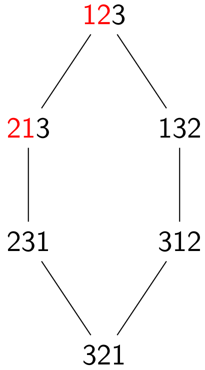
A lattice on decreasing trees: the metasylvester lattice

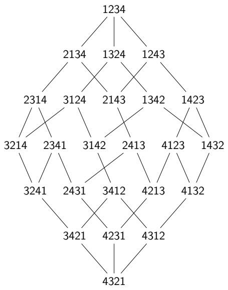
Viviane Pons

Université Paris-Sud

SLC, 07/09/2015

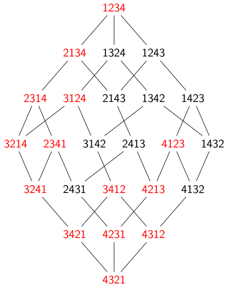


Weak order on permutations



Weak Order

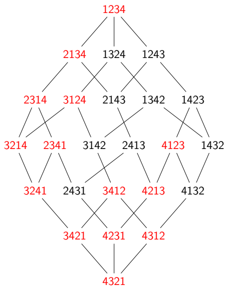
permutations

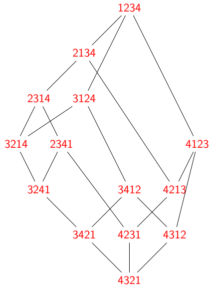


Weak Order

permutations

132-avoiding permutations





Weak Order

permutations

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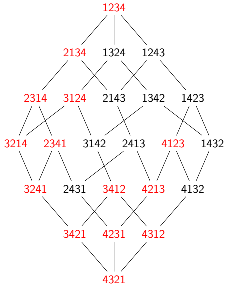
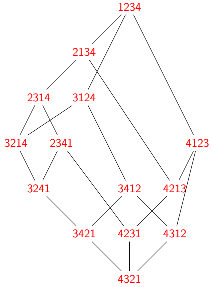


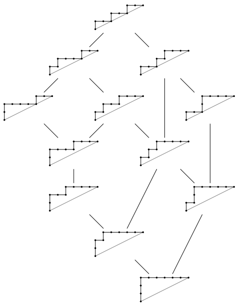
Weak Order

permutations



Tamari Lattice (1962)

Catalan objects:
132-avoiding permutations,
Dyck paths, binary trees,
triangulations, ...

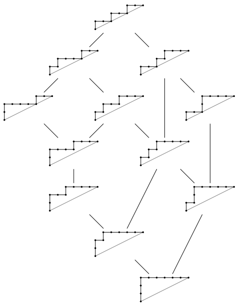


m-Tamari Lattice (2012)*

m-Catalan objects:
m-Dyck paths, $(m+1)$-ary trees

* Bergeron, Préville-Ratelle

??



m-permutations?

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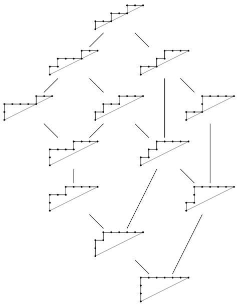

Metasylvester lattice

m-permutations?

