Difference operators for partitions and some applications

(joint work with Guo-Niu HAN)

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Plan of Talk

Definition

- A partition is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. The integer $\mid \lambda \mid = \sum_{1 \le i \le r} \lambda_i$ is called the size of the λ.

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- hook product of λ: $H(\lambda) = \prod_{\square \in \lambda} h_\square$.

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7	5	2	1
4	2		
3	1		
1			

Figure: The Young diagram of the partition $(4, 2, 2, 1)$, together with the hook lengths of the corresponding boxes.

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1	4	5	9
2	6		
3	7		
8			

Figure: An SYT of shape $(4, 2, 2, 1)$.

Theorem (Frame, Robinson and Thrall)

$$f_\lambda = \frac{n!}{H_\lambda}$$

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RSK algorithm or representation of finite groups $\Rightarrow$

$$\sum_{|\lambda|=n} f_\lambda^2 = n!$$

and therefore

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 = 1.$$

Theorem (Nekrasov and Okounkov 2003, Han 2008)

$$\sum_{n\geq 0}\left(\sum_{|\lambda|=n} f_\lambda^2 \prod_{\square\in\lambda}(t+h_\square^2)\right)\frac{x^n}{n!^2} = \prod_{i\geq 1}(1-x^i)^{-1-t}.$$

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First proved by Nekrasov and Okounkov. Rediscovered and generalized by Han with a more elementary proof.

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 g(\lambda) = ??.$$

Han

- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^2 = \frac{3n^2-n}{2}.$
- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^4 = \frac{40n^3-75n^2+41n}{6}.$
- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^6 = \frac{1050n^4-4060n^3+5586n^2-2552n}{24}.$

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Conjecture (Han 2008)

$$P(n) = \frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^{2k}$$

is always a polynomial of n for every $k \in \mathbf{N}$.

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Proved and generalized by Stanley.

Theorem (Stanley 2010)

Let F be a symmetric function. Then

$$P(n) = \frac{1}{n!} \sum_{|\lambda|=n} f_\lambda^2 F(h_\square^2 : \square \in \lambda)$$

is a polynomial of n.

Remark. Han-Stanley Theorem is a corollary of our main result.

Definition

Let λ be a partition and g be a function defined on partitions. Difference operators D and D^{-} are defined by

$$Dg(\lambda) = \sum_{\lambda^{+}} g(\lambda^{+}) - g(\lambda)$$

and

$$D^{-}g(\lambda) = \mid \lambda \mid g(\lambda) - \sum_{\lambda^{-}} g(\lambda^{-}),$$

where λ^{+} ranges over all partitions obtained by adding a box to λ and λ^{-} ranges over all partitions obtained by removing a box from λ.

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$$Dg(\lambda) = \sum_{\lambda^+} g(\lambda^+) - g(\lambda)$$

and

$$D^- g(\lambda) = |\lambda| \, g(\lambda) - \sum_{\lambda^-} g(\lambda^-),$$

where λ^+ ranges over all partitions obtained by adding a box to λ and λ^- ranges over all partitions obtained by removing a box from λ.
Let $D^0 g = g$ and $D^{k+1} g = D(D^k g)$ for $k \geq 0$.

Main result (Han and Xiong 2015)

Suppose that F is a symmetric function. Then there exists some $r \in \mathbf{N}$ such that $D^r(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}) = 0$ for every partition λ.

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Theorem (Han and Xiong 2015)

Suppose that g is a function defined on partitions and μ is a given partition. Then we have

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} g(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu)$$

and

$$D^n g(\mu) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \sum_{|\lambda/\mu|=k} f_{\lambda/\mu} g(\lambda).$$

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Our results $\Rightarrow$ Han-Stanley theorem, (skew) marked hook formula, Okada-Panova hook length formula, and Fujii-Kanno-Moriyama-Okada content formula...

When $Dg = 0$ or $D^{-}g = 0$?

