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SL_2-TILINGS DO NOT EXIST IN HIGHER DIMENSIONS (MOSTLY)

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Abstract. We define a family of generalizations of SL_2-tilings to higher dimensions called $\boldsymbol{\epsilon}$-SL_2-tilings. We show that, in each dimension 3 or greater, $\boldsymbol{\epsilon}$-SL_2-tilings exist only for certain choices of $\boldsymbol{\epsilon}$. In the case that they exist, we show that they are essentially unique and have a concrete description in terms of odd Fibonacci numbers.

1. Introduction

An SL_2-tiling of the plane (see Definition 1) is, loosely speaking, obtained by assigning a positive integer to each integral point of the plane in such a way that all the 2×2 matrices formed by squares of adjacent entries have determinant 1. This definition was introduced by I. Assem, C. Reutenauer and D. Smith in [1], and was used by these authors to obtain explicit formulas describing cluster variables in cluster algebras of type A. SL_2-tilings can also be viewed as T-systems of type A_1 (e.g. see [3, Remark 2.1]), where all variables are evaluated in positive integers.

The fact that SL_2-tilings of the plane even exist may be a little surprising in itself. On the contrary, already in [1] it was shown that there are infinitely many of them. More recently C. Bessenrodt, T. Holm and P. Jørgensen [2] classified all such tilings using triangulations of some suitable infinity-gon.

In this note, we introduce a higher-dimensional analogue of SL_2-tilings: integers will be assigned to points in $\mathbb{Z}^n$, for $n \geq 2$, and 2×2 matrices of adjacent entries in every slice will be required to satisfy a determinantal identity.

Although our definition seems like a natural generalization of the notion of SL_2-tilings, our main result (Theorem 12) states that, if $n \geq 3$, then the objects it defines almost never exist.

2. SL_2-Tilings of the Plane

The aim of this note is to study higher-dimensional analogues of the following object.

Definition 1 ([1])**.** *A bi-infinite array* $(a_{ij})_{i,j\in\mathbb{Z}}$ *with* $a_{ij} \in \mathbb{Z}_{>0}$ *is called an* SL_2-tiling of $\mathbb{Z}^2$ *if the entries satisfy the relation*

$$a_{i,j+1}a_{i+1,j} - a_{ij}a_{i+1,j+1} = 1. \tag{1}$$

A bi-infinite array $(b_{ij})_{i,j\in\mathbb{Z}}$ *with* $b_{ij} \in \mathbb{Z}_{>0}$ *is called an* anti-SL_2-tiling of $\mathbb{Z}^2$ *if the entries satisfy the relation*

$$b_{i,j+1}b_{i+1,j} - b_{ij}b_{i+1,j+1} = -1. \tag{2}$$

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The notion of an anti-SL_2-tiling is not actually giving anything new as shown by the following lemma, however this notion will be useful for our considerations in higher dimensions.

Lemma 2. *If $(a_{ij})_{i,j\in\mathbb{Z}}$ is an SL_2-tiling, then, by taking $b_{ij} = a_{i,-j}$, one obtains an anti-SL_2-tiling.*

One should think of the difference between SL_2-tilings and anti-SL_2-tilings as viewing the lattice $\mathbb{Z}^2$ "from above" or "from below." The following result from [1] was our starting point.

Theorem 3 ([1])**.** *There exist infinitely many SL_2-tilings of $\mathbb{Z}^2$.*

In fact, it is shown in [1] that any admissible frontier of 1's in the lattice can be completed into a unique SL_2-tiling. An interpretation of all possible SL_2-tilings was later given in [2] in terms of triangulations of a polygon with infinitely many vertices.

The following anti-SL_2-tiling will be relevant in our higher dimensional analysis. We will call it the *staircase* anti-SL_2-tiling of $\mathbb{Z}^2$.