m-Tamari Lattice (2012)*

m-Catalan objects:
m-Dyck paths, $(m+1)$-ary trees

* Bergeron, Préville-Ratelle

Sylvester congruence*

$$ac \dots b \equiv ca \dots b, \qquad a \leq b < c.$$

*Hivert, Novelli, Thibon – 2005

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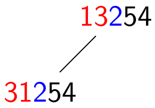
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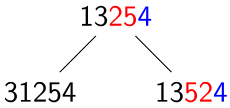
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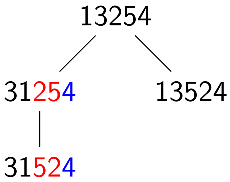
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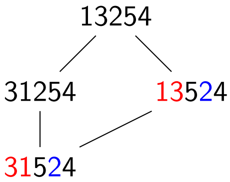
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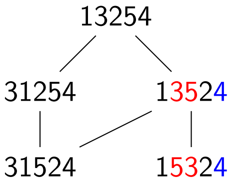
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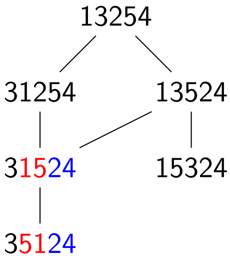
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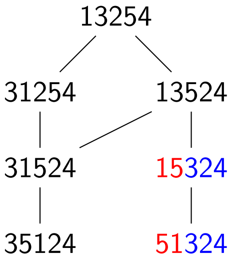
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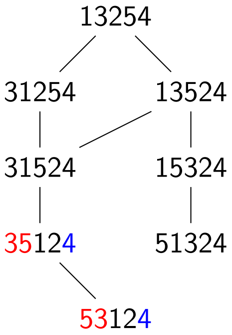
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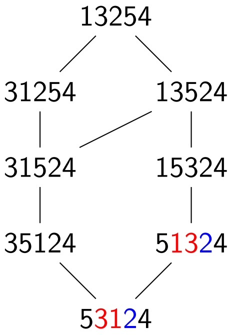
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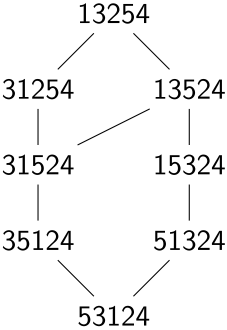
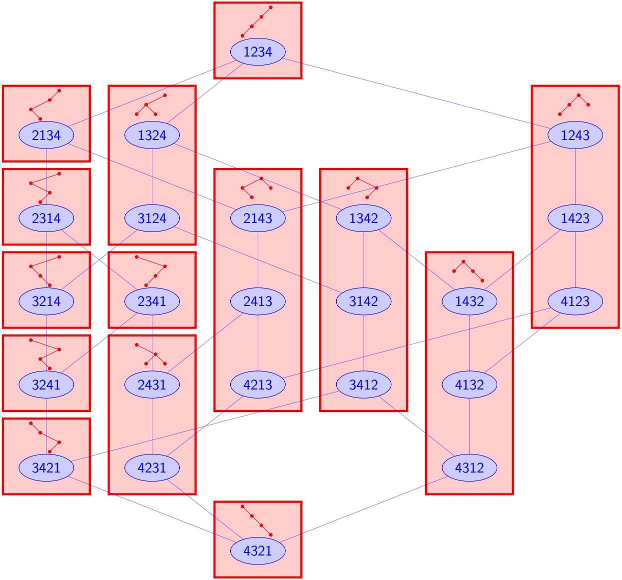


Number of sylvester classes: Catalan
$\rightarrow$ in bijection with Binary trees through the binary search tree insertion algorithm


1234
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3214
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2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

- Permutations : $n!$
- Binary trees: $\frac{1}{n+1}\binom{2n}{n}$

- Permutations : $n!$
- Decreasing binary trees: $n!$
- Binary trees: $\frac{1}{n+1}\binom{2n}{n}$

Permutation: 215634

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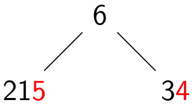
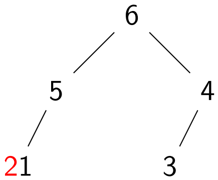
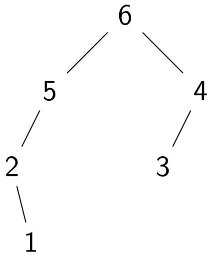
Permutation: 215634



