$	V:= Set of vertices, E:= set of oriented edges (arcs).



Simple acyclic digraphs: definition

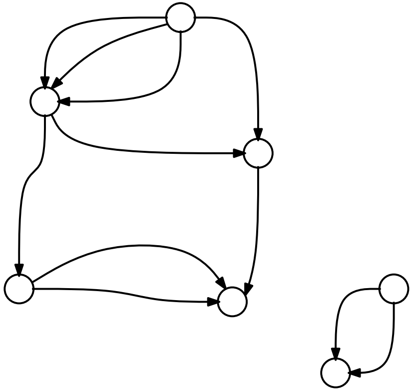
$G = (V, E)$	V:= Set of vertices, E:= set of oriented edges (arcs).



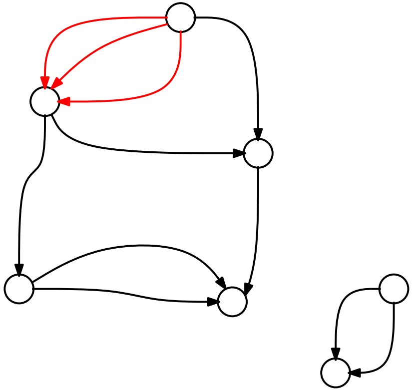
Acyclic :=

It is a digraph with no directed cycles.

Simple acyclic digraphs: definition

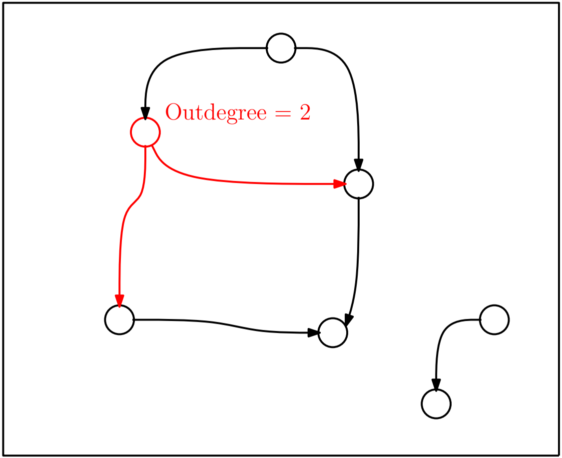
$G = (V, E)$	$V :=$ Set of vertices, $E :=$ set of oriented edges (arcs).
	Acyclic := It is a digraph with no directed cycles.

Simple acyclic digraphs: definition

$G = (V, E)$	V:= Set of vertices, E:= set of oriented edges (arcs).
	Acyclic := It is a digraph with no directed cycles. Simple := no more than one arc between two different vertices.

Simple acyclic digraphs: definition

$G = (V, E)$	V:= Set of vertices, E:= set of oriented edges (arcs).



Acyclic :=

It is a digraph with no directed cycles.

Simple :=

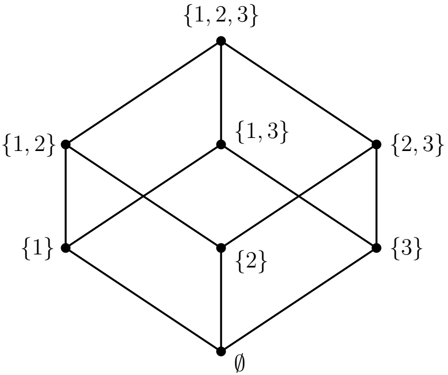
no more than one arc between two different vertices.

Outdegree of a vertex z :=

number of arcs which have z as starting point.

Some definitions about posets

$\mathcal{P} = (P, \leq)$	"$\leq$" is a *reflexive*, *antisymmetric* and *transitive* binary relation.

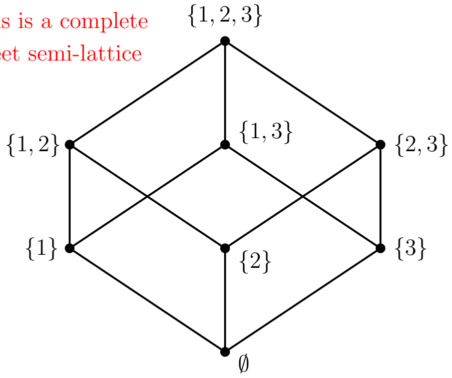


Subsets of $\{1, 2, 3\}$ ordered by inclusion.

Some definitions about posets

$\mathcal{P} = (P, \leq)$	"$\leq$" is a *reflexive*, *antisymmetric* and *transitive* binary relation.

This is a complete meet semi-lattice



Subsets of $\{1, 2, 3\}$ ordered by inclusion.

