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TORUS FIXED POINTS IN SCHUBERT VARIETIES AND NORMALIZED MEDIAN GENOCCHI NUMBERS

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Abstract. We give a new proof for the fact that the number of torus fixed points for the degenerate flag variety is equal to the normalized median Genocchi number, using the identification with a certain Schubert variety. We further study the torus fixed points for the symplectic degenerate flag variety and develop a combinatorial model, symplectic Dellac configurations, to parametrize them. The number of these symplectic fixed points is conjectured to be a median Euler number.

Introduction

We consider the Schubert variety X_{τ_n} associated to the Weyl group element

$$\tau_n := (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_k s_{k+1} \cdots s_{2k-2}) \cdots (s_3 s_4) s_2 \in \mathfrak{S}_{2n}$$

in the partial flag variety SL_{2n}/P, where P is the standard parabolic subalgebra associated to the simple roots $\{\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}\}$. Then there is a natural action of a $(2n-1)$-dimensional torus T_{2n-1}, and we are mainly interested in the fixed points $X_{\tau_n}^{T_{2n-1}}$ of this torus action. It is well known that the fixed points are parametrized by Weyl groups elements which are less than or equal to τ_n in the Bruhat order (modulo the stabilizer of the parabolic subalgebra; in this case, the subgroup generated by $s_1, s_3, \ldots, s_{2n-1}$). Our first result is the following.

Theorem A. *There is an explicit bijection* $\mathbf{b}$ *from Dellac configurations* DC_n *(Definition 1) of* $2n$ *columns and* n *rows to* $X_{\tau_n}^{T_{2n-1}}$*. Hence the number of torus fixed points is equal to the normalized median Genocchi number* h_n *(see Section 1 for definition).*

Here is an example of the Dellac configuration corresponding to a fixed point for $n = 3$:

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$\mapsto \quad \sigma = 124536$

We also consider Schubert varieties of the symplectic flag variety, e.g., the Schubert variety $X^{sp}_{\overline{\tau}_{2n}}$ corresponding to the element (of the symplectic Weyl group)

$$\overline{\tau}_{2n} := (r_{2n} \cdots r_{n+1}) \cdots (r_{2n} r_{2n-1} r_{2n-2})(r_{2n} r_{2n-1}) r_{2n} (r_n \cdots r_{2n-2}) \cdots (r_4 r_5 r_6)(r_3 r_4) r_2$$

in the symplectic partial flag variety. In this case, there is a natural action of T_{2n} on the Schubert variety, and we are again interested in the fixed points of this torus action. To parametrize them similar to the non-symplectic case, we introduce symplectic Dellac configurations (Definition 2). These are Dellac configurations with $4n$ columns and $2n$ rows, which are invariant under the involution mapping the i-th row to the $(2n-i+1)$-st row. Here is our second result.

Theorem B. *The torus fixed points in* $X^{sp}_{\overline{\tau}_{2n}}$ *are parametrized by the symplectic Dellac configurations* SpDC_{2n}.

We conjecture that the number of symplectic Dellac configurations is equal to a normalized median Euler number (cf. [K97]).

We should explain here why we are interested in these particular Schubert varieties. E. Feigin [Fei11] defined the degenerate flag variety

$$\mathcal{F}l^a_n := \{(U_1, \ldots, U_{n-1}) \in \prod_{i=1}^{n-1} \mathrm{Gr}_i(\mathbb{C}^n) \mid \mathrm{pr}_{i+1} U_i \subset U_{i+1}\},$$

where pr_i is the endomorphism of $\mathbb{C}^n$ setting the i-th coordinate to be zero. This is in fact a flat degeneration of the classical flag variety $\mathcal{F}l_n$. Moreover it was shown in [CFR12, CLL15] that there is an action of T_{2n-1} on $\mathcal{F}l^a_n$. The symplectic degenerate flag variety $(\mathcal{F}l^a_{2n})^{sp}$ has been defined in [FFiL14] in a similar way.

The degenerate flag variety is one of the main objects in the framework of PBW filtrations and degenerations on universal enveloping algebras of simple Lie algebras (see [FFoL11a, FFoL11b, FFoL13, FFR15, Hag14, Fou14, Fou15, CFR12] for various aspects). Here, one obtains *degenerate flag varieties* $\mathcal{F}l^a(\lambda)$ as highest weight orbits of PBW degenerate modules. In [Fei11, FFiL14], it has been shown that these highest weight orbits have an interpretation as a variety of certain flags.

Recently, it was shown in [CL15] that these degenerate flag varieties are in fact our particular Schubert varieties.

