A geometric approach for the upper bound theorem for Minkowski sums of convex polytopes

Eleni Tzanaki

joint work with Menelaos Karavelas

University of Crete

75th Sèminaire Lotharingien de Combinatoire, Bertinoro

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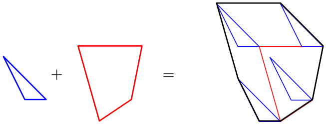
Minkowski sum

- Given two sets P_1 and P_2, their Minkowski sum is defined as

$$P_1 + P_2 = \{p + q \,|\, p \in P_1, q \in P_2\}.$$



- If P_1 and P_2 are convex, then $P_1 + P_2$ is also convex
  - In particular, if P_1 and P_2 are convex polytopes, so is $P_1 + P_2$.

- For the convex polytope case, $f_k(P_1 + P_2)$ is maximized if P_1 and P_2 are in *general position* (cf. [Fukuda & Weibel 2007]).

The general problem

- Let $P_{[r]} = P_1 + P_2 + \cdots + P_r$ be the Minkowski sum of r convex d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$ with $n_1, \ldots, n_r$ vertices, respectively.

Question

What is the maximum number of k-faces $f_k(P_{[r]})$ of $P_{[r]}$, for $0 \leq k \leq d-1$?

- In other words we seek to find a function $F_{k,d}(n_1, \ldots, n_r)$ such that, for all possible $P_1, P_2, \ldots, P_r$, we have

$$f_k(P_{[r]}) \leq F_{k,d}(n_1, \ldots, n_r)$$

and $F_{k,d}(n_1, \ldots, n_r)$ is as small as possible (ideally: *tight*).

Previous work – Early approaches

- *Zonotope bounds* (cf. [Gritzmann & Sturmfels 1993]):

$$f_l(P_1 + P_2 + \cdots + P_r) \leq 2\binom{n}{l} \sum_{j=0}^{d-1-l} \binom{n-l-1}{j},$$

  where n is the number of non-parallel edges of the r polytopes.
- The *trivial* bound (cf. [Fukuda & Weibel 2007]): for $d \geq 2$ and $r \geq 2$:

$$f_k(P_1 + P_2 + \cdots + P_r) \leq \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \ldots + s_r = k + r}} \prod_{i=1}^{r} \binom{n_i}{s_i}, \qquad 0 \leq k \leq d-1.$$

  - Tight for $d \geq 4$, $r \leq \lfloor \frac{d}{2} \rfloor$ and $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$.
- Bounds on vertices:

$$f_0(P_1 + P_2 + \cdots + P_r) \leq \prod_{i=1}^{r} n_i, \qquad 2 \leq r \leq d-1.$$

  - For $r \geq d$, the above bound cannot be attained (cf. [Sanyal 2009]).
  - Tight upper bounds for $r \geq d$ have been shown in [Weibel 2012].

Previous work – Recent approaches

- Bounds for two polytopes in any dimension (cf. [Karavelas & T. 2012]):

The UBTM for two d-polytopes in $\mathbb{R}^d$

Let P_1, P_2 be d-polytopes, $d \geq 2$, with $n_j \geq d+1$ vertices, $j = 1, 2$. Then:

$$f_{k-1}(P_1 + P_2) \leq f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \sum_{j=1}^{2} \binom{n_j - d - 2 + i}{i},$$

where $1 \leq k \leq d$, and $C_d(n)$ stands for the cyclic d-polytope with n vertices. These bounds are tight.

- Result extended to three polytopes in [Karavelas, Konaxis & T. 2013].
- Problem fully resolved in [Adiprasito & Sanyal 2014] using techniques from Combinatorial Commutative Algebra.