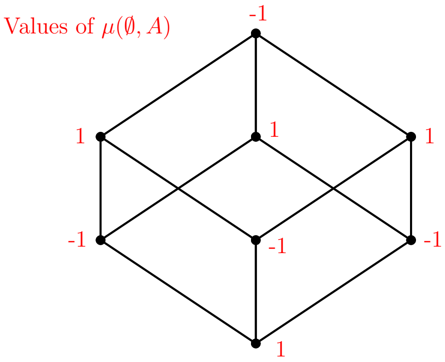
Complete meet semi-lattice :=

Any subset $S \subseteq P$ has an infimum (*i.e.* there exists z such that: $\forall y \in S$, $z \leq y$, and if $\forall y \in S$, $x \leq y$ then $x \leq z$)

Some definitions about posets

$\mathcal{P} = (P, \leq)$

"$\leq$" is a *reflexive*, *antisymmetric* and *transitive* binary relation.



Subsets of $\{1, 2, 3\}$ ordered by inclusion.

Complete meet semi-lattice :=

Any subset $S \subseteq P$ has an infimum (*i.e.* there exists z such that: $\forall y \in S,\ z \leq y$, and if $\forall y \in S,\ x \leq y$ then $x \leq z$)

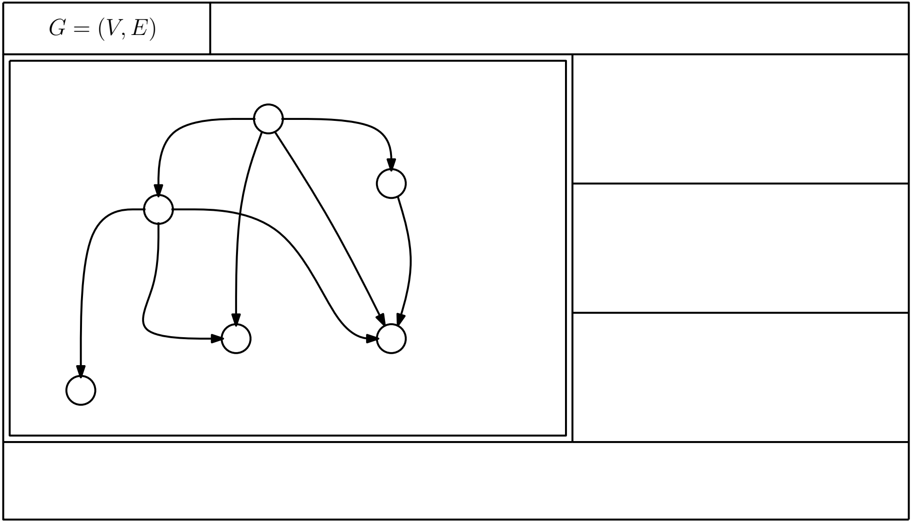
Möbius function of $\mathcal{P}$:=

Recursively defined by:

1) $\forall x \in P,\ \mu(x, x) = 1$;

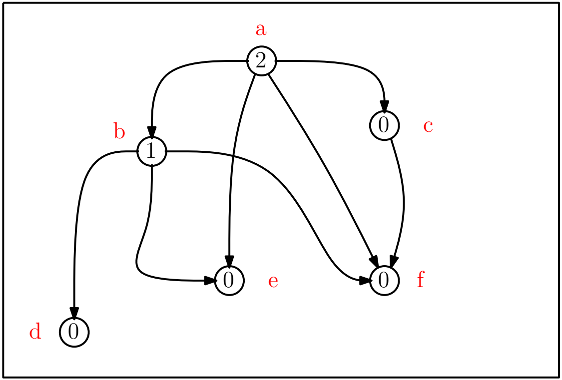
2) $\mu(x, y) = -\sum_{x \leq c < y} \mu(x, c)$.

The peeling process



The peeling process

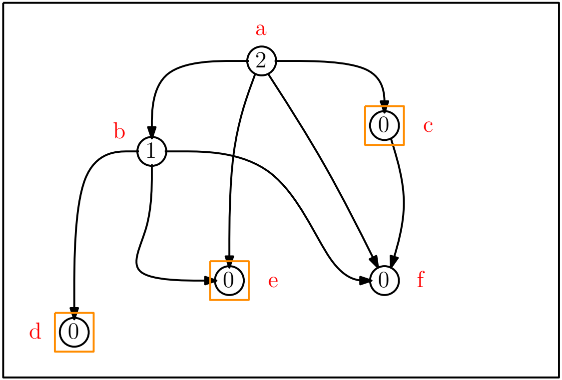
$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



$L = [\,]$

The peeling process

$G = (V, E)$	Consider $\theta : V \rightarrow \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :

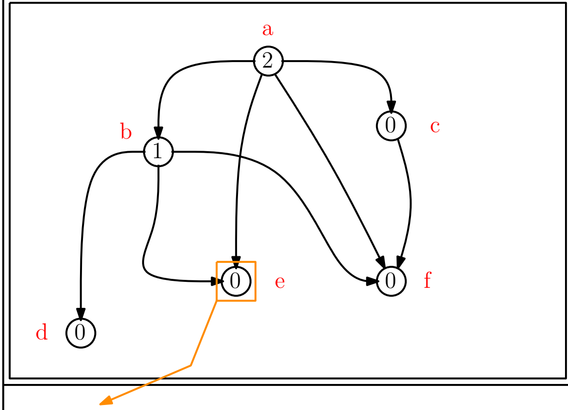
1) $\theta_z = 0$

2) if $(y, z) \in E$, then $\theta_y \neq 0$.

