Enumeration of minimal acyclic automata
Parking functions

Séminaire Lotharingien de Combinatoire '75

Jean-Baptiste PRIEZ

Laboratoire de Recherche en Informatique
Faculté des Sciences d'Orsay
Université Paris-Sud

LABORATOIRE DE RECHERCHE EN INFORMATIQUE

UNIVERSITÉ PARIS SUD

Tuesday, September 8th 2015

Acyclic automata

Generalized parking functions

Bijection: (n-i) ADFA–parking functions

Minimal automata

Acyclic deterministic finite automata

Let Σ be an alphabet of k symbols.

An *acyclic automaton* of n states on Σ: (i, A, δ)

- $i \in [n]$ an initial state ,
- $A \subset [n]$ a set of accepting states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\}$ a transition function

such that

$$\forall q \in [n]\,,\ \forall w \in \Sigma^+\,, \qquad \delta^*(q, w) \neq q.$$

Acyclic deterministic finite automata

Let Σ be an alphabet of k symbols.

An *acyclic automaton* of n states on Σ: (i, A, δ)

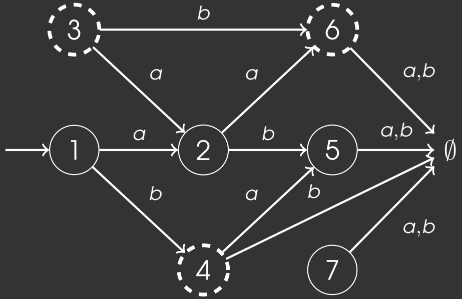
- $i \in [n]$ an initial state ,
- $A \subset [n]$ a set of accepting states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\}$ a transition function

such that

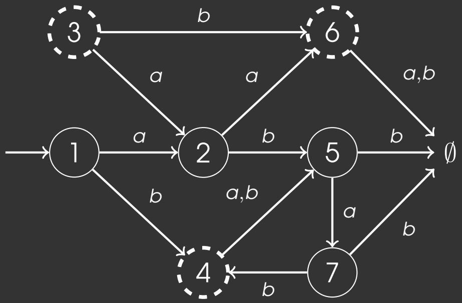
$$\forall q \in [n]\,,\ \forall w \in \Sigma^+\,, \qquad \delta^*(q, w) \neq q.$$

where $\emptyset$ is the *absorbing state* and δ^* is the transition function extended to words.

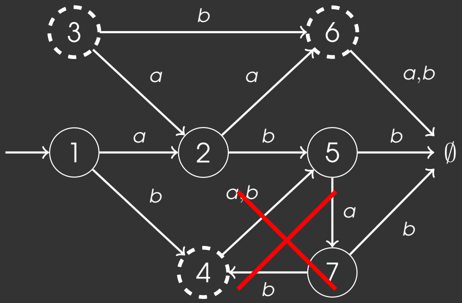
Acyclic deterministic finite automata (2)



Acyclic deterministic finite automata (2)



Acyclic deterministic finite automata (2)



Liskovets, 2003

<u>*Theorem*</u>:

$$\mathcal{T}_k(n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} (n-s+1)^{ks}\, \mathcal{T}_k(n-s)$$

with $\mathcal{T}_k(0) = 1$.

LISKOVETS, 2003

Theorem :

$$\mathcal{T}_k(n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} (n-s+1)^{ks} \, \mathcal{T}_k(n-s)$$

with $\mathcal{T}_k(0) = 1$.

Particular case : J. KUNG & C. YAN, 2003

$$\mathcal{P}(\chi; n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \chi(n-s+1)^{s} \, \mathcal{P}(\chi; n-s)$$

with $\chi : \mathbb{N} \to \mathbb{N}$ a non-decreasing map.

LISKOVETS, 2003

Theorem:

$$\mathcal{T}_k(n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} (n-s+1)^{ks}\, \mathcal{T}_k(n-s)$$

with $\mathcal{T}_k(0) = 1$.

Particular case: J. KUNG & C. YAN, 2003

$$\mathcal{P}(\chi; n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \chi(n-s+1)^{s}\, \mathcal{P}(\chi; n-s)$$

with $\chi : \mathbb{N} \to \mathbb{N}$ a non-decreasing map.

Bijection: Transition fct $\longleftrightarrow$ Generalized parking fct
with $\chi(m) := m^k$.

Liskovets, 2003

Theorem:

$$\mathcal{T}_k(n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} 2^s (n-s+1)^{ks} \, \mathcal{T}_k(n-s)$$

with $\mathcal{T}_k(0) = 1$.

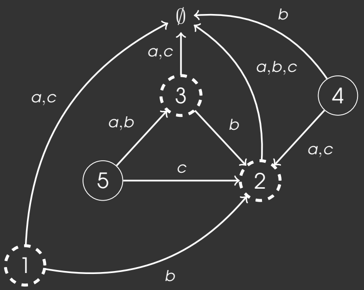
Particular case: J. Kung & C. Yan, 2003

$$\mathcal{P}(\chi; n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \chi(n-s+1)^s \, \mathcal{P}(\chi; n-s)$$

with $\chi : \mathbb{N} \to \mathbb{N}$ a non-decreasing map.

Bijection: Transition fct $\longleftrightarrow$ Generalized parking fct
with $\chi(m) := 2m^k$.

Bijection?



$\updownarrow$?