Theorem

For any partition λ, we have

$$D(\frac{1}{H_\lambda}) = 0$$

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$$(|\lambda| + 1)f_\lambda = \sum_{\lambda^+} f_{\lambda^+} \Rightarrow \sum_{\lambda^+} \frac{1}{H_{\lambda^+}} - \frac{1}{H_\lambda} = 0 \Rightarrow D(\frac{1}{H_\lambda}) = 0.$$

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$$f_\lambda = \sum_{\lambda^-} f_{\lambda^-} \Rightarrow \frac{|\lambda|}{H_\lambda} - \sum_{\lambda^-} \frac{1}{H_{\lambda^-}} = 0 \Rightarrow D^-(\frac{1}{H_\lambda}) = 0.$$

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Theorem

$D^{-}g(\lambda) = 0$ for every $\lambda \Rightarrow g(\lambda) = \frac{a}{H_\lambda}$ for some constant a.

When $Dg = 0$ or $D^- g = 0$?

Theorem

$D^- g(\lambda) = 0$ for every $\lambda \Rightarrow g(\lambda) = \frac{a}{H_\lambda}$ for some constant a.

Remark. When $Dg(\lambda) = 0$ for every λ, it is not easy to determine $g(\lambda)$. For example, actually we can show

$$D\left(\frac{\sum_{\square\in\lambda}(h_\square^2 - 1) - 3\binom{|\lambda|}{2}}{H_\lambda}\right) = 0.$$

Some properties of D and D^-

Theorem

Let λ be a partition. Suppose that g_1, g_2 are functions defined on partitions and $a_1, a_2 \in \mathbf{R}$. Then we have

$$D(a_1 g_1 + a_2 g_2)(\lambda) = a_1 D g_1(\lambda) + a_2 D g_2(\lambda)$$

and

$$D^-(a_1 g_1 + a_2 g_2)(\lambda) = a_1 D^- g_1(\lambda) + a_2 D^- g_2(\lambda).$$

Some properties of D and D^-

Theorem

For any function g defined on partitions, we have

$$D(\frac{g(\lambda)}{H_\lambda}) = \sum_{\lambda^+} \frac{g(\lambda^+) - g(\lambda)}{H_{\lambda^+}}$$

and

$$D^-(\frac{g(\lambda)}{H_\lambda}) = \sum_{\lambda^-} \frac{g(\lambda) - g(\lambda^-)}{H_{\lambda^-}}.$$

Some properties of D and D^-

For product of two functions:

Theorem

$$D(\frac{g_1(\lambda)g_2(\lambda)}{H_\lambda}) = g_1(\lambda)D(\frac{g_2(\lambda)}{H_\lambda}) + g_2(\lambda)D(\frac{g_1(\lambda)}{H_\lambda}) + \sum_{\lambda^+} \frac{(g_1(\lambda^+) - g_1(\lambda))(g_2(\lambda^+) - g_2(\lambda))}{H_{\lambda^+}}$$

and

$$D^-(\frac{g_1(\lambda)g_2(\lambda)}{H_\lambda}) = g_1(\lambda)D^-(\frac{g_2(\lambda)}{H_\lambda}) + g_2(\lambda)D^-(\frac{g_1(\lambda)}{H_\lambda}) - \sum_{\lambda^-} \frac{(g_1(\lambda) - g_1(\lambda^-))(g_2(\lambda) - g_2(\lambda^-))}{H_{\lambda^-}}.$$

Some properties of D and D^-

For product of several functions:

Theorem

Suppose that $g_1, g_2, \cdots, g_r$ are functions defined on partitions. Let $[r] = \{1, 2, \cdots, r\}$ and $\Delta_j(\lambda, \mu) = g_j(\mu) - g_j(\lambda)$ for $1 \le j \le r$. Then we have

$$D\left(\frac{\prod_{1\le j\le r} g_j(\lambda)}{H_\lambda}\right) = \sum_{\lambda^+} \sum_{\substack{A\cup B=[r]\\ A\cap B=\emptyset\\ A\neq\emptyset}} \frac{\prod_{k\in A} \Delta_k(\lambda, \lambda^+) \prod_{l\in B} g_l(\lambda)}{H_{\lambda^+}}$$

and

$$D^-\left(\frac{\prod_{1\le j\le r} g_j(\lambda)}{H_\lambda}\right) = -\sum_{\lambda^-} \sum_{\substack{A\cup B=[r]\\ A\cap B=\emptyset\\ A\neq\emptyset}} \frac{\prod_{k\in A} \Delta_k(\lambda, \lambda^-) \prod_{l\in B} g_l(\lambda)}{H_{\lambda^-}}.$$

Corners of partitions

For a partition λ, the outer corners are the boxes which can be removed to get a new partition λ^-. Let $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ be the coordinates of outer corners such that $\alpha_1 > \alpha_2 > \cdots \alpha_m$. Let $y_j = \beta_j - \alpha_j$ be the contents of outer corners for $1 \le j \le m$. We set $\alpha_{m+1} = \beta_0 = 0$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1) \ldots, (\alpha_{m+1}, \beta_m)$ the inner corners of λ. Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \le i \le m$.

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$$\sum_{0 \le i \le m} x_i = \sum_{1 \le j \le m} y_j.$$

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Theorem

$$\sum_{0 \le i \le m} x_i = \sum_{1 \le j \le m} y_j.$$

$$\sum_{0 \le i \le m} x_i^2 - \sum_{1 \le j \le m} y_j^2 = 2 \mid \lambda \mid .$$

An example

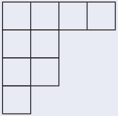

Figure: The Young diagrams of the partition $(4, 2, 2, 1)$.

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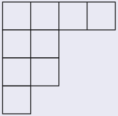

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Outer corners: $(4, 1),\ (3, 2),\ (1, 4)$.
$\{y_j\} = \{-3, -1, 3\}$.

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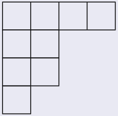

Figure: The Young diagrams of the partition $(4, 2, 2, 1)$.

Outer corners: $(4, 1)$, $(3, 2)$, $(1, 4)$.
$\{y_j\} = \{-3, -1, 3\}$.
inner corners: $(4, 0)$, $(3, 1)$, $(1, 2)$, $(0, 4)$.
$\{x_i\} = \{-4, -2, 1, 4\}$.

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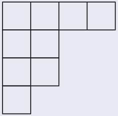

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$\sum_{0 \le i \le m} x_i = -1 = \sum_{1 \le j \le m} y_j$.

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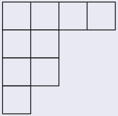

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inner corners: $(4, 0),\ (3, 1),\ (1, 2),\ (0, 4)$.
$\{x_i\} = \{-4, -2, 1, 4\}$.
$\sum_{0 \le i \le m} x_i = -1 = \sum_{1 \le j \le m} y_j$.
$\sum_{0 \le i \le m} x_i^2 - \sum_{1 \le j \le m} y_j^2 = 18 = 2 \cdot 9$.

On $q_\gamma(\lambda)$

Define

$$q_k(\lambda) = \sum_{0 \le i \le m} {x_i}^k - \sum_{1 \le j \le m} {y_j}^k$$

and

$$q_\gamma(\lambda) = \prod_{1 \le l \le t} q_{\gamma_l}(\lambda)$$

for the partition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$.

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$q_0(\lambda) = 1$, $q_1(\lambda) = 0$, $q_2(\lambda) = 2 \mid \lambda \mid$.

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for the partition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$.
$q_0(\lambda) = 1$, $q_1(\lambda) = 0$, $q_2(\lambda) = 2 \mid \lambda \mid$.
The idea to study x_i, y_j and $q_\gamma(\lambda)$ comes from Kerov, Okounkov and Olshanski.

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$.