Our result

Theorem [Karavelas & T. 2015]

Let $P_1, \ldots, P_r$ be r d-polytopes in $\mathbb{R}^d$ with $n_i \geq d+1$ vertices, $1 \leq i \leq r$.
Then, for $r < d$ and all $1 \leq k \leq d$, we have:

$$f_{k-1}(P_1 + \cdots + P_r) \leq \sum_{\emptyset \subset R \subseteq [r]} (-1)^{r-|R|} f_{k+r}\big(C_{d+r-1}(n_R)\big) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{i}{k-d+1+i} \sum_{\emptyset \subset R \subset [r]} \Phi_{i,d}^{(r-|R|)}(\boldsymbol{n}_R)$$

$C_\delta(\nu)$ is the cyclic δ-polytope with ν vertices, $n_R = \sum_{i \in R} n_i$, $\boldsymbol{n}_R = (n_i : i \in R)$ and $\Phi_{k,d}^{(m)}(\boldsymbol{n}_R)$ is defined by:

$$\Phi_{k,d}^{(0)}(\boldsymbol{n}_R) = \begin{cases} \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + k}{k}, & 0 \leq k \leq \lfloor \frac{d+|R|-1}{2} \rfloor \\ \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - 1 - k}{d - |R| - 1 - k} + \sum_{\emptyset \subset S \subset R} \Phi_{d+|R|-1-k,d}^{(|R|-|S|)}(\boldsymbol{n}_S), & k > \lfloor \frac{d+|R|-1}{2} \rfloor \end{cases}$$

$$\Phi_{k,d}^{(m)}(\boldsymbol{n}_R) = \Phi_{k,d}^{(m-1)}(\boldsymbol{n}_R) - \Phi_{k-1,d}^{(m-1)}(\boldsymbol{n}_R), \quad m > 1$$

This bound is tight.

Our approach

- We consider the *Cayley polytope* of $P_1, \ldots, P_r$ and we adapt the steps of McMullen's proof for the UBT
  - simplicial polytopes
  - shellings

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- Given a d-polytope P
  - $\boldsymbol{f}(P) = \big(f_{-1}(P), f_0(P), \ldots, f_{d-1}(P)\big)$, where $f_k(P) = \#$ of k-faces of P
  - $\boldsymbol{h}(P) = \big(h_0(P), h_1(P), \ldots, h_d(P)\big)$

    where

$$h_k(P) := \textstyle\sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k} f_{i-1}(P), \quad 0 \le k \le d.$$

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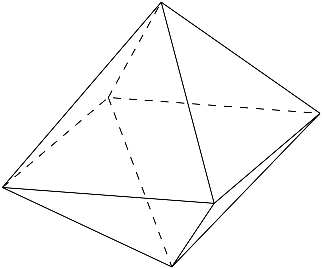
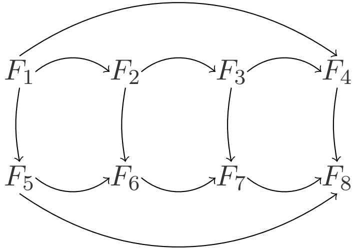
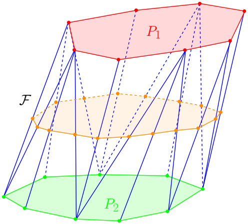
- For simplicial polytopes: $h_k(P)$ counts the number of vertices of the *oriented dual graph* of P, of in-degree k

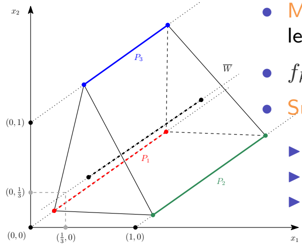




$$h = (1, 3, 3, 1)$$

The Cayley embedding & the Cayley trick





- Cayley embedding: Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{r-1}$ be the (standard) affine basis of $\mathbb{R}^{r-1}$. We embed each P_i in $\mathbb{R}^{d+r-1}$ using the inclusion $\mu_i(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{e}_{i-1})$
- Cayley polytope: $\mathcal{C}_{[r]} = \operatorname{conv}(P_1, \ldots, P_r)$
- Cayley trick: the Minkowski sum $P_1 + \cdots + P_r$ is the intersection of $\mathcal{C}_{[r]}$ with the d-flat $\overline{W}$ of $\mathbb{R}^{d+r-1}$