Theorem (Cerulli Irelli–Lanini).
(1) *In the* $\mathfrak{sl}_n$*-case, the degenerate flag variety* $\mathcal{F}l^a_n$ *is isomorphic to the Schubert variety* X_{τ_n}. *Moreover, the isomorphism* $\zeta : \mathcal{F}l^a_n \xrightarrow{\sim} X_{\tau_n}$ *is* T_{2n-1}*-equivariant.*
(2) *In the* $\mathfrak{sp}_{2n}$*-case, the degenerate symplectic flag variety is isomorphic to* $X^{sp}_{\overline{\tau}_{2n}}$, *and again the isomorphism* $\zeta^{sp} : X^{sp}_{\overline{\tau}_{2n}} \xrightarrow{\sim} (\mathcal{F}l^a_{2n})^{sp}$ *is torus-equivariant.*

The torus fixed points of the degenerate flag variety in type A_n have been studied in [Fei11]. In that paper, an explicit bijection $\mathbf{f}$ with the set of Dellac configurations has been provided. Hence it was shown that the number of torus fixed points is equal to a normalized median Genocchi number.

Combining the theorem by Cerulli Irelli and Lanini with Theorem A, we obtain another proof of this fact, using the classical set up of Schubert varieties only. Moreover, we can show that the following diagram commutes (there, α denotes the natural identification of $W^J_{\leq \tau_n}$ with $X^{T_{2n-1}}_{\tau_n}$)

$$\begin{array}{ccc} (\mathcal{F}l^a_n)^{T_n} & \xrightarrow{\mathbf{f}} & \mathrm{DC}_n \\ \downarrow \zeta & & \downarrow \mathbf{b} \\ X^{T_{2n-1}}_{\tau_n} & \xleftarrow{\alpha} & W^J_{\leq \tau_n} \end{array} .$$

In the symplectic case, the map $\mathbf{f}$ is not present, mainly because the construction of symplectic Dellac configurations has not been seen in the literature before. Nevertheless, we obtain a similar picture, namely the torus fixed points in the symplectic degenerate flag variety are parametrized by SpDC_{2n}. We should mention here that E. Feigin (via the symplectic degenerate flag variety in [FFiL14]) as well as G. Cerulli Irelli (via the quiver Grassmannian in [CFR12]) also conjectured the number of torus fixed points to be a normalized median Euler number.

This paper is organized as follows. In Section 1, we prove our first theorem for $\mathfrak{sl}_n$, while, in Section 2, we consider the symplectic case. In Section 3 we relate our results to the framework of degenerate flag varieties.

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1. Symmetric groups and Median Genocchi numbers

1.1. Let $W = \mathfrak{S}_{2n}$ be the symmetric group generated by $S = \{s_1, s_2, \ldots, s_{2n-1}\}$ where $s_i = (i, i+1)$. Let $J = \{s_1, s_3, \ldots, s_{2n-1}\} \subset S$ and W_J be the subgroup generated by J, and let W^J be the set of minimal representatives of right cosets of W_J in W. We define

$$\tau_n = (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_k s_{k+1} \cdots s_{2k-2}) \cdots (s_3 s_4) s_2 \in W.$$

Then, for $t = 1, 2, \ldots, 2n$, we have

$$\tau_n(t) = \begin{cases} k, & t = 2k-1; \\ n+k, & t = 2k. \end{cases} \tag{1.1}$$

By construction, τ_n is a representative of minimal length in W/W_J, so $\tau_n \in W^J$. We define

$$W_{\leq \tau_n} = \{w \in W \mid \ w \leq \tau_n\}, \quad W^J_{\leq \tau_n} = \{w \in W^J \mid \ w \leq \tau_n\},$$

where $\leq$ is the Bruhat order.

Definition 1. A Dellac configuration C is a board of $2n$ columns and n rows with $2n$ marked cells such that:

1. each column contains exactly one marked cell;
2. each row contains exactly two marked cells;
3. if the (i, j)-cell is marked, then $i \leq j \leq n + i$.

Let DC_n denote the set of such configurations.

It is worth pointing out that the definition of a Dellac configuration given above differs from that in [Fei11] by a rotation of the board by 90°.

The cardinality h_n of the set DC_n is called the n-th normalized median Genocchi number (see [Fei11, Fei12] and the references therein). Consider the polynomials defined recursively by: $H_0(x) = 1$,

$$H_n(x) = \frac{1}{2}(x+1)((x+1)H_{n-1}(x+1) - xH_{n-1}(x)).$$

Then it is proved in [DR94] that $h_n = H_n(1)$.

The following theorem is originally proved by Cerulli Irelli and Lanini in [CL15] as a corollary of their main result and a result of Feigin [Fei11] (see Remark 4 for details).

Theorem 1. *For any integer $n \geq 1$, we have $h_n = \#W^J_{\leq \tau_n}$.*

In this section, we provide a purely combinatorial proof of the theorem, in terms of a bijection.

1.2. **Rook configurations.** Consider a board of n rows and columns. A rook configuration R is a filling of the cells by n marks such that each row and each column have exactly one mark. Let $\mathcal{R}_n$ denote the set of all rook configurations. There is a bijection

$$\varphi : \mathcal{R}_n \xrightarrow{\sim} \mathfrak{S}_n \tag{1.2}$$

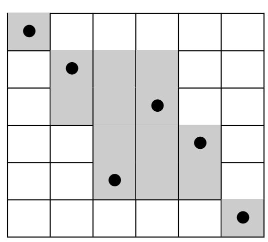
sending a rook configuration R to the permutation σ_R satisfying $\sigma_R(i) = j$ if and only if the cell (i, j) is marked in R, for $i = 1, 2, \ldots, n$. For $\sigma \in \mathfrak{S}_n$, we denote $R_\sigma := \varphi^{-1}(\sigma)$.