Let $S(\emptyset, r) = 0$ and $S(\lambda, r) = \sum_{\square\in\lambda}\prod_{i=1}^{r}(h_\square^2 - i^2)$ for $|\lambda| \geq 1$.

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Let $S(\emptyset, r) = 0$ and $S(\lambda, r) = \sum_{\square\in\lambda} \prod_{i=1}^{r} (h_\square^2 - i^2)$ for $|\lambda| \geq 1$.

Step 1: We want to show that there exist some $b_\gamma \in \mathbf{Q}$ such that

$$S(\lambda, r) = \sum_{|\gamma|\leq 2r+2} b_\gamma q_\gamma(\lambda)$$

for every partition λ.

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Suppose that f is a function defined on integers. Let

$$F_1(n) = \sum_{k=1}^{n} f(k)$$

and

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Outline of proof of $D^r\left(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}\right) = 0.$

For $0 \le j < i \le m$, let

$$B_{ij} = \{(a, b) \in \lambda : \alpha_{i+1} + 1 \le a \le \alpha_i, \beta_j + 1 \le b \le \beta_{j+1}\}.$$

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For $0\le j<i\le m$, let

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Then

$$\lambda=\bigcup_{0\le j<i\le m}B_{ij}$$

and thus

$$\sum_{\square\in\lambda}f(h_\square)=\sum_{0\le j<i\le m}\sum_{\square\in B_{ij}}f(h_\square).$$

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Then

$$\lambda = \bigcup_{0 \le j < i \le m} B_{ij}$$

and thus

$$\sum_{\square \in \lambda} f(h_\square) = \sum_{0 \le j < i \le m} \sum_{\square \in B_{ij}} f(h_\square).$$

The multiset of hook lengths of B_{ij} are

$$\bigcup_{a = x_i - y_{j+1}}^{x_i - x_j - 1} \{a, a-1, a-2, \ldots, a-(x_i - y_i - 1)\}.$$

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda})=0$.

This means that

$$\begin{aligned}\sum_{\square\in B_{ij}} f(h_\square) &= \sum_{a=x_i-y_{j+1}}^{x_i-x_j-1} \sum_{b=0}^{x_i-y_i-1} f(a-b)\\ &= \sum_{a=x_i-y_{j+1}}^{x_i-x_j-1} (F_1(a)-F_1(a-x_i+y_i))\\ &= F_2(x_i-x_j-1)+F_2(y_i-y_{j+1}-1)\\ &\quad -F_2(x_i-y_{j+1}-1)-F_2(y_i-x_j-1).\end{aligned}$$

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This means that

$$\begin{aligned}
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&= \sum_{a=x_i-y_{j+1}}^{x_i-x_j-1} (F_1(a) - F_1(a - x_i + y_i)) \\
&= F_2(x_i - x_j - 1) + F_2(y_i - y_{j+1} - 1) \\
&\quad - F_2(x_i - y_{j+1} - 1) - F_2(y_i - x_j - 1).
\end{aligned}$$

Replace $\sum_{\square\in\lambda} f(h_\square)$ by $S(\lambda, r)$. There exist some $b_k \in \mathbf{Q}$ such that for every partition λ, we have

$$\begin{aligned}
S(\lambda, r) &= \sum_{1\le k\le r+1} b_k (\sum_{0\le i\le j\le m} (x_i - x_j)^{2k} + \sum_{1\le i\le j\le m} (y_i - y_j)^{2k} \\
&\quad - \sum_{0\le i\le m} \sum_{1\le j\le m} (x_i - y_j)^{2k}).
\end{aligned}$$

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$.