  $$\overline{W} = \{\tfrac{1}{r}\boldsymbol{e}_0 + \tfrac{1}{r}\boldsymbol{e}_1 + \cdots + \tfrac{1}{r}\boldsymbol{e}_{r-1}\} \times \mathbb{R}^d$$
- Mixed faces: $\mathcal{F}_{[r]}$ is the set of faces in $\mathcal{C}_{[r]}$ having at least one vertex from each P_i, $\quad 1 \leq i \leq r$
- $f_k(\mathcal{F}_{[r]}) = f_{k-r+1}(P_1 + \cdots + P_r)$, for all $0 \leq k \leq d$
- Substructure of $\mathcal{C}_{[r]}$: For $\emptyset \subset R \subseteq [r]$
  - $\mathcal{C}_R$: the Cayley polytope of P_i, $i \in R$
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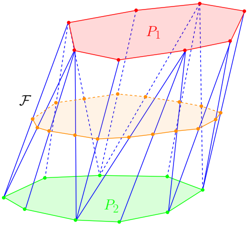

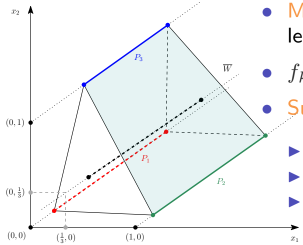

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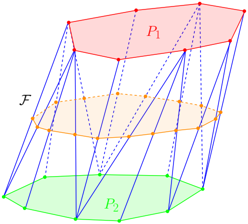

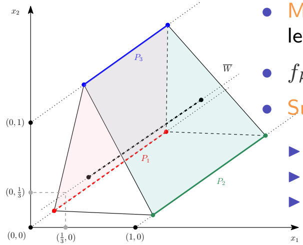

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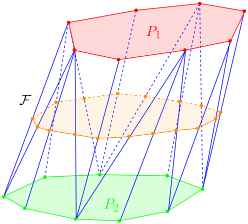

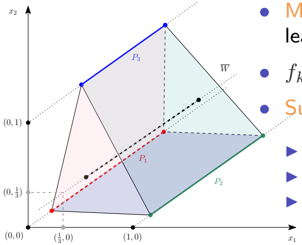

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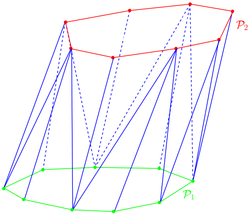
Simplicialization of $\mathcal{C}_R$

- WLOG assume that
  - each P_i is a simplicial d-polytope
  - all faces in $\mathcal{F}_R$, $\emptyset \subset R \subseteq [r]$ are simplices
- Perform repeated stellar subdivisions to triangulate each $\mathcal{C}_R$, $\emptyset \subset R \subset [r]$:



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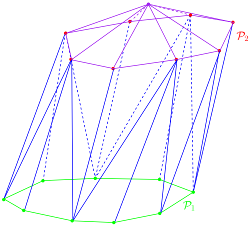

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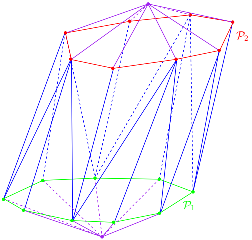

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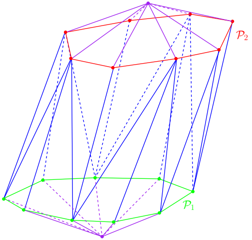