Generalized parking function

Acyclic automata

Generalized parking functions

- Parking function
- Generalization

Bijection: (n-i) ADFA–parking functions

Minimal automata

Parking function

KONHEIM and WEISS, 1966

Combinatorial object : Modeling hashes

$$f : [n] \to \mathbb{N}_+ \qquad \text{such that} \qquad \#f^{-1}([k]) \geqslant k$$

for any $k \in [n]$.

Parking function

KONHEIM and WEISS, 1966

Combinatorial object : Modeling hashes

$$f : [n] \to \mathbb{N}_+ \qquad \text{such that} \qquad \#f^{-1}([k]) \geqslant k$$

for any $k \in [n]$.

Interests :

- (Labeled) rooted trees,
- Prüfer sequences,
- Hyperplan arrangements
- n! conjecture
- ...

Parking function

KONHEIM and WEISS, 1966

Combinatorial object : Modeling hashes

$$f : [n] \to \mathbb{N}_+ \qquad \text{such that} \qquad \# f^{-1}([k]) \geqslant k$$

for any $k \in [n]$.

Interests :

- (Labeled) rooted trees,
- Prüfer sequences,
- Hyperplan arrangements
- n! conjecture
- ...

Parking functions on $\{1, 2, 3\}$: $\mathcal{F}[3]$

$$111, 112, 121, 211, 122, 212, 221, 113, 131, 311,$$
$$123, 132, 312, 213, 231, 321$$

where f denotes $f(1)f(2)f(3)$.

Generalized parking function

STANLEY and PITMAN, 2002; KUNG and YAN, 2003.

χ-parking function: Let $\chi : \mathbb{N}_+ \rightarrow \mathbb{N}$ be non-decreasing.

$$f : [n] \rightarrow \mathbb{N}_+ \qquad \text{such that} \qquad \# f^{-1}([\chi(k)]) \geqslant k$$

for any $k \in [n]$.

Generalized parking function

STANLEY and PITMAN, 2002; KUNG and YAN, 2003.

χ-parking function: Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

$$f : [n] \to \mathbb{N}_+ \qquad \text{such that} \qquad \#f^{-1}([\chi(k)]) \geqslant k$$

for any $k \in [n]$.

Example: $\mathcal{F}_\chi[2]$ with $\chi : m \mapsto m^2$

$$11, 12, 21, 13, 31, 14, 41$$

Generalized parking function

STANLEY and PITMAN, 2002; KUNG and YAN, 2003.

χ-parking function: Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

$$f : [n] \to \mathbb{N}_+ \qquad \text{such that} \qquad \#f^{-1}([\chi(k)]) \geqslant k$$

for any $k \in [n]$.

Example: $\mathcal{F}_\chi[2]$ with $\chi : m \mapsto m^2$

$$11, 12, 21, 13, 31, 14, 41$$

Notation: $\mathcal{F}_\chi \simeq \mathcal{F}_{m^2}$

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Let f be a χ-parking function:

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Let f be a χ-parking function:

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Let f be a χ-parking function:

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Let f be a χ-parking function:

Intuition

Let $\chi : \mathbb{N}_+ \to \mathbb{N}$ be non-decreasing.

Let f be a χ-parking function:

Generalized parking function (2)

Equivalent definition: P. and VIRMAUX, 2015

$(Q_i)_{i \in \mathbb{N}_+}$ a sequence of disjoint subsets of $[n]$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k \,, \qquad \text{for any} \quad k \in [n] \,.$$

Generalized parking function (2)

Equivalent definition: P. and Virmaux, 2015

$$(Q_i)_{i\in\mathbb{N}_+} \text{ a sequence of disjoint subsets of } [n]$$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k\,, \qquad \text{for any} \quad k \in [n]\,.$$

$\mathcal{F}_{m^2}[2]$:

$$11,\ 12,\ 21,\ 13, 31, 14, 41$$

with $Q_i := f^{-1}(\{i\})$

Generalized parking function (2)

Equivalent definition : P. and Virmaux, 2015

$$(Q_i)_{i \in \mathbb{N}_+} \text{ a sequence of disjoint subsets of } [n]$$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k\,, \qquad \text{for any} \quad k \in [n]\,.$$

$\mathcal{F}_{m^2}[2]$:

$$(12|\cdot|\cdot|\cdot),\ 12,\ 21,\ 13, 31, 14, 41$$

with $Q_i := f^{-1}(\{i\})$

Generalized parking function (2)

Equivalent definition: P. and VIRMAUX, 2015

$$(Q_i)_{i\in\mathbb{N}_+} \text{ a sequence of disjoint subsets of } [n]$$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k\,, \qquad \text{for any} \quad k \in [n]\,.$$

$\mathcal{F}_{m^2}[2]$:

$$(12|\cdot|\cdot|\cdot),\ (1|2|\cdot|\cdot),\ 21,\ 13, 31, 14, 41$$

with $Q_i := f^{-1}(\{i\})$

Generalized parking function (2)

Equivalent definition: P. and VIRMAUX, 2015

$$(Q_i)_{i \in \mathbb{N}_+} \text{ a sequence of disjoint subsets of } [n]$$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k\,, \qquad \text{for any} \quad k \in [n]\,.$$