$L = [\,]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

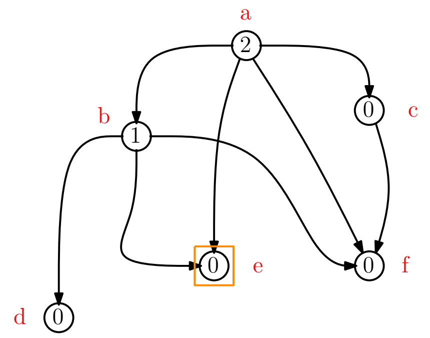
Choose one of these vertices and add it to the list.

$L = [e]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :

1) $\theta_z = 0$

2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

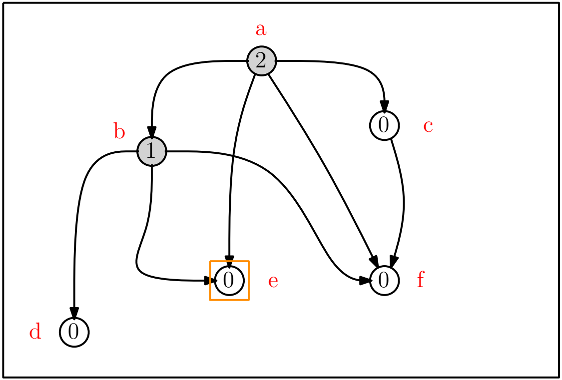
Step 3 :

Peeling process !

$L = [e]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :
Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :
Choose one of these vertices and add it to the list.

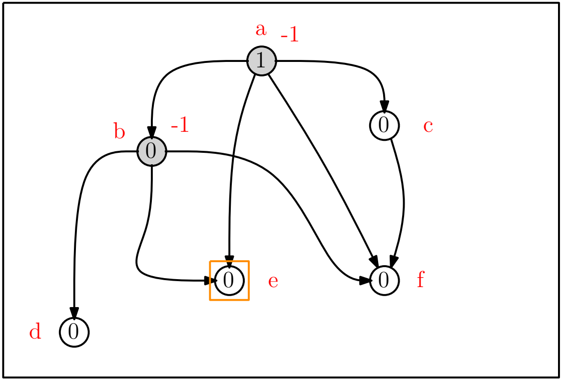
Step 3 :
Peeling process !

$L = [e]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

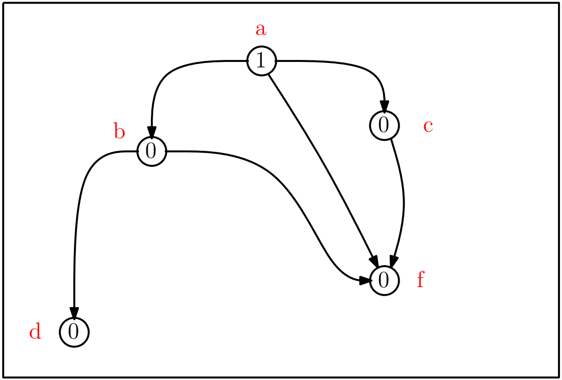
Step 3 :

Peeling process !

$L = [e]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

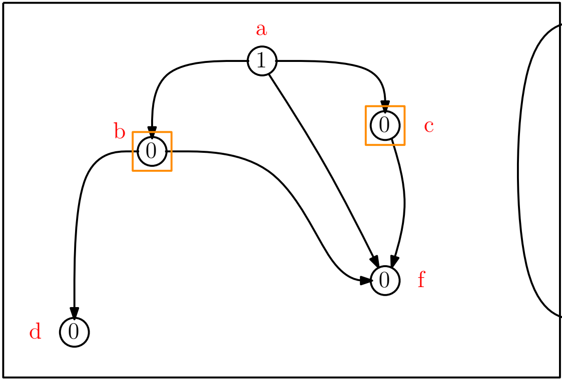
Peeling process !

$L = [e]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :

1) $\theta_z = 0$

2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

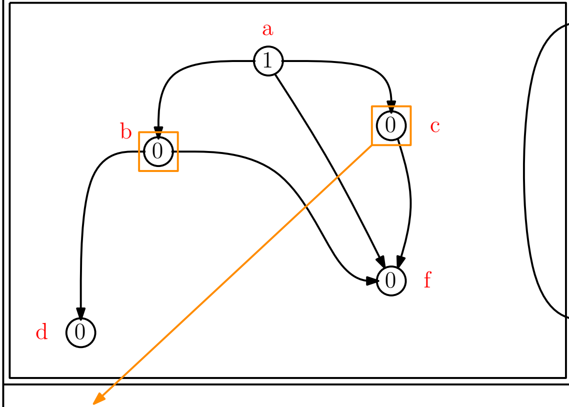
Peeling process !