Let R be a rook configuration. The convex hull of the marked cells in R is the smallest right-aligned skew Ferrers board containing all marks in R.

From now on we consider $\mathfrak{S}_{2n}$: R_{τ_n} is a board of $2n$ columns and rows. A restricted rook configuration with respect to τ_n is a rook configuration such that all marked cells in the board are contained in the convex hull (which is called the right hull in [Sjo07]) of the marked cells in R_{τ_n}. Let $R_{\leq \tau_n}$ denote the set of all restricted rook configurations with respect to τ_n.

Example 1. We consider an example where $n = 3$. Then $\tau_3 = 142536$, and the shadowed area is the convex hull of the marked cells in R_{τ_3}. We fix $\sigma = 124536$. The rook configuration of σ is (given by the dots):

$R_\sigma =$



R_σ is the restricted rook configuration with respect to τ_3.

It is clear that τ_n avoids the patterns 4231, 35142, 42513, and 351624. The following result is a special case of Theorem 4 in [Sjo07].

Theorem 2 ([Sjo07])**.** *The restriction of φ on $R_{\leq \tau_n}$ gives a bijection $R_{\leq \tau_n} \xrightarrow{\sim} W_{\leq \tau_n}$.*

1.3. **From rook configurations to Dellac configurations.** We define two maps $\mathbf{m} : R_{\leq \tau_n} \to \mathrm{DC}_n$, called the *melt map*, and $\mathbf{b} : \mathrm{DC}_n \to R_{\leq \tau_n}$, called the *blow-up map.*

Let $R \in R_{\leq \tau_n}$ be a restricted rook configuration. Consider a board C_R of $2n$ columns and n rows defined by: the cell (k, l) of C_R is marked if and only if either the cell $(2k-1, l)$ or the cell $(2k, l)$ is marked in R. Intuitively, the k-th row of C_R is obtained by merging the $(2k-1)$-st and the $2k$-th row in R.

Lemma 1. *The board C_R is a Dellac configuration.*

Proof. By the definition of a rook configuration, each row of C_R has exactly two marked cells, and each column of C_R has exactly one marked cell. Moreover, when R is restricted with respect to τ_n, then, by (1.1), C_R has the following property: if the cell (r, s) in C_R is marked, then $r \leq s \leq n + r$. □

By using the lemma, we obtain a well-defined melt map

$$\mathbf{m}(R) := C_R.$$

Example 2. Let $\sigma = 124536$ be the permutation in Example 1. The corresponding Dellac configuration via the melt procedure is given by:

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Let $C \in \mathrm{DC}_n$ be a Dellac configuration. A board R_C of $2n$ rows and columns is associated to C in the following way: the cells (i, j) and (i, k) with $j < k$ are marked in C if and only if the cells $(2i - 1, j)$ and $(2i, k)$ are marked in R_C. Intuitively, the i-th row in C is split into two rows, where the first row gets the first marked point, while the second row gets the second.

Lemma 2. *The board R_C is a restricted rook configuration with respect to τ_n.*

Proof. Conditions (1) and (2) in the definition of the Dellac configuration guarantees that R_C is a rook configuration. The condition (3) means that R_C is restricted with respect to τ_n. □

By defining $\mathbf{b}(C) = R_C$, the blow-up map is well-defined by Lemma 2.

Lemma 3. *The following statements hold:*

(1) *the map* $\mathbf{b}$ *is injective with* $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(W^J_{\leq \tau_n})$*;*
(2) *we have* $\mathbf{m} \circ \mathbf{b} = id$*.*

Proof. By construction, the only thing to be proved is $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(W^J_{\leq \tau_n})$. This relation indeed holds by the following description of W^J:

$$W^J = \{\sigma \in W \mid \ \sigma(2k - 1) < \sigma(2k) \text{ for } 1 \leq k \leq n\}.$$ □

As an application of these maps, we provide a bijective proof of Theorem 1.

Proof of Theorem 1. By Lemma 3, the blow-up map $\mathbf{b}$ induces a bijection $\mathrm{DC}_n \xrightarrow{\sim} W^J_{\leq \tau_n}$. Since $|\,\mathrm{DC}_n\,| = h_n$, we proved $h_n = \#W^J_{\leq \tau_n}$. □

Remark 1. The normalized median Genocchi numbers h_n count a combinatorial structure in $\mathfrak{S}_{2n+2}$ called normalized Dumont permutations. Although *a posteriori* there exists a bijection between the normalized Dumont permutations and $W^J_{\leq \tau_n}$, our approach is different from the one in [K97], see also [Fei11].