$\mathcal{F}_{m^2}[2]$:

$$(12|\cdot|\cdot|\cdot),\ (1|2|\cdot|\cdot),\ (2|1|\cdot|\cdot),\ 13, 31, 14, 41$$

with $Q_i := f^{-1}(\{i\})$

Generalized parking function (2)

Equivalent definition: P. and VIRMAUX, 2015

$$(Q_i)_{i\in\mathbb{N}_+} \text{ a sequence of disjoint subsets of } [n]$$

such that

$$\sum_{i=1}^{\chi(k)} \#Q_i \geqslant k\,, \qquad \text{for any} \quad k \in [n]\,.$$

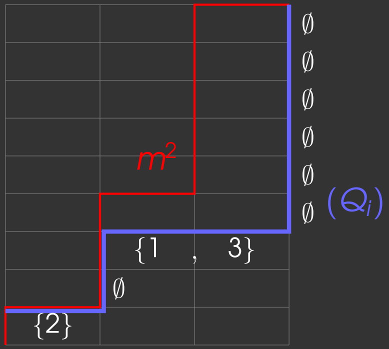
$\mathcal{F}_{m^2}[2]$:

$$(12|\cdot|\cdot|\cdot),\ (1|2|\cdot|\cdot),\ (2|1|\cdot|\cdot),$$
$$(1|\cdot|2|\cdot),\ (2|\cdot|1|\cdot),\ (1|\cdot|\cdot|2),\ (2|\cdot|\cdot|1)$$

with $Q_i := f^{-1}(\{i\})$

Graphic representation

Let $(Q_i) := (\ 2|\cdot|13|\cdot|\cdot|\cdot|\cdot|\cdot|\cdot\ )$ be a $\mathcal{F}_{m^2}$-structure on $\{1,2,3\}$.



Acyclic automata

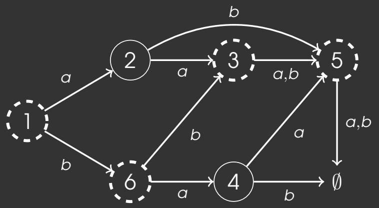
Generalized parking functions

Bijection: (n-i) ADFA–parking functions

Minimal automata

Goal

Explicit bijection :



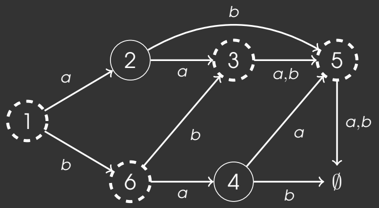
$\longleftrightarrow$

$$(Q_i)$$

$$Q_2 = \{5\}\,,\ Q_{24} = \{6\}\,,$$
$$Q_5 = \{4\}\,,\ Q_{27} = \{2\}\,,$$
$$Q_8 = \{3\}\,,\ Q_{72} = \{1\}\,.$$

Goal

Explicit bijection :



$\longleftrightarrow$

(Q_i)

$$Q_2 = \{5\}\,,\; Q_{24} = \{6\}\,,$$
$$Q_5 = \{4\}\,,\; Q_{27} = \{2\}\,,$$
$$Q_8 = \{3\}\,,\; Q_{72} = \{1\}\,.$$

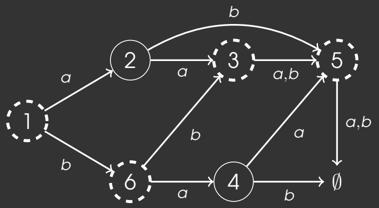
(non-initial) ADFA (A, δ) over an alphabet of k symbols

$\updownarrow$

$2m^k$-parking functions

Goal

~~*Explicit bijection*~~ *Mine the bijection* :



$\longleftrightarrow$

(Q_i)

$$Q_2 = \{5\}\,,\; Q_{24} = \{6\}\,,$$
$$Q_5 = \{4\}\,,\; Q_{27} = \{2\}\,,$$
$$Q_8 = \{3\}\,,\; Q_{72} = \{1\}\,.$$

(non-initial) ADFA (A, δ) over an alphabet of k symbols

$\updownarrow$

$2m^k$-parking functions

Acyclic automata

Generalized parking functions

Bijection: (n-i) ADFA–parking functions

Minimal automata
- Co-reachability
- Simplicity
- Reachability
- Extended automaton
- Conclusion

Co-reachability

Let (A, δ) be a (non-initial) ADFA.

Co-reachable :

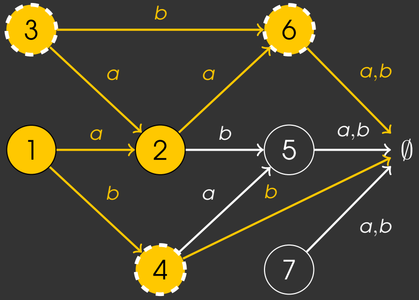
$$\forall q \in [n], \exists w \in \Sigma^*, \qquad \delta^*(q, w) \in A.$$

Co-reachability

Let (A, δ) be a (non-initial) ADFA.

Co-reachable :

$$\forall q \in [n]\,, \exists w \in \Sigma^*\,, \qquad \delta^*(q, w) \in A\,.$$

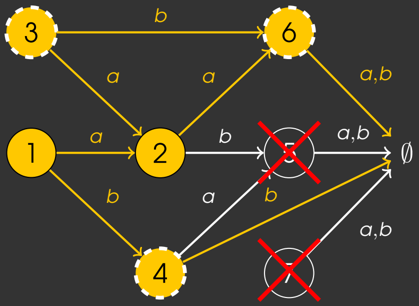


Co-reachability

Let (A, δ) be a (non-initial) ADFA.