Example 4. *Consider the anti-SL_2-tiling $(a_{ij})_{i,j\in\mathbb{Z}}$ of $\mathbb{Z}^2$ with $a_{ij} = 1$ if $i + j \in \{0, 1\}$. Using (2) and the well-known recursion $F_{2r-1}F_{2r+3} = F_{2r+1}^2 + 1$ $(r \geq 1)$ for the odd Fibonacci numbers, it is easy to see that*

$$a_{ij} = \begin{cases} F_{2r-1} & \text{if } i + j = r \geq 1; \\ F_{-2r+1} & \text{if } i + j = r \leq 0; \end{cases}$$

where we number the Fibonacci numbers as:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	$\cdots$
1	1	2	3	5	8	13	$\cdots$

The following figure is a portion of this tiling. Note the bold frontier of 1's; it is an "infinite staircase".

1	**1**	2	5	13	34	89	233
2	**1**	**1**	2	5	13	34	89
5	2	**1**	**1**	2	5	13	34
13	5	2	**1**	**1**	2	5	13
34	13	5	2	**1**	**1**	2	5
89	34	13	5	2	**1**	**1**	2
233	89	34	13	5	2	**1**	**1**
610	233	89	34	13	5	2	**1**

3. SL_2-Tilings in Higher Dimensions

For a fixed integer $n \geq 2$, denote vectors in $\mathbb{Z}^n$ by $\boldsymbol{i} = (i_1 \ldots, i_n)$ and let $\boldsymbol{e}_k$ be the k-th unit vector in the same lattice. A *signature matrix* is a symmetric $n \times n$ matrix $\boldsymbol{\epsilon} = (\epsilon_{k\ell})$ with $\epsilon_{k\ell} = \pm 1$ whenever $k \neq \ell$ and $\epsilon_{kk} = -1$.

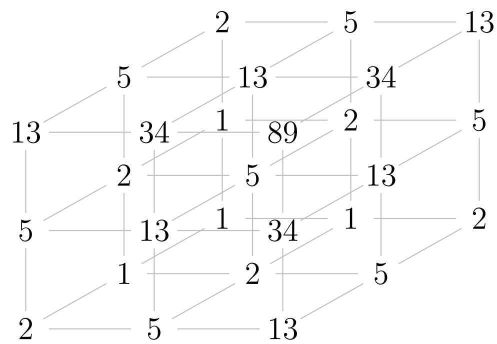
Definition 5. *Fix a signature matrix $\boldsymbol{\epsilon}$. An array $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ with $a_{\boldsymbol{i}} \in \mathbb{Z}_{>0}$ is called an* $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^n$ *if for all k and ℓ with $k \neq \ell$ we have*

$$a_{\boldsymbol{i}+\boldsymbol{e}_\ell} a_{\boldsymbol{i}+\boldsymbol{e}_k} - a_{\boldsymbol{i}} a_{\boldsymbol{i}+\boldsymbol{e}_k+\boldsymbol{e}_\ell} = \epsilon_{k\ell}. \tag{3}$$

The requirement on the diagonal entries of signature matrices might seem arbitrary right now because they do not play any role in the above definition; we will see later on that it is indeed a consistent choice.

In the case $n = 2$ there are only two possible signature matrices recovering the notions of SL_2-tilings and anti-SL_2-tilings of $\mathbb{Z}^2$.

The following is a portion of an $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^3$ with all entries of $\boldsymbol{\epsilon}$ equal to -1.



We will say that two $n \times n$ signature matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are *equivalent* if any $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^n$ can be made into an $\boldsymbol{\epsilon}'$-SL_2-tiling of $\mathbb{Z}^n$ by applying some linear transformation to its indices (i.e., elements of $\mathbb{Z}^n$). By Lemma 2, all signature matrices with $n = 2$ are equivalent. The situation is more complicated when $n \geq 3$ since not all signature matrices are equivalent. On the other hand, there are easy ways of constructing signature matrices equivalent to a given signature matrix $\boldsymbol{\epsilon}$.