Compare the coefficients of z^{2k} $(1 \le k \le r+1)$ on both sides of

$$
\begin{aligned}
&(\sum_{0\le i\le m} \exp(x_i z) - \sum_{1\le j\le m} \exp(y_j z))(\sum_{0\le i\le m} \exp(-x_i z) - \sum_{1\le j\le m} \exp(-y_j z)) \\
= &\sum_{0\le i\le m}\sum_{0\le j\le m} \exp((x_i - x_j)z) + \sum_{1\le i\le m}\sum_{1\le j\le m} \exp((y_i - y_j)z) \\
&- \sum_{0\le i\le m}\sum_{1\le j\le m} \exp((x_i - y_j)z) - \sum_{0\le i\le m}\sum_{1\le j\le m} \exp((y_j - x_i)z).
\end{aligned}
$$

There exist some $b_\gamma \in \mathbf{Q}$ such that

$$S(\lambda, r) = \sum_{|\gamma|\le 2r+2} b_\gamma q_\gamma(\lambda)$$

for every partition λ.

Outline of proof of $D^r(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}) = 0$.

Step 2 : Suppose that δ is a partition. Then there exist some $b_\gamma \in \mathbf{Q}$ such that

$$D(\frac{q_\delta(\lambda)}{H_\lambda}) = \sum_{|\gamma| \leq |\delta| - 2} b_\gamma \frac{q_\gamma(\lambda)}{H_\lambda}$$

for every partition λ.

Outline of proof of $D^r(\frac{F(h_\square^2 : \square\in\lambda)}{H_\lambda}) = 0$.

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For convenience, we just show the case $\delta = (k)$ here.

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for every partition λ.
For convenience, we just show the case $\delta = (k)$ here.
Let $\lambda^{k+} = \lambda \cup (\alpha_{k+1}+1, \beta_k+1)$. First we have

$$\frac{\prod_{\square\in\lambda^{k+}} g(h_\square)}{\prod_{\square\in\lambda} g(h_\square)} = g(1) \prod_{0\le i\le k-1} \frac{g(x_k - x_i)}{g(x_k - y_{i+1})} \prod_{k+1\le i\le m} \frac{g(x_i - x_k)}{g(y_i - x_k)}.$$

In particular, we have

$$\frac{H_{\lambda^{k+}}}{H_\lambda} = \frac{\prod\limits_{\substack{0\le i\le m\\ i\ne k}} (x_k - x_i)}{\prod\limits_{1\le j\le m} (x_k - y_j)}.$$

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$.

Let

$$g_1(\lambda) = \sum_{0\le i\le m} g(x_i) - \sum_{1\le j\le m} g(y_j).$$

Then

$$D(\frac{g_1(\lambda)}{H_\lambda}) = \sum_{0\le i\le m} \frac{g(x_i+1)+g(x_i-1)-2g(x_i)}{H_{\lambda^{i+}}}.$$

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$$D(\frac{g_1(\lambda)}{H_\lambda}) = \sum_{0 \le i \le m} \frac{g(x_i+1) + g(x_i-1) - 2g(x_i)}{H_{\lambda^{i+}}}.$$

In particular, let $g(z) = z^k$ and $g_1(\lambda) = q_k(\lambda)$, then we obtain

$$D(\frac{q_k(\lambda)}{H_\lambda}) = \sum_{0 \le i \le m} \frac{2\sum_{1 \le l \le \frac{k}{2}} \binom{k}{2l} x_i^{k-2l}}{H_{\lambda^{i+}}}.$$

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$.

Let

$$g(z) = \prod_{1\leq j\leq m}(1-y_j z) - \sum_{0\leq i\leq m}\frac{H_\lambda}{H_{\lambda^{i+}}}\prod_{\substack{0\leq l\leq m\\ l\neq i}}(1-x_l z)$$

be a polynomial of z with degree m.

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be a polynomial of z with degree m. Then we obtain

$$\begin{aligned} g(\frac{1}{x_t}) &= \prod_{1\le j\le m}(1 - \frac{y_j}{x_t}) - \frac{H_\lambda}{H_{\lambda^{t+}}} \prod_{\substack{0\le l\le m\\ l\ne t}} (1 - \frac{x_l}{x_t}) \\ &= \prod_{1\le j\le m}(1 - \frac{y_j}{x_t}) - \frac{\prod_{1\le j\le m}(x_t - y_j)}{\prod_{\substack{0\le l\le m\\ l\ne t}}(x_t - x_l)} \cdot \prod_{\substack{0\le l\le m\\ l\ne t}} (1 - \frac{x_l}{x_t}) \\ &= 0. \end{aligned}$$

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This means that $g(z)$ has at least $m+1$ roots and therefore $g(z)=0$.