Permutation: 215634



Permutation: 215634



m-permutations

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A m-permutation of size n is a permutation of the word

$$1^m 2^m \dots n^m$$

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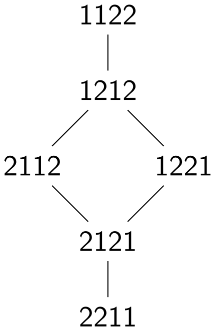
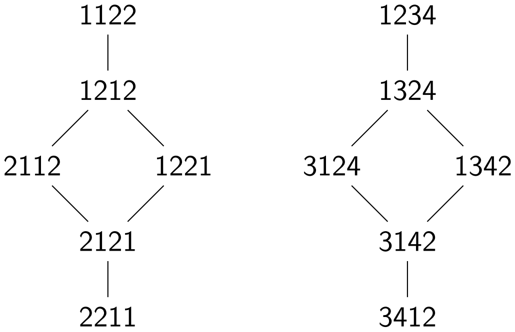
$$1^m 2^m \dots n^m$$

Example: 3244123515 is a 2-permutation of size 5.

The m-permutations lattice



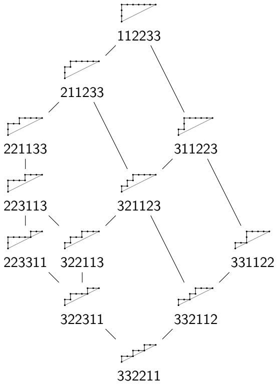
The m-permutations lattice



Sylvester classes?

Sylvester classes?

m-Tamari lattice



m-permutations	$\frac{(mn)!}{2^n}$ 1, 6, 90, 2520, 113400	m-permtuations lattice
m-Catalan objects: $(m+1)$-ary trees m-Dyck paths	$\frac{1}{mn+1}\binom{(m+1)n}{n}$ 1, 3, 12, 55, 273	m-Tamari lattice

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Decreasing $(m+1)$-ary trees		
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Decreasing $(m+1)$-ary trees	$\prod_{k=0}^{n-1}(1+kn)$ 1, 3, 15, 105, 945	
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Metasylvester congruence*

$$
\begin{aligned}
ac \dots a &\equiv ca \dots a & (a < c),\\
b \dots ac \dots b &\equiv b \dots ca \dots b & (a < b < c).
\end{aligned}
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Example:

$$1212 \rightarrow 2112$$

$$321132 \rightarrow 321312$$

*Novelli, Thibon – 2014

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Example:

$$1212 \rightarrow 2112$$
$$321132 \rightarrow 321312$$

The metasylvester classes are in bijection with decreasing $(m+1)$-ary trees.*

*Novelli, Thibon – 2014

Proposition (Novelli, Thibon)

Each metasylvester class contains a single element avoiding the pattern $a \dots b \dots a$ with $a < b$, it is the maximal element of the class in the m-permutation lattice.

Example:

211323	does not avoid the pattern.
322113	avoids the pattern.

Bijection metasylvester classes – decreasing trees

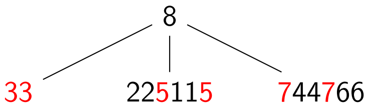
3382251158744766

Bijection metasylvester classes – decreasing trees

338225115874476 6

338225115874476 6

Bijection metasylvester classes – decreasing trees



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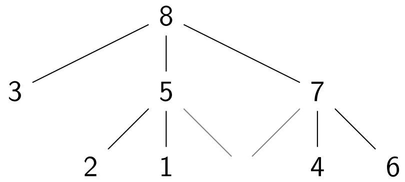
Bijection metasylvester classes – decreasing trees



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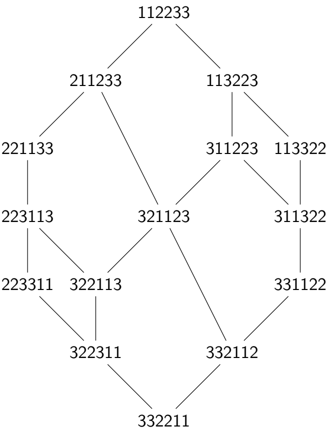
Theorem