Co-reachable :

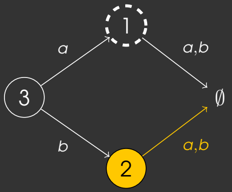
$$\forall q \in [n], \exists w \in \Sigma^*, \qquad \delta^*(q, w) \in A.$$



Yep but. . .

$$(Q_i) := \left(\begin{array}{c||c|c|c||c|c|c|c|c} 2 & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right) \begin{array}{l} \notin A \\ \in A \end{array}$$

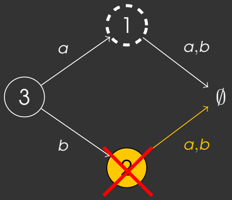
a	$\emptyset$	$\emptyset$	2	2	$\emptyset$	2	1	1	1
b	$\emptyset$	2	$\emptyset$	2	1	1	$\emptyset$	2	1



Yep but…

$(Q_i) :=$

$\notin A$	~~2~~	·	·	·	·	3	·	·	·	·
$\in A$	1	·	·	·	·	·	·	·	·	·
a	∅	∅	2	2	∅	2	1	1	1	
b	∅	2	∅	2	1	1	∅	2	1	



Co-reachability (2)

Let (Q_i) be a $2m^k$-parking function.

Let Θ be the associated (n-i) ADFA.

Co-reachability (2)

Let (Q_i) be a $2m^k$-parking function.

Let Θ be the associated (n-i) ADFA.

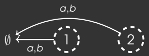
Fact : If $Q_1 = \emptyset$ then Θ is *co-reachable*.

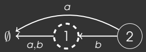
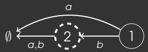
Theorem :

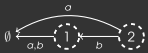
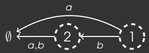
(n-i) co-reachable ADFA $\simeq$ $(2m^k - 1)$-parking functions

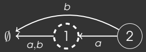
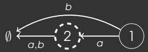
Co-reachability (3)

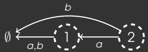
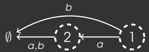
The 13 $(2m^2-1)$-parking functions on $\{1,2\}$ and the associated (n-i) co-reachable ADFA :

$(12|| \cdot | \cdot | \cdot | \cdot | \cdot | \cdot) \rightarrow$  ,

$(1||2| \cdot | \cdot | \cdot | \cdot | \cdot) \rightarrow$  ,  $\leftarrow (2||1| \cdot | \cdot | \cdot | \cdot | \cdot)$

$(1|| \cdot |2| \cdot | \cdot | \cdot | \cdot) \rightarrow$  ,  $\leftarrow (2|| \cdot |1| \cdot | \cdot | \cdot | \cdot)$

$(1|| \cdot | \cdot |2| \cdot | \cdot | \cdot) \rightarrow$  ,  $\leftarrow (2|| \cdot | \cdot |1| \cdot | \cdot | \cdot)$

$(1|| \cdot | \cdot | \cdot |2| \cdot | \cdot) \rightarrow$  ,  $\leftarrow (2|| \cdot | \cdot | \cdot |1| \cdot | \cdot)$

$(1|| \cdot | \cdot | \cdot | \cdot |2| \cdot) \rightarrow$  ,  $\leftarrow (2|| \cdot | \cdot | \cdot | \cdot |1| \cdot)$

$(1|| \cdot | \cdot | \cdot | \cdot | \cdot |2) \rightarrow$  ,  $\leftarrow (2|| \cdot | \cdot | \cdot | \cdot | \cdot |1)$

Simplicity

Let (A, δ) be a (n-i) ADFA.

Right language :

$$L_q = \{ w \in \Sigma^* \quad | \quad \delta^*(q, w) \in A \}$$

Simplicity

Let (A, δ) be a (n-i) ADFA.

Right language :

$$L_q = \{ \; w \in \Sigma^* \quad | \quad \delta^*(q, w) \in A \; \}$$

$\rightsquigarrow$ *Simple* : $\quad (A, \Sigma)$ is *simple* *iff* $\quad L_q \neq L_{q'}$

for each distinct $q, q' \in [n]$.

Simplicity

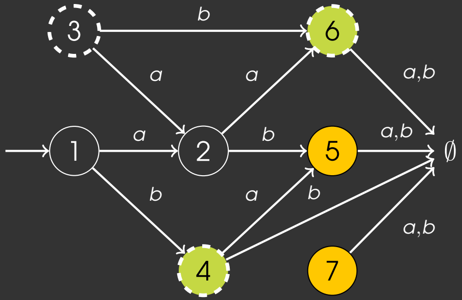
Let (A, δ) be a (n-i) ADFA.

Right language:

$$L_q = \{\ w \in \Sigma^* \quad | \quad \delta^*(q, w) \in A\ \}$$

$\rightsquigarrow$ *Simple*: (A, Σ) is *simple* *iff* $L_q \neq L_{q'}$

for each distinct $q, q' \in [n]$.