Lemma 6. *Let $\boldsymbol{\epsilon} = (\epsilon_{k\ell})$ be any signature matrix and write $\boldsymbol{\epsilon}^{(r)}$ for the matrix obtained from $\boldsymbol{\epsilon}$ by changing the sign of all the entries in row r and column r, leaving the diagonal entries fixed. That is, $\boldsymbol{\epsilon}^{(r)} = (\epsilon'_{k\ell})$ where $\epsilon'_{k\ell} = -\epsilon_{k\ell}$ if exactly one of k and ℓ equals r and $\epsilon'_{k\ell} = \epsilon_{k\ell}$ otherwise. If $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ is an $\boldsymbol{\epsilon}$-SL_2-tiling, then, by taking $b_{\boldsymbol{i}} = a_{\boldsymbol{i}-2i_r\boldsymbol{e}_r}$, one obtains an $\boldsymbol{\epsilon}^{(r)} - \mathrm{SL}_2$-tiling.*

Proof. Indeed, all the relations (3) not involving the index r are satisfied for $(b_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ because they are satisfied for $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$. However, the relations involving the index r pick up a sign when passing from $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ to $(b_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$, as desired. □

Definition 7. *If $\boldsymbol{\epsilon}$ is a signature matrix such that $\epsilon_{k\ell} = 1$ (respectively $\epsilon_{k\ell} = -1$ whenever $k \neq \ell$, we refer to an $\boldsymbol{\epsilon}$-SL_2-tiling as an* SL_2-tiling *(respectively* anti-SL_2-tiling*) of $\mathbb{Z}^n$.*

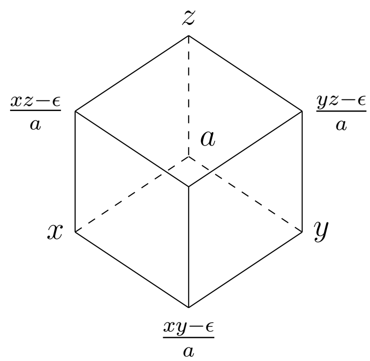
Lemma 8. *Let $n \geq 3$ and assume $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ is either an SL_2-tiling or an anti-SL_2-tiling of $\mathbb{Z}^n$. Then for any $r \in \mathbb{Z}$ the set $\{a_{\boldsymbol{i}} : \sum_{j=1}^n i_j = r\}$ consists of a single element.*

Proof. We will show that the value of $a_{\boldsymbol{i}+\boldsymbol{e}_k}$ depends only on $a_{\boldsymbol{i}}$. This is enough to conclude our claim because any vector in $\mathbb{Z}^n$ whose entries sum up to a given integer r can be obtained from any other with the same property via "zig-zagging" up and down by simultaneously adding and subtracting unit vectors.

Pick any three distinct indices $j, k, \ell \in [1, n]$. To prove our claim we compute $a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_k+\boldsymbol{e}_\ell}$ in terms of $a_{\boldsymbol{i}}, a_{\boldsymbol{i}+\boldsymbol{e}_j}, a_{\boldsymbol{i}+\boldsymbol{e}_k}, a_{\boldsymbol{i}+\boldsymbol{e}_\ell}$ in three different ways. For simplicity of notation we set

$$\epsilon_{jk} = \epsilon_{j\ell} = \epsilon_{k\ell} = \epsilon, \qquad a_{\boldsymbol{i}} = a, \qquad a_{\boldsymbol{i}+\boldsymbol{e}_j} = x, \qquad a_{\boldsymbol{i}+\boldsymbol{e}_k} = y, \qquad a_{\boldsymbol{i}+\boldsymbol{e}_\ell} = z.$$

The following picture will be useful.



Using (3) three times, we get

$$a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_k} = \frac{xy-\epsilon}{a}, \qquad a_{\boldsymbol{i}+\boldsymbol{e}_k+\boldsymbol{e}_\ell} = \frac{yz-\epsilon}{a}, \qquad a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_\ell} = \frac{xz-\epsilon}{a}.$$

Three more applications of (3) then lead to

$$a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_k+\boldsymbol{e}_\ell} = \begin{cases} \frac{a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_k} a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_\ell} - \epsilon}{a_{\boldsymbol{i}+\boldsymbol{e}_j}} = \frac{xyz}{a^2} - \epsilon\frac{y+z}{a^2} - \epsilon\frac{a^2-\epsilon}{a^2 x} \\ \frac{a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_k} a_{\boldsymbol{i}+\boldsymbol{e}_k+\boldsymbol{e}_\ell} - \epsilon}{a_{\boldsymbol{i}+\boldsymbol{e}_k}} = \frac{xyz}{a^2} - \epsilon\frac{x+z}{a^2} - \epsilon\frac{a^2-\epsilon}{a^2 y} \\ \frac{a_{\boldsymbol{i}+\boldsymbol{e}_j+\boldsymbol{e}_\ell} a_{\boldsymbol{i}+\boldsymbol{e}_k+\boldsymbol{e}_\ell} - \epsilon}{a_{\boldsymbol{i}+\boldsymbol{e}_\ell}} = \frac{xyz}{a^2} - \epsilon\frac{x+y}{a^2} - \epsilon\frac{a^2-\epsilon}{a^2 z} \end{cases}$$