$L = [e]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :

1) $\theta_z = 0$

2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

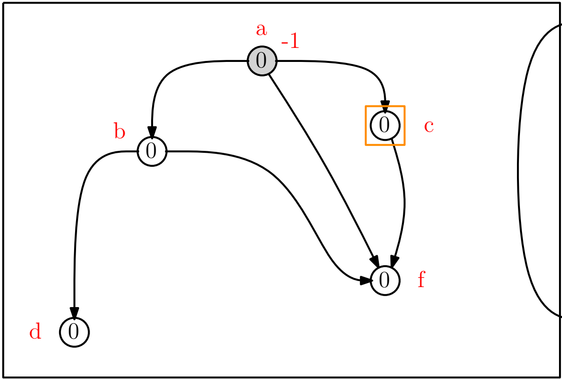
Step 3 :

Peeling process !

$L = [e, c]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

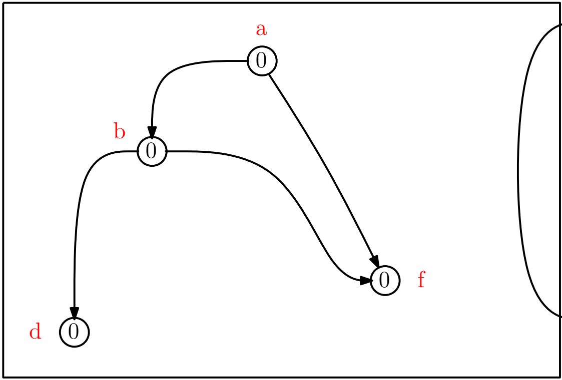
Step 3 :

Peeling process !

$L = [e, c]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

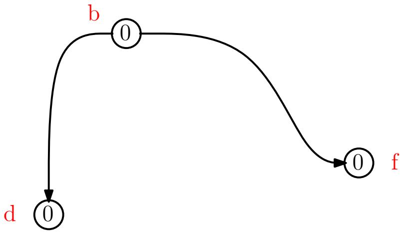
Step 3 :

Peeling process !

$L = [e, c]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

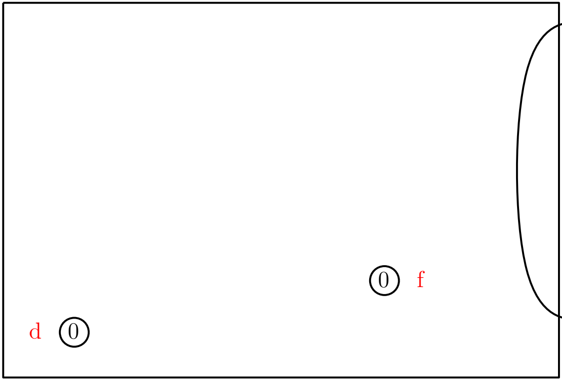
Peeling process !

$L = [e, c, a]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

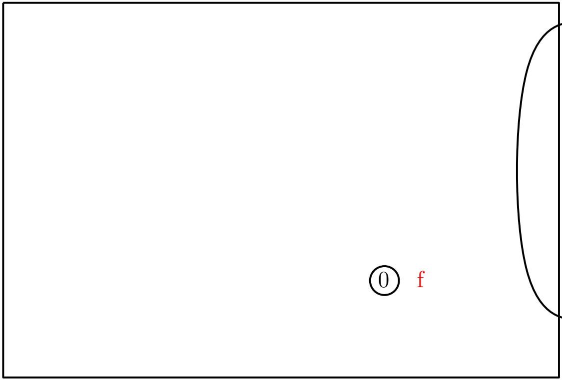
Peeling process !

$L = [e, c, a, b]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.



Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

Peeling process !

$L = [e, c, a, b, d]$

The peeling process

$G = (V, E)$	Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.

Step 1 :

Consider each $z \in V$ such that :
1) $\theta_z = 0$
2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

Peeling process !

$L = [e, c, a, b, d, f]$

The peeling process

$G = (V, E)$

Consider $\theta : V \to \mathbb{N}$ such that $\forall z \in V$, $0 \leq \theta_z \leq$ outdegree of z.

Step 1 :

Consider each $z \in V$ such that :

1) $\theta_z = 0$

2) if $(y, z) \in E$, then $\theta_y \neq 0$.

Step 2 :

Choose one of these vertices and add it to the list.

Step 3 :

Peeling process !

$L = [e, c, a, b, d, f]$ ⟵ Such a sequence is called a *peeling sequence* of (G, θ).

Initial sections of a peeling sequence and definition of $\mathcal{P}(\mathcal{G})$

- Consider $L = [e, c, a, b, d, f]$ the previous peeling sequence. The initial sections of L are the following sets

$$L_0 = \emptyset,\ L_1 = \{e\},\ L_2 = \{e, c\}, \ldots,\ L_6 = \{e, c, a, b, d, f\}.$$

Clearly, two different peeling sequences give rise to different initial sections (at least some of them).