2. Symplectic case

2.1. **Notations.** Let $\widetilde{W} = \mathfrak{S}_{4n}$ be the symmetric group acting on $\{1, 2, \dots, 4n\}$, and $\widetilde{J} = \{s_1, s_3, \dots, s_{4n-1}\}$. Let ι be the involution of $\widetilde{W}$ defined by

$$\iota(\sigma)(k) = 4n + 1 - \sigma(4n + 1 - k) \text{ for } \sigma \in \widetilde{W} \text{ and } 1 \leq k \leq 4n.$$

The Weyl group W of the symplectic group Sp_{4n} with generators $\{r_1, r_2, \dots, r_{2n}\}$ can be embedded into $\widetilde{W}$ via the map $\kappa : W \to \widetilde{W}$, $r_i \mapsto s_i s_{4n-i}$ for $1 \leq i \leq 2n - 1$ and $r_{2n} \mapsto s_{2n}$. The image of κ consists of the set $\widetilde{W}^\iota$ of ι-fixed elements in W. Let $J = \{r_1, r_3, \dots, r_{2n-1}\}$. We define

$$\overline{\tau}_{2n} = (r_{2n} \cdots r_{n+1}) \cdots (r_{2n}r_{2n-1}r_{2n-2})(r_{2n}r_{2n-1})r_{2n}(r_n \cdots r_{2n-2}) \cdots (r_4r_5r_6)(r_3r_4)r_2 \in W.$$

It is observed in [CLL15] that $\kappa(\overline{\tau}_{2n}) = \tau_{2n}$.

By Corollary 8.1.9 in [GTM05] (notice the differences between the indices here and those in the reference), the restriction of κ to $W_{\leq \overline{\tau}_{2n}}$ gives a bijection

$$\alpha : W_{\leq \overline{\tau}_{2n}} \xrightarrow{\sim} (\widetilde{W}_{\leq \tau_{2n}})^\iota.$$

By passing to right cosets, α induces a bijection $\alpha' : W^J_{\leq \overline{\tau}_{2n}} \xrightarrow{\sim} (\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}})^\iota$.

2.2. Symplectic Dellac configurations.

Definition 2. A symplectic Dellac configuration C is a board of $4n$ columns and $2n$ rows with $4n$ marked cells such that:

(1) each column contains exactly one marked cell;
(2) each row contains exactly two marked cells;
(3) if the (i, j)-cell is marked, then $i \leq j \leq 2n + i$;
(4) for $1 \leq i, j \leq 2n$, the (i, j)-cell is marked if and only if the $(2n-i+1, 4n-j+1)$-cell is marked.

Let SpDC_{2n} denote the set of such configurations and e_n its cardinality.

We have $e_1 = 1, e_2 = 2, e_3 = 10, e_4 = 98, e_5 = 1594$. Consider the sequence of polynomials defined by recursion: $E_0(x) = 1$,

$$E_n(x) = \frac{1}{2}(x + 1)((x + 2)E_{n-1}(x + 2) - xE_{n-1}(x)).$$

Conjecture 1. *For any $n \geq 0$, we have $e_{n+1} = E_n(1)$.*

Remark 2. Giovanni Cerulli Irelli and Evgeny Feigin kindly informed us that they have also a similar conjecture.

If this conjecture were true, these numbers e_n coincide with the numbers r_n in [RZ96] (see A098279 in OEIS), where the continued fraction expansion of the corresponding generating function is given (Théorème 29 in *loc. cit.*).

2.3. **Main result.** The main result of this section is the following.

Theorem 3. *For any integer $n \geq 1$, we have $e_n = \#W^J_{\leq \overline{\tau}_{2n}}$.*

Proof. We prove the theorem by establishing a bijection between $W^J_{\leq \overline{\tau}_{2n}}$ and SpDC_{2n}, following the strategy in the proof of Theorem 1.

A *symplectic rook configuration* C is a board of $4n$ columns and rows with $4n$ marked points satisfying:

(1) C is a rook configuration;
(2) for $1 \leq i \leq 4n$ and $1 \leq j \leq 2n$, the cell (i, j) is marked if and only if the cell $(4n+1-i, 4n+1-j)$ is marked.

The set of symplectic rook configurations is denoted by $\mathcal{SR}_{4n}$. Similarly to Section 1.2, we can define the restricted symplectic rook configurations with respect to τ_{2n}: $\mathcal{SR}_{\leq \tau_{2n}} := \mathcal{SR}_{4n} \cap \mathcal{R}_{\leq \tau_{2n}}$.

Consider the bijection $\varphi : \mathcal{R}_{4n} \xrightarrow{\sim} \mathfrak{S}_{4n}$ from (1.2).

