Sum of matrix entries of representations of the symmetric group and its asymptotics

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Partitions

A partition $\lambda \vdash n$ is a non increasing sequence of positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_l)$$

such that $\sum \lambda_i = n$

Example

$$\lambda = (3, 2) \vdash 5$$

$$\lambda = \ydiagram{3,2}$$

Representations

A representation of S_n is a morphism

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$$\pi^\lambda, \quad \dim \lambda := \dim V^\lambda$$

$$\chi^\lambda(\sigma) = tr(\pi^\lambda(\sigma)), \qquad \hat{\chi}^\lambda(\sigma) = \frac{tr(\pi^\lambda(\sigma))}{\dim \lambda}$$

Standard Young tableaux

1	2	8	9	12
3	5	10	13	
4	7			
6				
11				

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$$\begin{ytableau} 1 & 2 & 8 & 9 & 12 \\ 3 & 5 & 10 & 13 \\ 4 & 7 \\ 6 \\ 11 \end{ytableau}$$

$\dim\lambda :=$ number of SYT of shape λ

$$\lambda = (3,2) \Rightarrow \dim\lambda = 5$$

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}$$

Plancherel measure

$$\sum_{\lambda \vdash n} (\dim \lambda)^2 = n!$$

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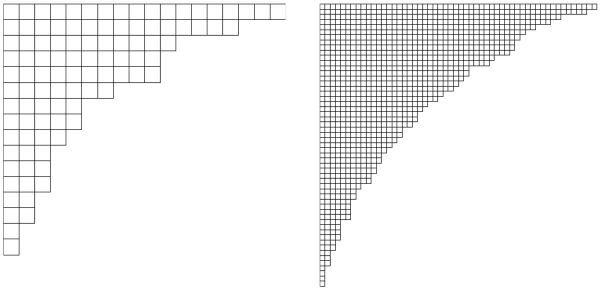
Probability on the set $\mathbb{Y}_n$ of partitions of n

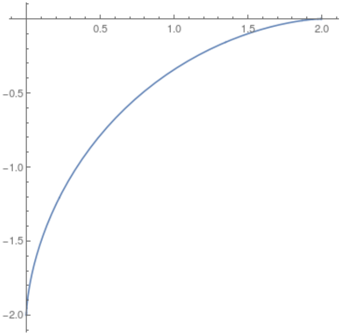
Limit shape

λ distributed with the Plancherel measure and renormalized, then



(a) (b)

Image from D. Romik "The Surprising Mathematics of Longest Increasing Subsequences"



$$\omega_x(\theta) = \left(1 + \frac{2\theta}{\pi}\right)\sin\theta + \frac{2}{\pi}\cos\theta$$

$$\omega_y(\theta) = \left(1 - \frac{2\theta}{\pi}\right)\sin\theta - \frac{2}{\pi}\cos\theta$$

Theorem (Kerov 1999)

$$n^{\frac{|\rho|-m_1(\rho)}{2}}\hat{\chi}^{\lambda}_{\rho} \to \prod_{k\geq 2} k^{m_k(\rho)/2}\mathcal{H}_{m_k(\rho)}(\xi_k)$$

Relations with random matrices

Rows $\lambda_1, \lambda_2, \lambda_3, \ldots$ of a random Young diagram

First, second, third, ... biggest eigenvalues of a Gaussian random Hermitian matrix

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Relations with random matrices

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First, second, third, ... biggest eigenvalues of a Gaussian random Hermitian matrix

Same first order asymptotics
Same joint fluctuation (Tracy-Widom law)

Similar tools: *moment method*, link with *free probability theory*

Signed distance

$$d_k(T) \;=\; \begin{array}{c} \text{length of northeast path from } k \text{ to } k+1 \\ \text{or } - \text{ length of southwest path from } k \text{ to } k+1 \end{array}$$

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$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \Rightarrow d_3(T) = -3$$

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$$(3,4)\ \begin{array}{|c|c|c|c|}\hline 1 & \color{red}{3} & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} \color{red}{4} \\ \cline{1-1}\end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline 1 & \color{red}{4} & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} \color{red}{3} \\ \cline{1-1}\end{array}$$

Young seminormal representation

$$\pi^{\lambda}((k,k+1))_{T,\tilde{T}} = \begin{cases} 1/d_k(T) & \text{if } T = \tilde{T} \\ \sqrt{1 - \frac{1}{d_k(T)^2}} & \text{if } (k,k+1)T = \tilde{T} \\ 0 & \text{else} \end{cases}$$

Example

$$\lambda = (3,2)$$

$$\pi^\lambda((2,4,3)) = \pi^\lambda((3,4)(2,3)) = \pi^\lambda((3,4))\pi^\lambda((2,3))$$

$$= \begin{bmatrix} -1/3 & \sqrt{8/9} & 0 & 0 & 0 \\ \sqrt{8/9} & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3/4} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & \sqrt{3/4} & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -\sqrt{2/9} & \sqrt{2/3} & 0 & 0 \\ \sqrt{8/9} & -1/6 & \sqrt{1/12} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & -\sqrt{3/4} & -1/2 \end{bmatrix}$$

$$0 \le u \le 1$$

Partial trace

$$PT_u^\lambda(\sigma) := \sum_{i \le u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}$$