The subposet formed by the maximal elements of the metasylvester classes is a lattice.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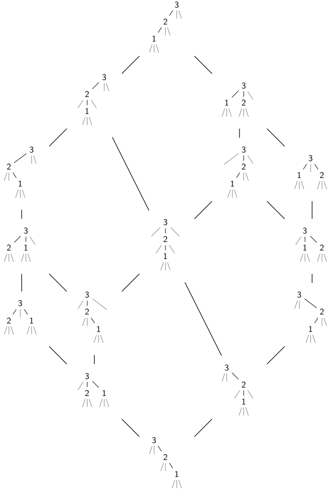

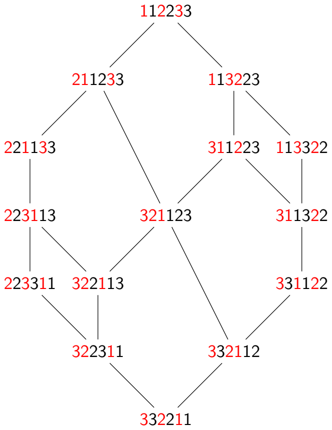
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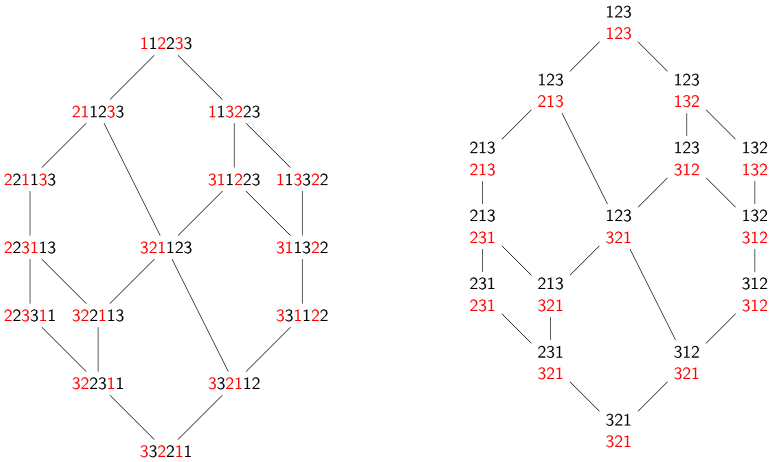
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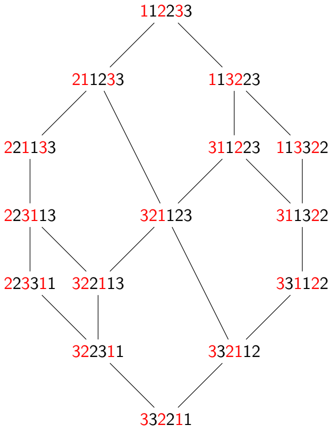
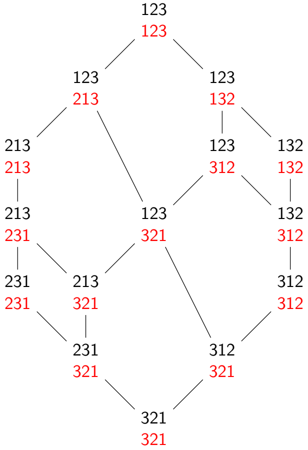
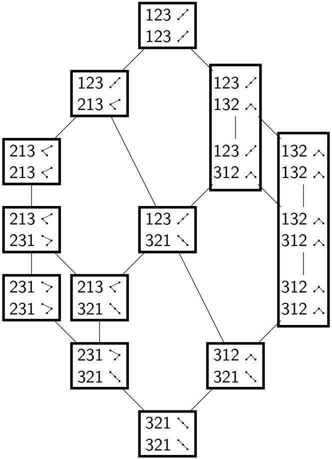




$\sigma^{-1}\mu$ avoids the pattern 231.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