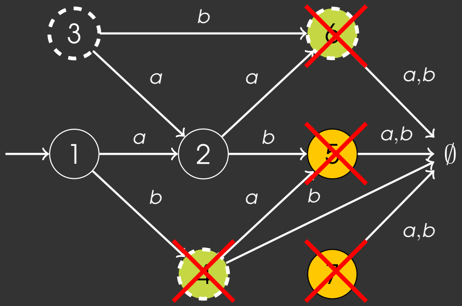
Simplicity

Let (A, δ) be a (n-i) ADFA.

Right language :

$$L_q = \{\ w \in \Sigma^* \quad | \quad \delta^*(q, w) \in A\ \}$$

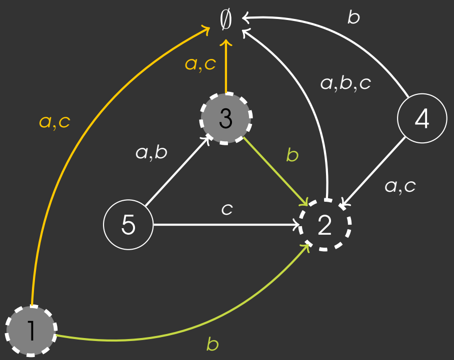
$\rightsquigarrow$ Simple : (A, Σ) is *simple* *iff* $L_q \neq L_{q'}$ for each distinct $q, q' \in [n]$.



And the condition of simplicity?

$(Q_i) \in \mathcal{F}_{2m^3-1}[5]$ *with* $Q_1 = \{2\}$, $Q_4 = \{1,3\}$, $Q_8 = \{4\}$ and $Q_{110} = \{5\}$

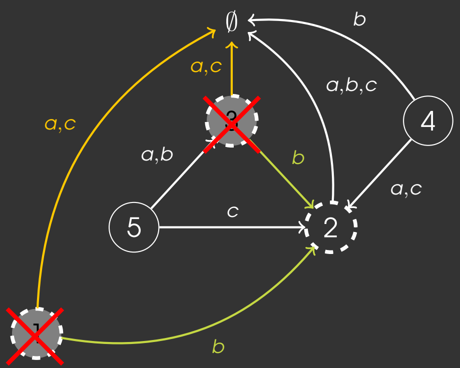
$\downarrow$



And the condition of simplicity?

$(Q_i) \in \mathcal{F}_{2m^3-1}[5]$ *with* $Q_1 = \{2\}$, $Q_4 = \{1, 3\}$, $Q_8 = \{4\}$ and $Q_{110} = \{5\}$

$\downarrow$



Simplicity (2)

Let Θ be (n-i) co-reachable ADFA associated to (Q_i).

Simplicity (2)

Let Θ be (n-i) co-reachable ADFA associated to (Q_i).

Theorem :

$$\Theta \text{ simple} \quad \textit{iff} \quad \#Q_i \leqslant 1\,, \qquad \text{for all} \quad i \in [n].$$

Simplicity (2)

Let Θ be (n-i) co-reachable ADFA associated to (Q_i).

Theorem :

$$\Theta \text{ simple } \quad \textit{iff} \quad \#Q_i \leqslant 1\,, \qquad \text{for all} \quad i \in [n].$$

Proposition : The number of (n-i) co-reachable and simple ADFA

$$\mathcal{S}_k(n) = \sum_{i=1}^{n} (-1)^{i-1}\, \mathcal{S}_k(n-i) \binom{2(n-i+1)^k - 1 - (n-i)}{i}$$

with $\mathcal{S}_k(0) = 1$.

Reachability

Let (i, A, δ) be an ADFA.

Reachable :

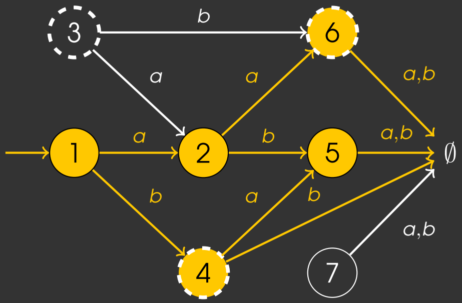
$$\forall q \in [n] \,,\ \exists w \in \Sigma^* \,, \qquad \delta^*(i, w) = q \,.$$

Reachability

Let (i, A, δ) be an ADFA.

Reachable :

$$\forall q \in [n]\,,\ \exists w \in \Sigma^*\,, \qquad \delta^*(i, w) = q\,.$$

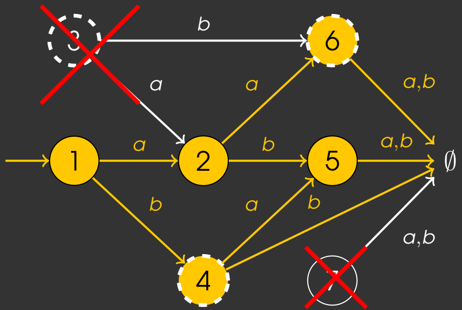


Reachability

Let (i, A, δ) be an ADFA.