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda})=0.$

Let

$$g(z)=\prod_{1\le j\le m}(1-y_jz)-\sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}\prod_{\substack{0\le l\le m\\ l\ne i}}(1-x_lz)$$

be a polynomial of z with degree m. Then we obtain

$$\begin{aligned} g(\frac{1}{x_t}) &= \prod_{1\le j\le m}(1-\frac{y_j}{x_t})-\frac{H_\lambda}{H_{\lambda^{t+}}}\prod_{\substack{0\le l\le m\\ l\ne t}}(1-\frac{x_l}{x_t}) \\ &= \prod_{1\le j\le m}(1-\frac{y_j}{x_t})-\frac{\prod_{1\le j\le m}(x_t-y_j)}{\prod_{\substack{0\le l\le m\\ l\ne t}}(x_t-x_l)}\cdot\prod_{\substack{0\le l\le m\\ l\ne t}}(1-\frac{x_l}{x_t}) \\ &= 0. \end{aligned}$$

This means that $g(z)$ has at least $m+1$ roots and therefore $g(z)=0$. Now we have

$$\sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}\cdot\frac{1}{1-x_iz}=\frac{\prod_{1\le j\le m}(1-y_jz)}{\prod_{0\le i\le m}(1-x_iz)}.$$

Outline of proof of $D^r(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}) = 0.$

This means that

$$\begin{aligned}\sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}\Big(\sum_{k\ge 0}(x_i z)^k\Big) &= \exp\Big(\sum_{1\le j\le m}\ln(1-y_j z)-\sum_{0\le i\le m}\ln(1-x_i z)\Big)\\ &= \exp\Big(\sum_{k\ge 1}\frac{q_k(\lambda)}{k}z^k\Big).\end{aligned}$$

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda})=0$.

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By comparing the coefficients of z^k on both sides, we obtain there exist some $b_\gamma\in\mathbf{Q}$ such that

$$\sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}x_i{}^k=\sum_{|\gamma|\le k}b_\gamma q_\gamma(\lambda)$$

for every partition λ.

Outline of proof of $D^r(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}) = 0$.

But we have

$$D(\frac{q_k(\lambda)}{H_\lambda}) = \sum_{0 \le i \le m} \frac{2 \sum_{1 \le l \le \frac{k}{2}} \binom{k}{2l} {x_i}^{k-2l}}{H_{\lambda^{i+}}}.$$

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Thus there exist some $b_\gamma \in \mathbf{Q}$ such that

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For $q_\delta(\lambda)$, the proof is similar.

Outline of proof of $D^r(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}) = 0$.

Step 3 : $F(h_\square^2 : \square \in \lambda)$ could be written as a linear combination of some $\binom{|\lambda|}{j} S(\lambda, \gamma)$ and $\binom{|\lambda|}{k}$ with coefficients independent of λ.

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$.

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Therefore by Step 1 $F(h_\square^2 : \square \in \lambda)$ could be written as a linear combination of some $q_\gamma(\lambda)$.

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By Step 2 $D^r(\frac{q_\gamma(\lambda)}{H_\lambda}) = 0$ for some $r \in \mathbf{N}$.

Outline of proof of $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0.$

Step 3 : $F(h_\square^2 : \square \in \lambda)$ could be written as a linear combination of some $\binom{|\lambda|}{j} S(\lambda, \gamma)$ and $\binom{|\lambda|}{k}$ with coefficients independent of λ.
Therefore by Step 1 $F(h_\square^2 : \square \in \lambda)$ could be written as a linear combination of some $q_\gamma(\lambda)$.
By Step 2 $D^r(\frac{q_\gamma(\lambda)}{H_\lambda}) = 0$ for some $r \in \mathbf{N}$.
Then $D^r(\frac{F(h_\square^2:\square\in\lambda)}{H_\lambda}) = 0$ for some $r \in \mathbf{N}$.