Initial sections of a peeling sequence and definition of $\mathcal{P}(\mathcal{G})$

- Consider $L = [e, c, a, b, d, f]$ the previous peeling sequence. The initial sections of L are the following sets

$$L_0 = \emptyset,\ L_1 = \{e\},\ L_2 = \{e, c\}, \ldots,\ L_6 = \{e, c, a, b, d, f\}.$$

Clearly, two different peeling sequences give rise to different initial sections (at least some of them).

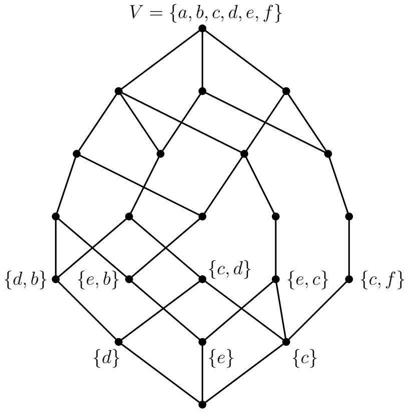
Definition

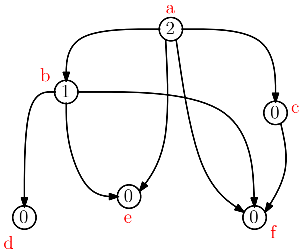
Set $\mathcal{G} = (G, \theta)$ a pair of a simple acyclic digraph and a "compatible" valuation on its vertices. We denote $IS(\mathcal{G})$ the set constituted of all the initial sections of all the peeling sequences of $\mathcal{G}$, and finally we denote $\mathcal{P}(\mathcal{G}) = (IS(\mathcal{G}), \subseteq)$ obtained by ordering $IS(\mathcal{G})$ by inclusion.

First result

Theorem, (V, 2014)

For all pair $\mathcal{G} = (G, \theta)$, the poset $\mathcal{P}(\mathcal{G})$ is a complete meet semi-lattice. Furthermore, if G is finite, it is a complete lattice with V as maximal element.





Möbius function of $\mathcal{P}(\mathcal{G})$

Definition

Set $A \in IS(\mathcal{G})$, we define:

- $\mathcal{N}(A) = \{z \in A \mid \theta_z = 0\}$;
- $\mathcal{F}(A) = \{z \in A \mid A \setminus \{z\} \in IS(\mathcal{G})\}$.

Möbius function of $\mathcal{P}(\mathcal{G})$

Definition

Set $A \in IS(\mathcal{G})$, we define:

- $\mathcal{N}(A) = \{z \in A \mid \theta_z = 0\}$;
- $\mathcal{F}(A) = \{z \in A \mid A \setminus \{z\} \in IS(\mathcal{G})\}$.

Theorem, V. 2014

We have the two following cases:

- if $\mathcal{N}(A) = \mathcal{F}(A)$, then $\mu(\emptyset, A) = (-1)^{|\mathcal{N}(A)|}$;
- otherwise, $\mu(\emptyset, A) = 0$.

The weak order on the symmetric group

The symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$, which exchange the integers i and $i+1$.

The weak order on the symmetric group

The symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$, which exchange the integers i and $i+1$.

- Each permutation $\sigma \in S_n$ can be written as a product of a minimal number of simple transpositions. This minimal number is denoted $\ell(\sigma)$ and is called the length of σ.

The weak order on the symmetric group

The symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$, which exchange the integers i and $i+1$.

- Each permutation $\sigma \in S_n$ can be written as a product of a minimal number of simple transpositions. This minimal number is denoted $\ell(\sigma)$ and is called the length of σ.
- We define the weak order $\leq_R$ on S_n as follows: we say that $\sigma \leq_R \omega$ if and only if there exists $s_{i_1}, \ldots, s_{i_k}$ such that $\omega = \sigma s_{i_1} \cdots s_{i_k}$, and $\ell(\sigma) + k = \ell(\omega)$.

Classical property

The poset $(S_n, \leq_R)$ is a complete lattice, and its möbius function takes values in $\{1, 0, -1\}$.

The weak order on the symmetric group

Definition/property

For all $\sigma \in S_n$, we define the *inversion set* of σ as:

$$\mathrm{Inv}(\sigma) = \{(a, b) \mid 1 \leq a < b \leq n,\ \sigma^{-1}(a) > \sigma^{-1}(b)\}.$$

Then for all σ and ω in S_n, $\sigma \leq_R \omega$ if and only if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega)$.

The weak order on the symmetric group

Definition/property

For all $\sigma \in S_n$, we define the *inversion set* of σ as:

$$\mathrm{Inv}(\sigma) = \{(a, b) \mid 1 \leq a < b \leq n,\ \sigma^{-1}(a) > \sigma^{-1}(b)\}.$$

Then for all σ and ω in S_n, $\sigma \leq_R \omega$ if and only if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega)$.