Reachable :

$$\forall q \in [n]\,,\ \exists w \in \Sigma^*\,, \qquad \delta^*(i, w) = q\,.$$



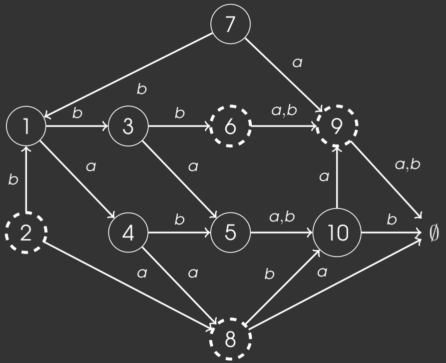
Minimality

Minimal automaton :

$$(i, A, \delta) \text{ minimal} \quad \textit{iff} \quad \begin{cases} \text{reachable,} \\ \text{co-reachable,} \\ \text{simple.} \end{cases}$$

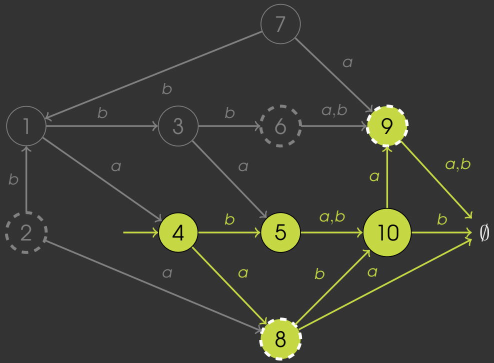
Reachability (2)

Let $\Theta := (A, \delta)$ (co-reachable and simple).



Reachability (2)

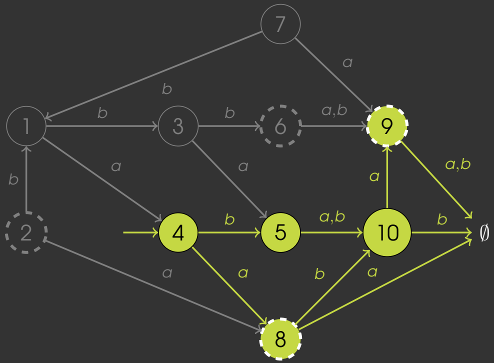
Let $\Theta := (A, \delta)$ (co-reachable and simple).



Reachable part : $\Theta^{(4)}$

Reachability (2)

Let $\Theta := (A, \delta)$ (co-reachable and simple).

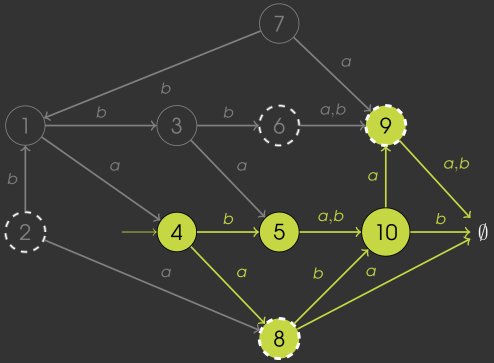


Reachable part : $\Theta^{(4)}$

Proposition : If Θ *co-reachable* and *simple* then $\Theta^{(q)}$ *minimal*.

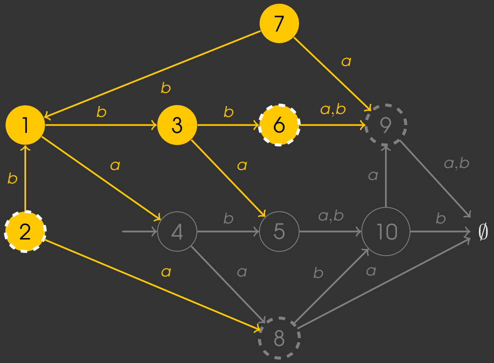
Extended automaton

Complement of $\Theta^{(4)}$: $\bar{\Theta}^{(4)}$



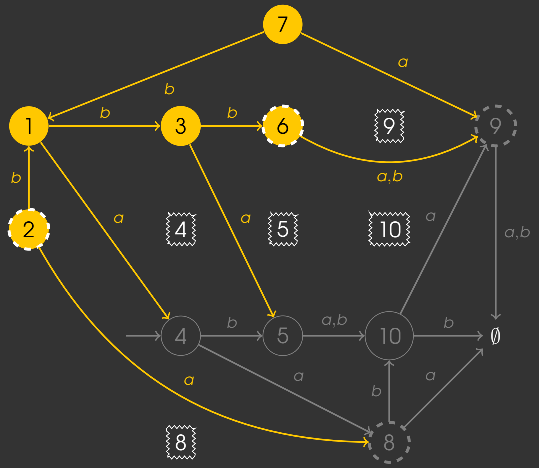
Extended automaton

Complement of $\Theta^{(4)}$: $\bar{\Theta}^{(4)}$



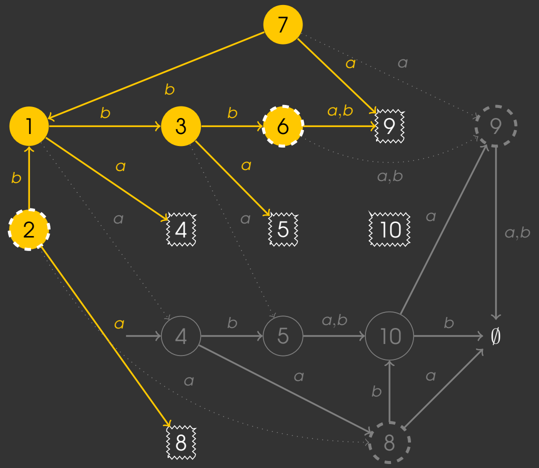
Extended automaton

Complement of $\Theta^{(4)}$: $\bar{\Theta}^{(4)}$



Extended automaton

Complement of $\Theta^{(4)}$: $\bar{\Theta}^{(4)}$



Extended automaton (2)

An *extended automaton* of n states on Σ: (A, T, δ)

- $A \subset [n]$ a set of accepting states,
- T a set of (extra-)absorbing states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\} \cup T$ an extended transition fct°

such that δ *acyclic*.