Corollary (Han and Xiong 2015)

Let μ be a given partition. Suppose that there exists some $r \in \mathbf{N}$ such that $D^r g(\lambda) = 0$ for every partition λ. Then

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} g(\lambda) = \sum_{0 \le i \le r-1} d_i \binom{n}{i}$$

is a polynomial of n, where $d_i = D^i g(\mu)$.

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Then $D^r\left(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}\right) = 0$ implies

Corollary (Han and Xiong 2015)

Suppose that μ is a given partition. Let F be a symmetric function and k be a integer. Then there exists some $r \in \mathbf{N}$ such that

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} D^k\left(\frac{F(h_\square^2 : \square \in \lambda)}{H_\lambda}\right) = \sum_{0 \le i \le r-k-1} D^{k+i} F(h_\square^2 : \square \in \mu) \binom{n}{i}$$

is a polynomial of n.

In particular, $k = 0 \Rightarrow$

Corollary (Han and Xiong 2015)

Suppose that μ is a given partition. Let F be a symmetric function and k be a integer. Then there exists some $r \in \mathbf{N}$ such that

$$\frac{1}{(n+\mid \mu \mid)!} \sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} F(h_\square^2 : \square \in \lambda) = \sum_{0 \le i \le r-1} D^i F(h_\square^2 : \square \in \mu) \binom{n}{i}$$

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$k = 0$ and $\mu = \emptyset$ implies

Corollary (Han-Stanley Theorem)

Suppose that μ is a given partition. Let F be a symmetric function and k be a integer. Then

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 F(h_\square^2 : \square \in \lambda)$$

is a polynomial of n.

$k = 0$ and $F = 1 \Rightarrow$

Corollary

$$\frac{1}{(n+ \mid \mu \mid)!} \sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} = \frac{1}{H_\mu}.$$

$k = 0$ and $F = 1 \Rightarrow$

Corollary

$$\frac{1}{(n+\mid \mu \mid)!} \sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} = \frac{1}{H_\mu}.$$

$k = 0$, $F = 1$ and $\mu = \emptyset$ implies

Corollary

$$\frac{1}{n!} \sum_{|\lambda|=n} f_\lambda^2 = 1.$$

Theorem

We have

$$H_\lambda D^r\left(\frac{S(\lambda,r)}{H_\lambda}\right) = \frac{(2r)!(2r+1)!}{r!((r+1)!)^2} \mid \lambda \mid .$$

and therefore

$$H_\lambda D^{r+1}\left(\frac{S(\lambda,r)}{H_\lambda}\right) = \frac{(2r)!(2r+1)!}{r!((r+1)!)^2}.$$

$$H_\lambda D^{r+2}\left(\frac{S(\lambda,r)}{H_\lambda}\right) = 0.$$

$$D^i\left(\frac{S(\emptyset,r)}{H_\emptyset}\right) = 0$$

for $0 \leq i \leq r$.

Theorem (Han and Xiong 2015)

Suppose that μ is a given partition. Let $S(\lambda, r) = \sum_{\square \in \lambda} \prod_{1 \leq j \leq r} (h_\square^2 - j^2)$ and $d_i = D^i(\frac{S(\mu,r)}{H_\mu})$. Then we have

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} D^i(\frac{S(\lambda, r)}{H_\lambda}) = \sum_{0 \leq k \leq r+1-i} d_{i+k} \binom{n}{k}$$

for $0 \leq i \leq r+1$.