Main idea

Find a pair $\mathcal{G} = (G, \theta)$ such that:

- the vertices of G are indexed by the couples of integers (a, b) such that $1 \leq a < b \leq n$;

The weak order on the symmetric group

Definition/property

For all $\sigma \in S_n$, we define the *inversion set* of σ as:

$$\mathrm{Inv}(\sigma) = \{(a, b) \mid 1 \leq a < b \leq n,\ \sigma^{-1}(a) > \sigma^{-1}(b)\}.$$

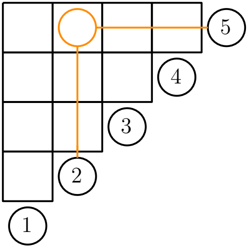
Then for all σ and ω in S_n, $\sigma \leq_R \omega$ if and only if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\omega)$.

Main idea

Find a pair $\mathcal{G} = (G, \theta)$ such that:

- the vertices of G are indexed by the couples of integers (a, b) such that $1 \leq a < b \leq n$;
- the elements of $IS(\mathcal{G})$ are precisely the set of the form $\mathrm{Inv}(\sigma)$, $\sigma \in S_n$.

The weak order on the symmetric group



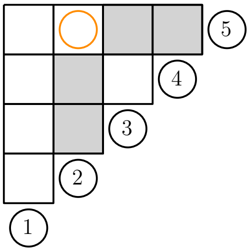
This box represents the couple $(2, 5)$

One can easilly represent the set

$$\{(a, b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

The weak order on the symmetric group



Hook based on the box $(2, 5)$

One can easilly represent the set

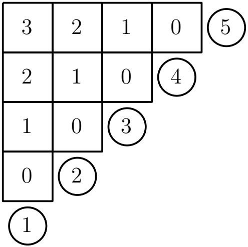
$$\{(a, b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The weak order on the symmetric group



Values of the valuation θ

One can easilly represent the set

$$\{(a,b) \mid 1 \le a < b \le n\}$$

as a staircase tableau.

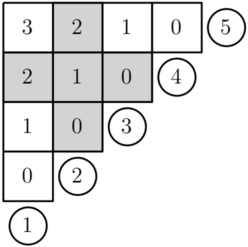
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\text{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\text{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:

$\text{Inv}([4, 1, 3, 5, 2]) = \{(1, 4), (2, 3), (2, 4), (2, 5), (3, 4)\}$

One can easilly represent the set

$$\{(a, b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

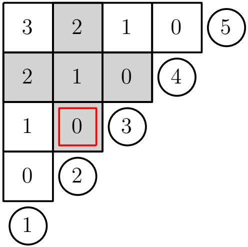
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\text{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\text{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:
$\text{Inv}([4,1,3,5,2]) = \{(1,4),(2,3),(2,4),(2,5),(3,4)\}$

One can easilly represent the set

$$\{(a,b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

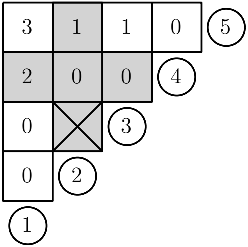
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\text{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:
$\mathrm{Inv}([4, 1, 3, 5, 2]) = \{(1, 4), (2, 3), (2, 4), (2, 5), (3, 4)\}$

One can easilly represent the set

$$\{(a, b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

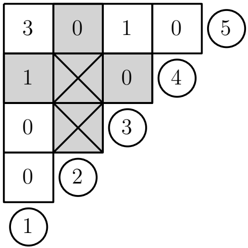
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\mathrm{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:

$\mathrm{Inv}([4,1,3,5,2]) = \{(1,4),(2,3),(2,4),(2,5),(3,4)\}$

One can easilly represent the set

$\{(a,b) \mid 1 \leq a < b \leq n\}$

as a staircase tableau.

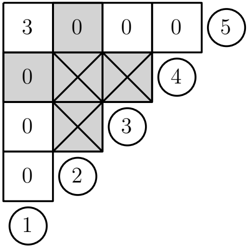
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\mathrm{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:
$\mathrm{Inv}([4,1,3,5,2]) = \{(1,4),(2,3),(2,4),(2,5),(3,4)\}$

One can easilly represent the set

$$\{(a,b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

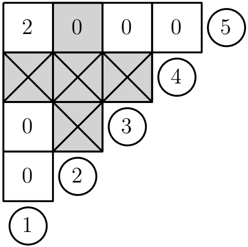
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\mathrm{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\mathrm{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:
$\mathrm{Inv}([4,1,3,5,2]) = \{(1,4),(2,3),(2,4),(2,5),(3,4)\}$

One can easilly represent the set

$$\{(a,b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

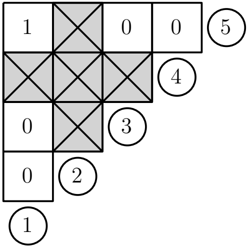
We implement an (implicit) digraph structure on this diagram.

We say that there is an arc from c to d iff d is in the hook based on c.

The outdegree of any box is an even number.