Extended automaton (2)

An *extended automaton* of n states on Σ: (A, T, δ)

- $A \subset [n]$ a set of accepting states,
- T a set of (extra-)absorbing states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\} \cup T$ an extended transition fct°

such that δ *acyclic*.

LISKOVET *(2003)*:

$$\mathcal{T}_k(n \quad) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \left(2(n-s+1 \qquad)^k\right)^s \mathcal{T}_k(n-s \quad)$$

Extended automaton (2)

An *extended automaton* of n states on Σ: (A, T, δ)

- $A \subset [n]$ a set of accepting states,
- T a set of (extra-)absorbing states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\} \cup T$ an extended transition fct°

such that δ *acyclic*.

LISKOVET *(2003)*:

$$\mathcal{T}_k(n,t) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \left(2(n-s+1+t)^k\right)^s \mathcal{T}_k(n-s,t)$$

Extended automaton (2)

An *extended automaton* of n states on Σ: (A, T, δ)

- $A \subset [n]$ a set of accepting states,
- T a set of (extra-)absorbing states,
- $\delta : [n] \times \Sigma \to [n] \cup \{\emptyset\} \cup T$ an extended transition fct°

such that δ *acyclic*.

LISKOVET *(2003)*:

$$\begin{aligned}\mathcal{T}_k(n,t) &= \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \left(2(n-s+1+t)^k\right)^s \mathcal{T}_k(n-s,t) \\ &= \mathcal{P}(\chi, n)\end{aligned}$$

with $\chi : m \mapsto 2(m+t)^k$.

Extended automaton (3)

Let $\Lambda := (A, T, \delta)$.

- *co-reachable* *iff* $\exists \Theta$ *co-reachable* and $\exists q$ st $\Lambda = \bar{\Theta}^{(q)}$

Lemma :

$$\text{Co-reachable extended automata} \simeq (2(m+t)^k - 1)\text{-parking functions}$$

Extended automaton (3)

Let $\Lambda := (A, T, \delta)$.

- *co-reachable* *iff* $\exists\Theta$ *co-reachable* and $\exists q$ st $\Lambda = \bar{\Theta}^{(q)}$

Lemma :

Co-reachable extended automata

$\simeq$

$(2(m+t)^k - 1)$-parking functions

- *simple* *iff* $\exists\Theta$ *simple* and $\exists q$ st $\Lambda = \bar{\Theta}^{(q)}$

Lemma :

Simple extended automata

$\simeq$

$(2(m+t)^k$)-parking functions (Q_i) with $\#Q_i \leqslant 1$

Extended automaton (3)

Let $\Lambda := (A, T, \delta)$.

- *co-reachable* *iff* $\exists \Theta$ *co-reachable* and $\exists q$ st $\Lambda = \bar{\Theta}^{(q)}$

Lemma:

$$\text{Co-reachable extended automata} \simeq (2(m+t)^k - 1)\text{-parking functions}$$

- *simple* *iff* $\exists \Theta$ *simple* and $\exists q$ st $\Lambda = \bar{\Theta}^{(q)}$

Lemma:

$$\text{Simple (co-reachable) extended automata} \simeq (2(m+t)^k\ (-1))\text{-parking functions } (Q_i) \text{ with } \#Q_i \leqslant 1$$

Theorem S-1

Let $\Theta = (i, A, \zeta)$ be a minimal ADFA.

Extended automata with constraint Θ :

$\mathcal{S}^{(\Theta)} = \{ \Lambda$ *co-reachable* and *simple* $\mid \Lambda \cdot \Theta$ *co-reachable* and *simple* $\}$

Theorem S-1

Let $\Theta = (i, A, \zeta)$ be a minimal ADFA.

Extended automata with constraint Θ :

$$\mathcal{S}^{(\Theta)} = \{\ \Lambda \textit{ co-reachable} \text{ and } \textit{simple} \mid \Lambda \cdot \Theta \textit{ co-reachable} \text{ and } \textit{simple}\ \}$$

Enumeration formula :

$$\mathcal{S}_k(n\quad) = \sum_{i=1}^{n} (-1)^{i-1}\, \mathcal{S}_k(n-i\quad) \binom{2(n-i+1\qquad)^k - 1\qquad - (n-i)}{i}$$

Theorem S-1

Let $\Theta = (i, A, \zeta)$ be a minimal ADFA.

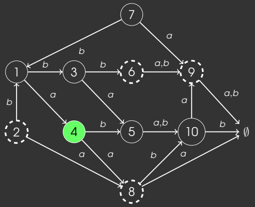
Extended automata with constraint Θ :

$$\mathcal{S}^{(\Theta)} = \{ \Lambda \textit{ co-reachable} \text{ and } \textit{simple} \mid \Lambda \cdot \Theta \textit{ co-reachable} \text{ and } \textit{simple} \}$$

Enumeration formula :

$$\mathcal{S}_k(n, t) = \sum_{i=1}^{n} (-1)^{i-1} \, \mathcal{S}_k(n-i, t) \binom{2(n-i+1+t)^k - 1 - t - (n-i)}{i}$$

Theorem: double counting



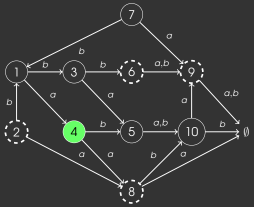
$$\simeq \left(\qquad \right)$$

$$n\,\mathcal{S}_k(n,0) =$$

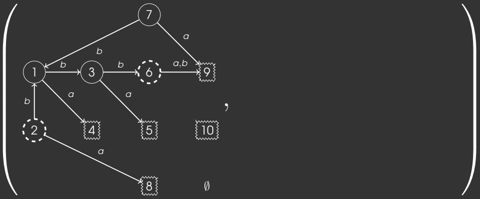
where

and k is the number of symbols of Σ.