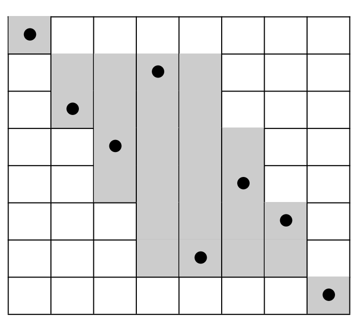
Lemma 4. (1) *The restriction of the map φ induces a bijection $\varphi' : \mathcal{SR}_{4n} \xrightarrow{\sim} \widetilde{W}^\iota = W$.*
(2) *The restriction of the map φ' induces a bijection $\psi : \mathcal{SR}_{\leq \tau_{2n}} \xrightarrow{\sim} (\widetilde{W}_{\leq \tau_{2n}})^\iota$.*

Proof. (1) Take a board R in $\mathcal{SR}_{4n}$. Condition (2) in its definition implies that $\varphi(R)$ is invariant under the involution ι. It suffices to show that φ' is surjective. Let $\sigma \in \widetilde{W}$. By definition of ι, the element σ is fixed by the involution ι if and only if $\sigma(4n+1-k) = 4n+1-\sigma(k)$ for all k with $1 \leq k \leq 4n$, i.e., for all i and j with $1 \leq i \leq 4n$ and $1 \leq j \leq 2n$, we have $\sigma(i) = j$ if and only if $\sigma(4n+1-i) = 4n+1-j$. This implies that $\varphi^{-1}(\sigma)$ is in $\mathcal{SR}_{4n}$.

(2) Since $\mathcal{SR}_{\leq \tau_{2n}} = \mathcal{SR}_{4n} \cap \mathcal{R}_{\leq \tau_{2n}}$ and $(\widetilde{W}_{\leq \tau_{2n}})^\iota = \widetilde{W}^\iota \cap \widetilde{W}_{\leq \tau_{2n}}$, the bijectivity of ψ follows from (1) and Theorem 2. □

Moreover, consider the restriction of the melt map $\mathbf{m} : \mathcal{R}_{\leq \tau_{2n}} \to \mathrm{DC}_{2n}$ to $\mathcal{SR}_{\leq \tau_{2n}}$. Since Condition (2) in the definition of the symplectic rook configuration translates into Condition (4) in the definition of the symplectic Dellac configuration under the melt map, $\mathbf{m}$ induces a map $\mathbf{m}' : \mathcal{SR}_{\leq \tau_{2n}} \to \mathrm{SpDC}_{2n}$.

Example 3. Let us consider an example where $n = 2$ and the permutation is given by the following rook configuration:



where the shadowed area is the convex hull of the marked cells in $R_{\overline{\tau}_4}$. It is straightforward to see that the rook configuration is fixed by ι and hence symplectic. The corresponding symplectic Dellac configuration via the melt map $\mathbf{m}$ is given by

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Continuation of the proof of Theorem 3. The restriction of the blow-up map $\mathbf{b} : \mathrm{DC}_{2n} \to \mathcal{R}_{\leq \tau_{2n}}$ to SpDC_{2n} gives a map $\mathbf{b}' : \mathrm{SpDC}_{2n} \to \mathcal{SR}_{\leq \tau_{2n}}$. By Lemma 3, $\mathbf{b}$ is injective with $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}})$. This implies that $\mathbf{b}'$ is injective with

$$\mathrm{im}(\mathbf{b}') = \varphi^{-1}(\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}} \cap \widetilde{W}^{\iota}) = \psi^{-1}((\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}})^{\iota})$$

and $\mathbf{m}' \circ \mathbf{b}' = \mathrm{id}$.

By the above argument, the blow-up map $\mathbf{b}'$ gives a bijection $\mathrm{SpDC}_{2n} \xrightarrow{\sim} (\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}})^{\iota}$. Composing with $(\varphi')^{-1}$, we get a bijection $\mathrm{SpDC}_{2n} \xrightarrow{\sim} W^{J}_{\leq \overline{\tau}_{2n}}$. □

3. Application to torus fixed points

We show how the construction in Section 1 is related to the study of the torus fixed points in the degenerate flag variety.

3.1. **Schubert varieties.** Let $\sigma_n \in \mathfrak{S}_{2n}$ be the permutation defined by

$$\sigma_n(r) = \begin{cases} k, & r = 2k; \\ n+1+r, & r = 2k+1. \end{cases} \tag{3.1}$$

We see that σ_n can be obtained by restricting $\tau_{n+1} \in S_{2n+2}$ to the set $\{2, \ldots, 2n+1\}$.

We denote by X_{σ_n} the Schubert variety corresponding to σ_n in the projective variety SL_n/P, where P is the standard parabolic subalgebra defined as the stabilizer of the highest weight line of weight $\varpi_1 + \varpi_3 + \cdots + \varpi_{2n-1}$. The maximal torus T_{2n-1} of SL_{2n} acts naturally on X_{σ_n}. Let $X_{\sigma_n}^{T_{2n-1}}$ be the set of torus fixed points.