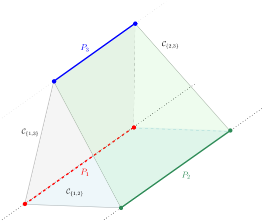

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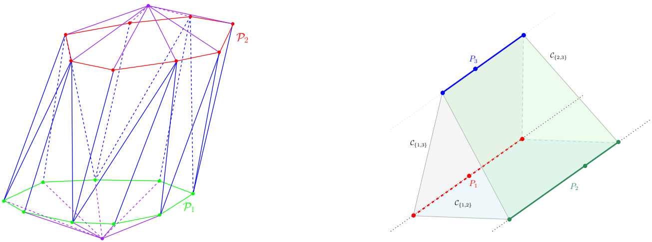

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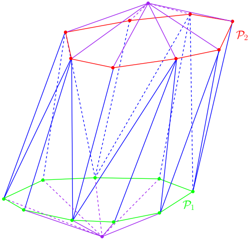

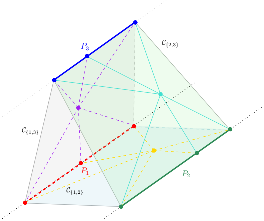

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Dehn-Sommerville equations

For all $\emptyset \subset R \subseteq [r]$ we have:

- $$f_k(\partial \mathcal{Q}_R) = \sum_{\emptyset \subset S \subseteq R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1}\, f_{k-i}(\mathcal{F}_S)$$
- $$f_k(\partial \mathcal{Q}_R) = f_k(\mathcal{K}_{[r]}) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|}\, f_{k-1-i}(\mathcal{K}_S),$$

where:

$$S^k_m := \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^m, \quad 0 \le k \le m$$

are the Stirling numbers of the second kind.

Dehn-Sommerville equations

For all $\emptyset \subset R \subseteq [r]$ we have:

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- $h_k(\partial \mathcal{Q}_R) = h_k(\mathcal{K}_R) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} E^i_{|R|-|S|} h_{k-1-i}(\mathcal{K}_S)$

where:

$$E^k_m = \sum_{i=0}^{k} (-1)^i \binom{m+1}{i} (k+1-i)^m, \qquad m \geq k+1 > 0,$$

are the Eulerian numbers.

Dehn-Sommerville equations

For all $\emptyset \subset R \subseteq [r]$ we have:

- $h_k(\partial \mathcal{Q}_R) = h_k(\mathcal{F}_R) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} E^i_{|R|-|S|} h_{k-i}(\mathcal{F}_S)$
- $h_k(\partial \mathcal{Q}_R) = h_k(\mathcal{K}_R) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} E^i_{|R|-|S|} h_{k-1-i}(\mathcal{K}_S)$

where:

$$E^k_m = \sum_{i=0}^{k} (-1)^i \binom{m+1}{i} (k+1-i)^m, \qquad m \geq k+1 > 0,$$

are the Eulerian numbers.

Lemma (DS for the Cayley Polytope)

$$h_{d+|R|-1-k}(\mathcal{F}_R) = h_k(\mathcal{K}_R), \quad \text{for all } 0 \leq k \leq d+|R|-1 \text{ and } \emptyset \subset R \subseteq [r]$$

Lemma (DS for simplicial polytopes)

$$h_k(P) = h_{d-k}(P), \quad \text{for all } 0 \leq k \leq d$$

Link/non-link relations (use shellings)

$$(k+1)h_{k+1}(\partial P) + (\dim(P) - k)h_k(\partial P) = \sum_{v \in \mathrm{vert}(P)} h_k(\partial P/v)$$

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Lemma

For all $v \in \mathrm{vert}(P_i)$ and all $0 \leq k \leq d+|R|-2$ we have:

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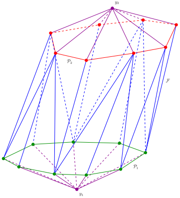