Theorem: double counting



$\simeq$



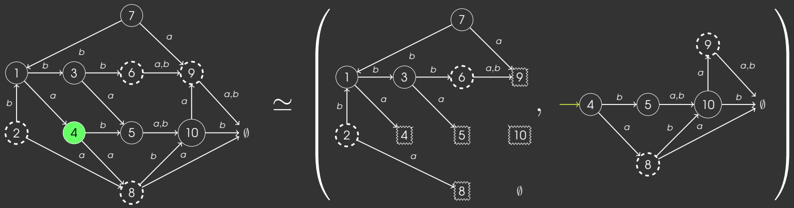
$$n\,\mathcal{S}_k(n,0) \quad = \quad \mathcal{S}_k(n-t,t)$$

where

- $\mathcal{S}_k(n,t)$ is the number extended ADFA of n states, t extra-absorbing states and $t+1$ constraints,

and k is the number of symbols of Σ.

Theorem: double counting



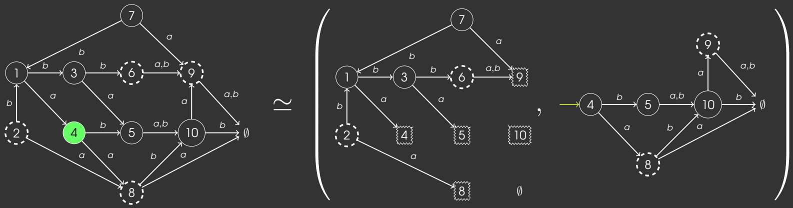
$$n\,\mathcal{S}_k(n,0) \quad = \quad \mathcal{S}_k(n-t,t) \qquad \mathcal{M}_k(t)$$

where

- $\mathcal{S}_k(n,t)$ is the number extended ADFA of n states, t extra-absorbing states and $t+1$ constraints,
- $\mathcal{M}_k(t)$ is the number of minimal ADFA of t states

and k is the number of symbols of Σ.

Theorem: double counting



$$n\,\mathcal{S}_k(n,0) = \sum_{t=1}^{n} \mathcal{S}_k(n-t,t) \quad \mathcal{M}_k(t)$$

where

- $\mathcal{S}_k(n,t)$ is the number extended ADFA of n states, t extra-absorbing states and $t+1$ constraints,
- $\mathcal{M}_k(t)$ is the number of minimal ADFA of t states

and k is the number of symbols of Σ.

Conclusion

Explicit bijection :

- $\mathcal{F}_{2m^k}$ $\longleftrightarrow$ (n-i) ADFA

Conclusion

Explicit bijection:

- $\mathcal{F}_{2m^k}$ $\longleftrightarrow$ (n-i) ADFA
- $\mathcal{F}_{2m^k-1}$ $\longleftrightarrow$ (n-i) *co-reachable* ADFA

Conclusion

Explicit bijection:

- $\mathcal{F}_{2m^k}$ $\longleftrightarrow$ (n-i) ADFA
- $\mathcal{F}_{2m^k-1}$ $\longleftrightarrow$ (n-i) *co-reachable* ADFA
- $\mathcal{F}_{2m^k-1}$ *simple* $\longleftrightarrow$ (n-i) *co-reachable* and *simple* ADFA

Conclusion

Explicit bijection :

- $\mathcal{F}_{2m^k}$ $\longleftrightarrow$ (n-i) ADFA
- $\mathcal{F}_{2m^k-1}$ $\longleftrightarrow$ (n-i) *co-reachable* ADFA
- $\mathcal{F}_{2m^k-1}$ *simple* $\longleftrightarrow$ (n-i) *co-reachable* and *simple* ADFA

Update :

$\mathcal{F}_{2(m+t)^k-1-t}$ *simple* $\leftrightarrow$ *extended co-reachable* and *simple* ADFA with *constraints*

Conclusion

Explicit bijection :

- $\mathcal{F}_{2m^k}$ $\longleftrightarrow$ (n-i) ADFA
- $\mathcal{F}_{2m^k-1}$ $\longleftrightarrow$ (n-i) *co-reachable* ADFA
- $\mathcal{F}_{2m^k-1}$ *simple* $\longleftrightarrow$ (n-i) *co-reachable* and *simple* ADFA

Update :

$\mathcal{F}_{2(m+t)^k-1-t}$ *simple* $\leftrightarrow$ *extended co-reachable* and *simple* ADFA with *constraints*

Enumeration : minimal ADFA

$$n\,\mathcal{S}_k(n,0) = \sum_{t=1}^{n} \mathcal{S}_k(n-t,t)\,\mathcal{M}_k(t)$$