It is a standard result that the torus fixed points $X_{\sigma_n}^{T_{2n-1}}$ can be identified with the quotient $W^J_{\leq \sigma_n}$, where $W = \mathfrak{S}_{2n}$ and $J = \{2, 4, \ldots, 2n-2\}$. For $\tau \in W^J_{\leq \sigma_n}$, the corresponding torus fixed point in $X_{\sigma_n}^{T_{2n-1}}$ is

$$\langle e_{\tau(1)} \rangle_{\mathbb{C}} \subset \langle e_{\tau(1)}, e_{\tau(2)}, e_{\tau(3)} \rangle_{\mathbb{C}} \subset \cdots \subset \langle e_{\tau(1)}, e_{\tau(2)}, \ldots, e_{\tau(2n-1)} \rangle_{\mathbb{C}} \in X_{\sigma_n},$$

where $e_1, e_2, \ldots, e_{2n}$ is a fixed basis of $\mathbb{C}^{2n}$.

3.2. **Degenerate flag varieties.** We fix a basis $\{f_1, f_2, \ldots, f_{n+1}\}$ of $\mathbb{C}^{n+1}$. Let $\mathcal{F}l^a_{n+1}$ be the degenerate flag variety of SL_{n+1} (see [Fei11] for details):

$$\mathcal{F}l^a_{n+1} = \{(V_1, V_2, \ldots, V_n) \in \prod_{i=1}^{n} \mathrm{Gr}_i(\mathbb{C}^{n+1}) \mid \ \mathrm{pr}_{i+1}(V_i) \subset V_{i+1} \text{ for } i = 1, 2, \ldots, n\},$$

where $\mathrm{pr}_i : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the linear projection along the line generated by f_i. By [CFR12], the torus T_{2n-1} acts on $\mathcal{F}l^a_{n+1}$. Let $(\mathcal{F}l^a_{n+1})^{T_{2n-1}}$ be the corresponding set of torus fixed points.

In [CL15], it is shown that there exists a T_{2n-1}-equivariant isomorphism of projective varieties $\zeta : \mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_{\sigma_n} \subset \mathrm{SL}_{2n}/P$. We are particularly interested in the image of torus fixed points under ζ.

Fix a basis $\{e_1, e_2, \dots, e_{2n}\}$ of $\mathbb{C}^{2n}$. For $i = 1, 2, \dots, n$, we write U_{n+i} for the coordinate subspace $\langle e_1, e_2, \dots, e_{n+i}\rangle \subset W$. The surjection $\pi_i : U_{n+i} \to \mathbb{C}^{n+1}$ is defined by

$$\pi_i(e_k) = \begin{cases} 0, & \text{if } 1 \leq k \leq i-1; \\ f_k, & \text{if } i \leq k \leq n+1; \\ f_{k-n-1}, & \text{if } n+2 \leq k \leq n+i. \end{cases} \tag{3.2}$$

Define $\zeta_i : \mathrm{Gr}_i(\mathbb{C}^{n+1}) \to \mathrm{Gr}_{2i-1}(\mathbb{C}^{2n})$ to be the concatenation of the maps

$$\mathrm{Gr}_i(\mathbb{C}^{n+1}) \to \mathrm{Gr}_{2i-1}(U_{n+i}) \to \mathrm{Gr}_{2i-1}(\mathbb{C}^{2n}), \quad U \mapsto \pi_i^{-1}(U) \mapsto \pi_i^{-1}(U).$$

Then $\zeta : \mathcal{F}l^a_{n+1} \to X_{\sigma_n}$ is given by $\prod_{i=1}^n \zeta_i$ (see Section 2 of [CL15] for details).

It is clear that the torus T_n of SL_{n+1} acts naturally on $\mathcal{F}l^a_{n+1}$. By results in Section 7.2 of [CFR12], any T_{2n-1}-fixed point in $\mathcal{F}l^a_{n+1}$ is in fact a T_n-fixed point. In [Fei11], an explicit bijection $\mathbf{f}$ between the T_{2n-1}-fixed points and Dellac configuration is provided.

3.3. **A commutative diagram.** As a summary, starting with a T_n-fixed point in $\mathcal{F}l^a_{n+1}$, there are two ways to obtain a Dellac configuration:

(1) via the bijection $\mathbf{f}$ given by [Fei11];
(2) consider this fixed point as a fixed point in the Schubert variety X_{σ_n}, hence identify it with an element in $W^J_{\leq \sigma_n}$, then melt the corresponding rook configuration to get a Dellac configuration.

It is natural to ask whether the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{F}l^a_{n+1})^{T_{2n-1}} = (\mathcal{F}l^a_{n+1})^{T_n} & \xrightarrow{\mathbf{f}} & \mathrm{DC}_{n+1} \\ \downarrow \beta & & \downarrow \mathbf{b} \\ X_{\tau_{n+1}}^{T_{2n+1}} = X_{\sigma_n}^{T_{2n-1}} & \xleftarrow{\alpha} & W^J_{\leq \tau_{n+1}} \end{array},$$

where the map α is given as follows: for $\sigma \in W^J_{\leq \tau_{n+1}}$, where $W = \mathfrak{S}_{2n+2}$, we define the map α as follows: $\alpha(\sigma)$ is the sequence of subspaces $W_1 \subset W_2 \subset \cdots \subset W_n$ such that W_i is the subspace of $\mathbb{C}^{2n}$ generated by $e_{\overline{\sigma}(1)}, e_{\overline{\sigma}(2)}, \dots, e_{\overline{\sigma}(2i-1)}$, where $\overline{\sigma}$ is the (well-defined) restriction of σ to $\mathfrak{S}_{2n}$. We can identify this element in $X_{\sigma_n}^{T_{2n-1}}$ with n subsets $J_1, \dots, J_n$ of $\{1, 2, \dots, 2n\}$ such that $J_i = \{\overline{\sigma}(1), \overline{\sigma}(2), \dots, \overline{\sigma}(2i-1)\}$.