We set $\theta_c = \frac{\mathrm{outdegree}(c)}{2}$.

The weak order on the symmetric group



Denote $\mathcal{A} = (G, \theta)$ the obtained pair.

We have that $IS(\mathcal{A}) = \{\text{Inv}(\sigma) \mid \sigma \in S_n\}$

Example:

$\text{Inv}([4,1,3,5,2]) = \{(1,4),(2,3),(2,4),(2,5),(3,4)\}$

One can easilly represent the set

$$\{(a,b) \mid 1 \leq a < b \leq n\}$$

as a staircase tableau.

We implement an (implicit) digraph structure on this diagram.

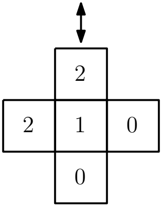
We say that there is an arc from c to d iff d is in the hook based on c.

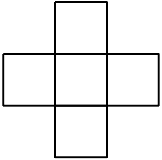
The outdegree of any box is an even number.

We set $\theta_c = \frac{\text{outdegree}(c)}{2}$.

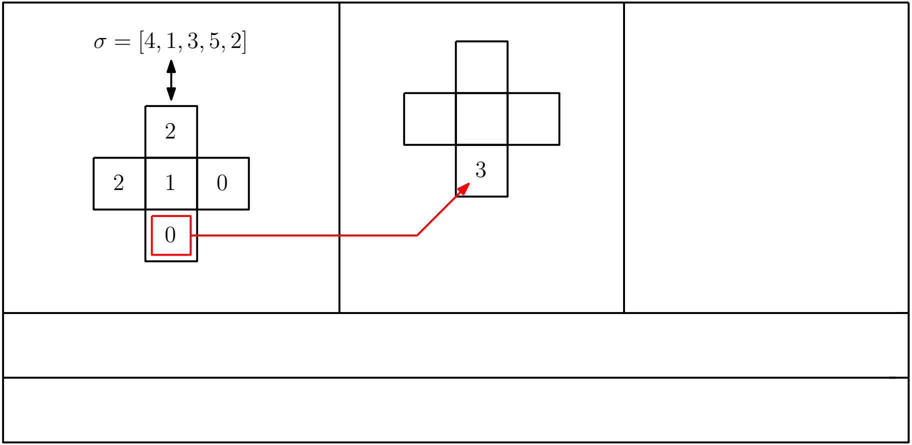
Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$



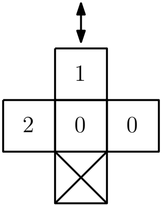


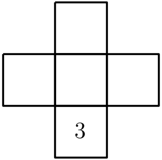
Link with the Stanley symmetric function



Link with the Stanley symmetric function

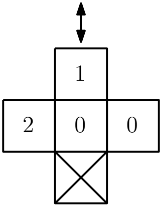
$\sigma = [4, 1, 3, 5, 2]$

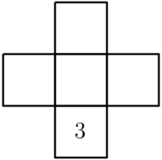




Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$

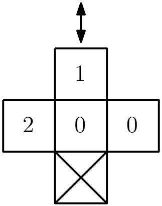


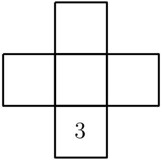


The next integer must be weakly bigger than the previous one.

Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$

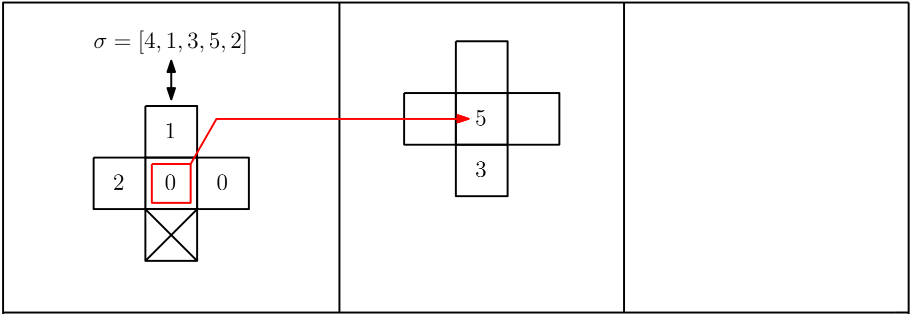




The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function

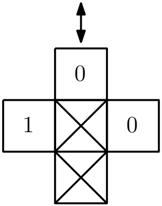


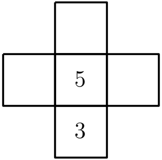
The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$



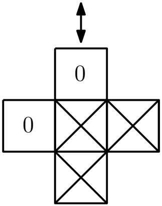


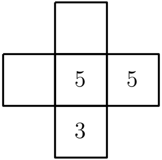
The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$

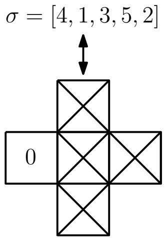


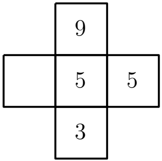


The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function



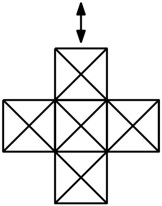


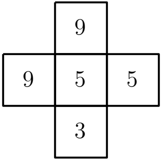
The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function