- We would like to refine Kerov's result

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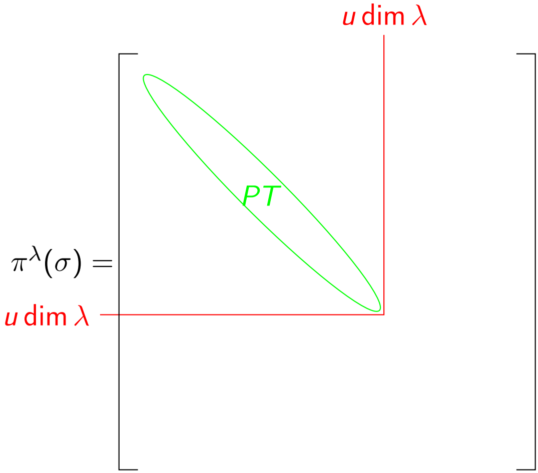
Partial trace

$$PT_u^{\lambda}(\sigma) := \sum_{i \leq u \dim \lambda} \frac{\pi^{\lambda}(\sigma)_{i,i}}{\dim \lambda}$$

- We would like to refine Kerov's result
- The partial trace has been studied in random matrix theory, *e.g.* for orthogonal random matrices

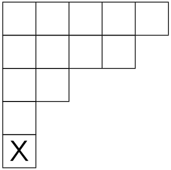
Visually

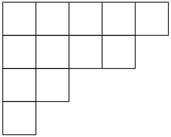


Decomposition of PT

$$\lambda = \ydiagram{5,4,2,1,1}$$

Decomposition of PT

$\lambda =$ 

$\mu_1 =$ 

Decomposition of PT

$$\lambda = \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & X \\ \ \\ \ \end{ytableau} \qquad \mu_1 = \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ \\ \ \end{ytableau} \qquad \mu_2 = \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ \\ \ \\ \ \end{ytableau} \qquad \cdots$$

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Proposition (DS)

$$PT_u^{\lambda}(\sigma) = \sum_{i<\bar{k}} \frac{\chi^{\mu_i}(\sigma)}{\dim\lambda} + \mathrm{Rem}$$

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Proposition (DS)

$$PT_u^{\lambda}(\sigma) = \sum_{i<\bar{k}} \frac{\chi^{\mu_i}(\sigma)}{\dim \lambda} + \mathrm{Rem}$$

$$\mathrm{Rem} = \sum_{i \leq \tilde{u} \dim \mu_{\bar{k}}} \frac{\pi^{\mu_{\bar{k}}}(\sigma)_{i,i}}{\dim \lambda} = \frac{\dim \mu_{\bar{k}}}{\dim \lambda} PT_{\tilde{u}}^{\mu_{\bar{k}}}(\sigma)$$

Proof

$$\pi^{\lambda}(\sigma) = \begin{bmatrix} \boxed{\pi^{\mu_1}(\sigma)} & & & \mathbb{0} \\ & \boxed{\pi^{\mu_2}(\sigma)} & & \\ & & \boxed{\pi^{\mu_3}(\sigma)} & \\ \mathbb{0} & & & \ddots \end{bmatrix}$$

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$$PT_u^\lambda(\sigma) = \sum_{j<\bar{k}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) + \mathrm{Rem}$$

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$$F_{sc}(c) n^{-\frac{|\rho| - m_{\mathbf{1}}(\rho)}{2}} \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k)$$

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$$A \cdot n^{-\frac{|\rho|-m_1(\rho)}{2}} B$$

Theorem (Kerov 1993)

$$\sum_{j<\bar{k}} \frac{\dim \mu_j}{\dim \lambda} \to A \text{ (deterministic)}$$

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The two objects are *asymptotically independent*

First, a definition

Contents

$$c(\square) := col(\square) - row(\square)$$

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0	1	2	3	4
-1	0	1	2	
-2	-1			
-3				
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Jucys-Murphy elements

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$$\pi^{\lambda}(J_k) = \begin{bmatrix} c_{T_1}(\boxed{k}) & & \mathbb{0} \\ & c_{T_2}(\boxed{k}) & \\ \mathbb{0} & & \ddots \end{bmatrix}$$

$$\chi^\lambda(J_2 + \ldots + J_n)$$

$$\chi^\lambda(J_2 + \ldots + J_n) = \sum_{i=2}^{n} \chi^\lambda(J_i)$$

$$\chi^{\lambda}(J_2 + \ldots + J_n) = \sum_{i=2}^{n} \chi^{\lambda}(J_i) = \sum_{i=2}^{n} \sum_{k=1}^{\dim \lambda} c_{T_k}(\boxed{i})$$