It remains to consider the restriction of the map ζ to fixed points. Here we have to include an extra twist, since the definition of the degenerate flag variety is slightly different in [Fei11] and [CL15]: let $(V_1, V_2, \dots, V_n) \in (\mathcal{F}l^a_{n+1})^{T_n}$. This flag can be identified (see [Fei11, Corollary 2.11]) with n subsets $I_1, I_2, \dots, I_n$ of $\{1, 2, \dots, n+1\}$ such that $\#I_k = k$ and, for $k = 1, 2, \dots, n$, $I_k \backslash \{k+1\} \subset I_{k+1}$.

We let $\kappa = (12 \cdots n+1)^{-1}$ be the inverse of the longest cycle in $\mathfrak{S}_{n+1}$. Suppose that $I_l = \{i_{l,1}, i_{l,2}, \dots, i_{l,l}\}$. We write $I^\kappa_l = \{\kappa(i_{l,1}), \kappa(i_{l,2}), \dots, \kappa(i_{l,l})\}$. Furthermore, we define a map $p_l : \{1, 2, \dots, n+l\} \to \{1, 2, \dots, n+1\}$ by

$$p_l(s) = \begin{cases} 0, & \text{if } 1 \le s \le l-1; \\ s, & \text{if } l \le k \le n+1; \\ s-n-1, & \text{if } n+2 \le k \le n+l. \end{cases} \tag{3.3}$$

Then $\beta((I_1, I_2, \dots, I_n)) = (T_1, T_2, \dots, T_n)$, where $T_l = p_l^{-1}(I_l^\kappa)$.

Theorem 4. *The diagram above commutes, i.e., $\zeta = \alpha \circ \mathbf{b} \circ \mathbf{f}$.*

The proof consists of a case-by-case examination. We only provide a sketch.

Proof. We pick $\mathbf{I} = (I_1, I_2, \dots, I_n) \in (\mathcal{F}l^a_{n+1})^{T_{n+1}}$. Recall that the map $\mathbf{f}$ is given in [Fei11, Proposition 3.1].

(1) Suppose that $l \notin I_{l-1}$. Then $I_l \backslash I_{l-1} = \{j\}$. We consider the case $j > l$: in the Dellac configuration $f(\mathbf{I})$, the cells (l, l) and (l, j) are marked. Then, by definition, $\sigma = \mathbf{b}(f(\mathbf{I}))$ satisfies $\sigma(2l-1) = l$ and $\sigma(2l) = j$. Hence, in $\alpha(\sigma)$, $J_l \backslash J_{l-1} = \{l-1, j-1\}$.

    We compute $\beta(\mathbf{I})$: it is clear that $I_l^\kappa \backslash I_{l-1}^\kappa = \{j-1\}$. Then $p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{l-1, j-1\}) = \{l-1, j-1\}$. Therefore $T_l \backslash T_{l-1} = \{l-1, j-1\}$, i.e., $J_l = T_l$.

    The case $j < l$ can be dealt with similarly.

(2) Suppose that $l \in I_{l-1}$ and $l \in I_l$. Then $I_l \backslash I_{l-1} = \{j\}$. We study the case $j < l$: in the corresponding Dellac configuration, the cells $(l, l+n+1)$ and $(l, j+n+1)$ are marked. The associated permutation $\sigma = \mathbf{b}(f(\mathbf{I}))$ satisfies $\sigma(2l-1) = j+n+1$ and $\sigma(2l) = l+n+1$. Hence, in $\alpha(\sigma)$, $J_l \backslash J_{l-1} = \{j+n, l+n\}$.

    For $\beta(\mathbf{I})$: $l \in I_{l-1} \cap I_l$ and $I_l \backslash I_{l-1} = \{j\}$ imply that $l-1 \in I_{l-1}^\kappa \cap I_l^\kappa$ and $I_l^\kappa \backslash I_{l-1}^\kappa = \{\kappa(j)\}$. Notice that, no matter whether $j = 1$ or $j > 1$, $p_l^{-1}(\kappa(j)) = j+n$. By the assumption $j < l$, we have

    $$p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{l-1, \kappa(j)\}) = \{j+n, l+n\},$$

    which establishes $J_l = T_l$.

    The case where $j > l$ can be dealt with similarly.