$\sigma = [4, 1, 3, 5, 2]$



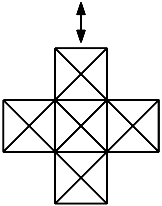


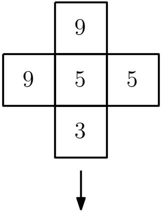
The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

Link with the Stanley symmetric function

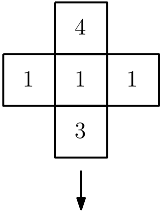
$\sigma = [4, 1, 3, 5, 2]$





monomial $x_3 x_5^2 x_9^2$

Another example:



monomial $x_1^3 x_3 x_4$

The next integer must be weakly bigger than the previous one.

A given integer cannot appear twice in the same column.

A similar thing can be done with the diagram of any permutation.

Link with the Stanley symmetric function

Définition

Set $\sigma \in S_n$, we denote $\mathrm{Tab}(\sigma)$ the set of all the tableaux obtained from the diagram of σ by the previous method. We denote x^T the monomial associated to $T \in \mathrm{Tab}(\sigma)$.

Link with the Stanley symmetric function

Définition

Set $\sigma \in S_n$, we denote $\mathrm{Tab}(\sigma)$ the set of all the tableaux obtained from the diagram of σ by the previous method. We denote x^T the monomial associated to $T \in \mathrm{Tab}(\sigma)$.

Definition

For all $\sigma \in S_n$, we define the Stanley symmetric function of σ as follows:

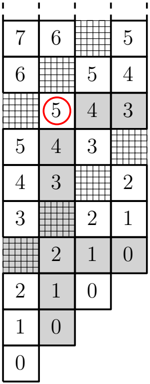
$$F_\sigma(x_1, x_2, \dots) = \sum_{(i_1,\dots,i_{\ell(\sigma)})\in\mathrm{Red}(\sigma)} \sum_{\substack{b_1\leq b_2\leq\dots\leq b_{\ell(\sigma)} \text{ integers} \\ b_j<b_{j+1} \text{ if } i_j<i_{j+1}}} x_{b_1}x_{b_2}\cdots x_{b_{\ell(\sigma)}}.$$

Théorème, V. 2014

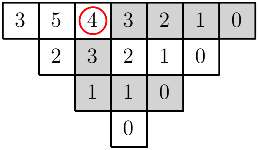
Set $\sigma \in S_n$ and F_σ the associated Stanley symmetric function, then:

$$F_\sigma(x_1, x_2, \dots) = \sum_{T\in\mathrm{Tab}(\sigma)} x^T.$$

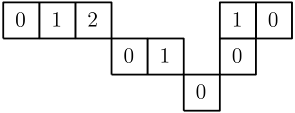
Some other examples



$(\widetilde{A}_4, \leq_R)$



$(B_4, \leq_R)$



"Flag weak order" on $G(2,3)$.

And other examples as the up-set (resp. down-set) lattice of any finite poset, ...

What's next ?

- Can we do the same thing with other Coxeter groups ?

What's next ?

- Can we do the same thing with other Coxeter groups ?
  - A general method can be implemented, based on the study of the root system associated with the Coxeter group + some geometric considerations.
  - This method provides a good candidate for the type D, but in general it gives rise to digraphs which contain cycles.

What's next ?

- Can we do the same thing with other Coxeter groups ?
  - A general method can be implemented, based on the study of the root system associated with the Coxeter group + some geometric considerations.
  - This method provides a good candidate for the type D, but in general it gives rise to digraphs which contain cycles.
- A generalization on more complicated digraphs ? With cycles or multi-edges ?

What's next ?

- Can we do the same thing with other Coxeter groups ?
  - A general method can be implemented, based on the study of the root system associated with the Coxeter group + some geometric considerations.
  - This method provides a good candidate for the type D, but in general it gives rise to digraphs which contain cycles.
- A generalization on more complicated digraphs ? With cycles or multi-edges ?
- Combinatorial properties of $\mathcal{P}(\mathcal{G})$ when $\mathcal{G} = (G, \theta)$ is balanced ? That is, the out-degree of any vertex of G is an even number, and the valuation is given by the out-degree divided by 2.

What's next ?

- Can we do the same thing with other Coxeter groups ?
  - A general method can be implemented, based on the study of the root system associated with the Coxeter group + some geometric considerations.
  - This method provides a good candidate for the type D, but in general it gives rise to digraphs which contain cycles.
- A generalization on more complicated digraphs ? With cycles or multi-edges ?
- Combinatorial properties of $\mathcal{P}(\mathcal{G})$ when $\mathcal{G} = (G, \theta)$ is balanced ? That is, the out-degree of any vertex of G is an even number, and the valuation is given by the out-degree divided by 2.

Thank you for your attention !