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Suppose that μ is a given partition. Let $S(\lambda, r) = \sum_{\square\in\lambda} \prod_{1\leq j\leq r} (h_\square^2 - j^2)$ and $d_i = D^i(\frac{S(\mu,r)}{H_\mu})$. Then we have

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for $0 \leq i \leq r+1$.

In particular $i = 0, \mu = \emptyset$ implies Okada-Panova hook length formula.

Corollary (Okada-Panova hook length formula 2008)

$$n! \sum_{|\lambda|=n} \frac{\sum_{\square\in\lambda} \prod_{i=1}^{r} (h_\square^2 - i^2)}{H_\lambda^2} = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1} \prod_{j=0}^{r} (n-j).$$

$r = 1$ implies Marked hook formula.

Corollary (Marked hook formula)

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^2 = \frac{3n^2-n}{2}.$$

$r = 1$ implies Marked hook formula.

Corollary (Marked hook formula)

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^2 = \frac{3n^2-n}{2}.$$

Skew marked hook formula (Han and Xiong 2015)

Let μ be a given partition. Then

$$\frac{1}{(n+\mid\mu\mid)!}\sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} \sum_{\square\in\lambda}(h_\square^2-1) = 3\sum_{0\le i\le 2} d_i \binom{n}{i}$$

is a polynomial of n, where $d_0 = \frac{\sum_{\square\in\mu}(h_\square^2-1)}{3H_\mu}$, $d_1 = \frac{|\mu|}{H_\mu}$, $d_2 = \frac{1}{H_\mu}$.

The content of the box (i, j) of a partition is defined by $c_{(i,j)} = j - i$.

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Let $C(\emptyset, r) = 0$ and $C(\lambda, r) = \sum_{\square \in \lambda} \prod_{i=0}^{r-1} (c_{\square}^2 - i^2)$ for $|\lambda| \geq 1$.

The content of the box (i,j) of a partition is defined by $c_{(i,j)} = j - i$.

Let $C(\emptyset, r) = 0$ and $C(\lambda, r) = \sum_{\square\in\lambda} \prod_{i=0}^{r-1} (c_\square^2 - i^2)$ for $|\lambda| \geq 1$.

We need the following result.

Theorem

$$H_\lambda D\left(\frac{C(\lambda, r)}{H_\lambda}\right) = \sum_{|\gamma|\leq 2r} b_\gamma q_\gamma(\lambda)$$

for some $b_\gamma \in \mathbf{Q}$. In particular,

$$D^{r+2}\left(\frac{C(\lambda, r)}{H_\lambda}\right) = 0.$$

Theorem

We have

$$H_\lambda D^r\left(\frac{C(\lambda, r)}{H_\lambda}\right) = \frac{(2r)!}{(r+1)!} \mid \lambda \mid .$$

and therefore

$$H_\lambda D^{r+1}\left(\frac{C(\lambda, r)}{H_\lambda}\right) = \frac{(2r)!}{(r+1)!}.$$

$$D^{r+2}\left(\frac{C(\lambda, r)}{H_\lambda}\right) = 0.$$

$$D^i\left(\frac{C(\emptyset, r)}{H_\lambda}\right) = 0$$

for $0 \leq i \leq r$.

Theorem (Han and Xiong 2015)

Suppose that μ is a given partition. Let $d_i = D^i(\frac{C(\mu,r)}{H_\mu})$. Then we have

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} D^i(\frac{C(\lambda,r)}{H_\lambda}) = \sum_{0\le k\le r+1-i} d_{i+k}\binom{n}{k}$$

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In particular, $i=0, \mu=\emptyset$ implies Fujii-Kanno-Moriyama-Okada content formula.

Theorem (Fujii-Kanno-Moriyama-Okada content formula 2008).

$$n! \sum_{|\lambda|=n} \frac{\sum_{\square\in\lambda}\prod_{i=0}^{r-1}(c_\square^2 - i^2)}{H_\lambda^2} = \frac{(2r)!}{(r+1)!^2}\prod_{j=0}^{r}(n-j).$$

Thank You for your listening!