(3) Suppose that $l \in I_{l-1}$ and $l \notin I_l$. Then there exist j_1 and j_2 such that $I_l \backslash I_{l-1} = \{j_1, j_2\}$. We assume that $j_1 < l$ and $j_2 > l$. In the corresponding Dellac configuration, the cells (l, j_1+n+1) and (l, j_2) are marked. Hence, in $\alpha(\mathbf{b}(f(\mathbf{I})))$, $J_l \backslash J_{l-1} = \{j_1+n, j_2-1\}$.

    For $\beta(\mathbf{I})$, we have

    $$p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{\kappa(j_1), j_2-1\}) = \{j_1+n, j_2-1\},$$

    therefore $J_l = T_l$.

    All other cases can be proved in the same way. □

Remark 3. A similar diagram without the map $\mathbf{f}$ exists in the symplectic case by changing

(1) the degenerate flag variety to the symplectic degenerate flag variety (see [FFiL14]);
(2) the Schubert variety of SL_{2n} by the Schubert variety in the symplectic group (see [CL15]);

(3) the Dellac configuration by the symplectic Dellac configuration;
(4) the set $W^J_{\leq \tau_{n+1}}$ by $W^J_{\leq \overline{\tau}_{2n+2}}$.

Remark 4. The original proof of Theorem 1 is given by showing that the composition $\alpha^{-1} \circ \beta \circ \mathbf{f}^{-1}$ is a bijection: that $\mathbf{f}$ is a bijection is shown in [Fei11]; by the main theorem of [CL15], β is a bijection; α is a well-known bijection. Our proof of the theorem uses the intuitive map $\mathbf{b}$ to avoid the geometrical proof.

References

[GTM05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups.* Graduate Texts in Mathematics, 231. Springer, New York, 2005. xiv+363 pp.

[CL15] G. Cerulli Irelli and M. Lanini, *Degenerate flag varieties of type A and C are Schubert varieties.* Int. Math. Res. Notices (2015), 6353–6374.

[CLL15] G. Cerulli Irelli, M. Lanini, and P. Littelmann, *Degenerate flag varieties and Schubert varieties: a characteristic free approach.* Pac. J. Math (to appear).

[CFR12] G. Cerulli Irelli, E. Feigin, and M. Reineke, *Quiver Grassmannians and degenerate flag varieties.* Algebra Number Theory 6 (2012), no. 1, 165–194.

[DR94] D. Dumont and A. Randrianarivony, *Dérangements et nombres de Genocchi.* Discrete Math. 132 (1994), no. 1–3, 37–49.

[FFR15] X. Fang, G. Fourier, and M. Reineke, *PBW-type filtration on quantum groups of type A_n.* J. Algebra 449 (2016) 321–345.

[Fei11] E. Feigin, *Degenerate flag varieties and the median Genocchi numbers.* Math. Res. Lett. 18 (2011), no. 6, 1163–1178.

[Fei12] E. Feigin, *The median Genocchi numbers, q-analogues and continued fractions.* European J. Combin. 33 (2012), no. 8, 1913–1918.

[FFiL14] E. Feigin, M. Finkelberg, and P. Littelmann, *Symplectic degenerate flag varieties.* Canad. J. Math. 66 (2014), no. 6, 1250–1286.

[FFoL11a] E. Feigin, G. Fourier, and P. Littelmann, *PBW filtration and bases for irreducible modules in type A_n*, Transformation Groups 16 (2011), no. 1, 71–89.

[FFoL11b] E. Feigin, G. Fourier, and P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*, Int. Math. Res. Notices 2011 (24), 5760–5784.

[FFoL13] E. Feigin, G. Fourier, and P. Littelmann, *Favourable modules: Filtrations, polytopes, Newton–Okounkov bodies and flat degenerations*, Transformation Groups, 2016, DOI: 10.1007/S00031-016-9389-2.

[Fou15] G. Fourier, *New homogeneous ideals for current algebras: filtrations, fusion products, and the Pieri rules*, Moscow Math. J. 15 (2015), no. 1, 49–72.

[Fou14] G. Fourier, *Marked poset polytopes: Minkowski sums, indecomposables, and unimodular equivalence*, J. Pure Appl. Algebra 220 (2016), no. 2, 606–620.

[Hag14] C. Hague, *Degenerate coordinate rings of flag varieties and Frobenius splitting*, Selecta Math. (N.S.) 20 (2014), no. 3, 823–838.

[K97] G. Kreweras, *Sur les permutations comptées par les nombres de Genocchi de 1-ière et 2-ième espèce.* European J. Combin. 18 (1997), no. 1, 49–58.

[RZ96] A. Randrianarivony and J. Zeng, *Une famille de polynômes qui interpole plusieurs suites classiques de nombres.* Adv. Appl. Math. 17 (1996), no. 1, 1–26.

[Sjo07] J. Sjöstrand, *Bruhat intervals as rooks on skew Ferrers boards.* J. Combinatorial Theory, Ser. A 114 (2007), 1182–1198.

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