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$$\binom{n}{2}\chi^\lambda(\tau) = \chi^\lambda(J_2 + \ldots + J_n) = \sum_{i=2}^{n} \chi^\lambda(J_i) = \sum_{i=2}^{n} \sum_{k=1}^{\dim\lambda} c_{T_k}(\boxed{i}) = \dim\lambda \sum_{\square\in\lambda} c(\square)$$

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$$c_\rho n^{\downarrow(|\rho|-m_1(\rho))}\hat{\chi}^\lambda_\rho = \prod_{i=1}^{l}\left(\sum_{\square \in \lambda} c(\square)^{\nu_i}\right) - \sum_{\tilde{\rho}<\rho} c_{\tilde{\rho}} n^{\downarrow(|\tilde{\rho}|-m_1(\tilde{\rho}))}\hat{\chi}^\lambda_{\tilde{\rho}}$$

where $\rho_i = \nu_i + 1$

Considering $\chi^\lambda(J_2 + \ldots + J_n)$ we get

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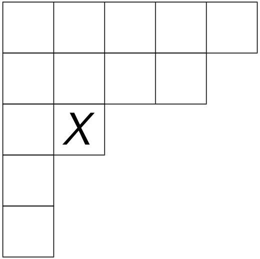
$$\hat{\chi}^\lambda(\sigma) n^{\frac{|\rho| - m_1(\rho)}{2}} \sim \prod_{i=1}^{l}\left(\sum_{\square \in \lambda} c(\square)^{\nu_i}\right)$$

$\mu \nearrow \lambda =$ 

$$\hat{\chi}^{\mu}(\sigma) n^{\frac{|\rho| - m_1(\rho)}{2}} \quad \sim \quad \prod_{i=1}^{l} \left(\sum_{\square \in \mu} c(\square)^{\nu_i} \right)$$

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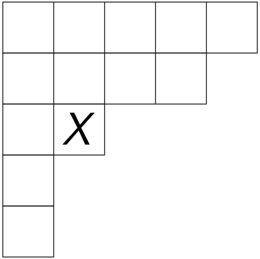

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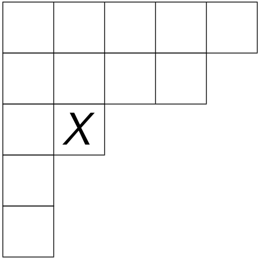
$$= \prod_{i=1}^{l} \left(\sum_{\square \in \lambda} c(\square)^{\nu_i} - c(\boxed{X})^{\nu_i} \right)$$

$\mu \nearrow \lambda =$ 

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$$\wr$$

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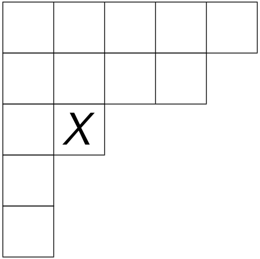

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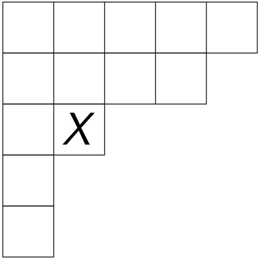
$\mu \nearrow \lambda =$ 

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$$\downarrow$$

$$A \cdot B$$

Telescopic sum

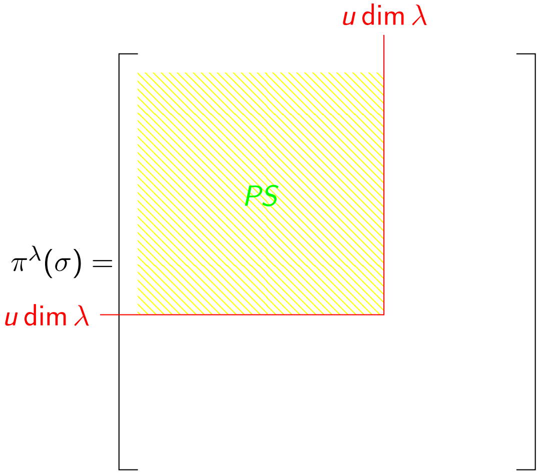
$$PT_u^\lambda(\sigma) = \sum_{j<\bar{k}_1} \frac{\dim \mu_j^{(1)}}{\dim \lambda} \hat{\chi}^{\mu_j^{(1)}}(\sigma) + \sum_{j<\bar{k}_2} \frac{\dim \mu_j^{(2)}}{\dim \lambda} \hat{\chi}^{\mu_j^{(2)}}(\sigma) + \dots$$

Unfortunately, I cannot prove convergence...

Partial sum

$$PS_u^{\lambda}(\sigma) := \sum_{i,j \leq u \dim \lambda} \frac{\pi^{\lambda}(\sigma)_{i,j}}{\dim \lambda}$$

Visually



Decomposition of PS

$$\sigma \in S_r$$

$$PS_u^{\lambda}(\sigma) = \sum_{j<\bar{k}} \frac{\dim \mu_j}{\dim \lambda} PS_1^{\mu_j}(\sigma) + \text{ Rem}$$

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And we have convergence $PS_u^\lambda(\sigma) \to u\mathbb{E}_{PL}^r \left[PS_1^{\cdot}(\sigma)\right]$

THANK YOU