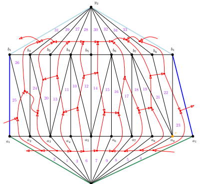

The recurrence relation for $\boldsymbol{h}(\mathcal{F})$

Lemma

For all $0 \leq k \leq d + |R| - 1$, and all $\emptyset \subset R \subseteq [r]$, we have:

$$h_{k+1}(\mathcal{F}_R) \leq \frac{n_R - (d+|R|-1)+k}{k+1} h_k(\mathcal{F}_R) + \sum_{i \in R} \frac{n_i}{k+1} g_k(\mathcal{F}_{R\setminus\{i\}}).$$

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Lemma (the recurrence for polytopes)

For every simplicial d-polytope P and all $0 \leq k \leq d$:

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Induction on $k \rightsquigarrow h_k(P) \leq \binom{n-d-1+k}{k}$

Upper bounds for $h_k(\mathcal{F})$ and $h_k(\mathcal{K})$

Lemma

For all $0 \leq k \leq d + |R| - 1$, we have:

- $h_k(\mathcal{F}_R) \leq \Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$,
- $h_k(\mathcal{K}_R) \leq \Psi_{k,d}(\boldsymbol{n}_R)$.

First equality holds if $\mathcal{C}_R$ is R-neighborly.
Second equality holds if, for all $\emptyset \subset S \subseteq R$, $\mathcal{C}_S$ is S-neighborly (*Minkowki-neighborly)*.

$\Phi^{(m)}_{k,d}(\boldsymbol{n}_R)$ and $\Psi_{k,d}(\boldsymbol{n}_R)$ are defined via the following conditions:

- $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + k}{k}$, $0 \leq k \leq \lfloor \frac{d+|R|-1}{2} \rfloor$,
- $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R) = \Phi^{(m-1)}_{k,d}(\boldsymbol{n}_R) - \Phi^{(m-1)}_{k-1,d}(\boldsymbol{n}_R)$, $m > 0$,
- $\Psi_{k,d}(\boldsymbol{n}_R) = \sum_{\emptyset \subset S \subseteq R} \Phi^{(|R|-|S|)}_{k,d}(\boldsymbol{n}_S)$,
- $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R)$,

where $\boldsymbol{n}_R = (n_i \, : \, i \in R)$.

Upper bounds for the Minkowski sum

Cayley trick:

$$f_{k-r}(P_{[r]}) = f_{k-1}(\mathcal{F}_{[r]}), \qquad r \leq k \leq d+r-1$$

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&= \ldots \\
&= \sum_{\emptyset \subset R \subseteq [r]} (-1)^{r-|R|} f_k\left(C_{d+r-1}(n_R)\right) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{j}{k-d-r+1+i} \sum_{\emptyset \subset R \subset [r]} \Phi_{i,d}^{(r-|R|)}
\end{aligned}$$

Tight bound construction

Theorem

There exist d-polytopes P_i, $1 \leq i \leq r$, for which the upper bounds are attained.

- Define the curves $(1 \leq i \leq r)$:

$$\gamma_i(t) := (0, \ldots, 0, \underbrace{t}_{i-th \text{ coordinate}}, 0, \ldots, 0, t^2, \ldots, t^{d-r+1}),$$

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  where $t_{i,j} := x_{i,j}\tau^i$ are chosen so that
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- For $\tau \to 0^+$ *sufficiently small*, the upper bounds are attained.

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- For $\tau \to 0^+$ *sufficiently small*, the upper bounds are attained.
- To make the construction full dimensional:

$$\widetilde{\gamma}_i(t;\zeta) := (\zeta t^{d-r+2}, \ldots, \zeta t^{d-r+i}, t, \zeta t^{d-r+i+2}, \ldots, \zeta t^{d+1}, t^2, t^3, \ldots, t^{d-r+1})$$

  - For $\zeta \to 0^+$ *sufficiently small*, the combinatorial structure does not change

Open problems

We are interested in devising tight upper bounds for the following three settings:

- When we are restricted to the class of simple polytopes
- When we know the numbers of facets of the polytopes, rather than their number of vertices.
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Thank you for your attention!