It follows that $\frac{x-y}{a^2} = \frac{a^2-\epsilon}{a^2 x} - \frac{a^2-\epsilon}{a^2 y}$ or $(xy + a^2 - \epsilon)(x - y) = 0$. But $xy + a^2 - \epsilon \geq 1$ since $a, x, y \geq 1$, hence $x = y$. Similarly $y = z$. The result then follows by iterating on all possible triples of distinct indices j, k, ℓ. □

We now come to our first main result: in dimension n, an "infinite staircase" of 1's yields the only possible anti-SL_2-tiling.

Theorem 9. *For $n \geq 3$, there exists a unique (up to translation) anti-SL_2-tiling of $\mathbb{Z}^n$. Any of its "two-dimensional slices" obtained by fixing all but two of the entries of $\boldsymbol{i}$ is a translation of the staircase anti-SL_2-tiling of $\mathbb{Z}^2$ from Example 4. In particular, all the integers appearing are odd Fibonacci numbers.*

Proof. Assume $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ is an anti-SL_2-tiling of $\mathbb{Z}^n$. Pick $\boldsymbol{i}$ with $a_{\boldsymbol{i}}$ minimal. Applying (3), we get

$$a_{\boldsymbol{i}+\boldsymbol{e}_1} a_{\boldsymbol{i}-\boldsymbol{e}_2} = a_{\boldsymbol{i}} a_{\boldsymbol{i}+\boldsymbol{e}_1-\boldsymbol{e}_2} + 1 = a_{\boldsymbol{i}}^2 + 1,$$

where we used Lemma 8 in the last equality. If $a_{\boldsymbol{i}} > 1$, this implies $a_{\boldsymbol{i}+\boldsymbol{e}_1} < a_{\boldsymbol{i}}$ or $a_{\boldsymbol{i}-\boldsymbol{e}_2} < a_{\boldsymbol{i}}$, contradicting minimality, so we must have $a_{\boldsymbol{i}} = 1$. In turn, again leveraging Lemma 8, this implies $\{a_{\boldsymbol{i}+\boldsymbol{e}_k}, a_{\boldsymbol{i}-\boldsymbol{e}_k}\} = \{1, 2\}$ for any $k \in [1, n]$. Without loss of generality (by replacing $\boldsymbol{i}$ by $\boldsymbol{i}+\boldsymbol{e}_1$ if needed) we will assume $a_{\boldsymbol{i}+\boldsymbol{e}_k} = 2$ and $\sum_{j=1}^n i_j = 1$. Then, applying (3) repeatedly, we see that $a_{\boldsymbol{i}'}$ with $\sum_{j=1}^n i'_j = r \geq 1$ is exactly the r^{th} odd Fibonacci number F_{2r-1} (see Example 4). Similarly one sees that $a_{\boldsymbol{i}'}$ with $\sum_{j=1}^n i'_j = r \leq 0$ is the odd Fibonacci number F_{-2r+1}. □

Proposition 10. *There does not exist any SL_2-tiling of $\mathbb{Z}^n$ for $n \geq 3$.*

Proof. Since any 3-dimensional slice of an SL_2-tiling of $\mathbb{Z}^n$ is an SL_2-tiling of $\mathbb{Z}^3$, it suffices to show that there is no SL_2-tiling of $\mathbb{Z}^3$. Assume $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^3}$ is an SL_2-tiling of $\mathbb{Z}^3$. Pick $\boldsymbol{i}$ with

$a_{\boldsymbol{i}}$ minimal. Applying (3), we get

$$a_{\boldsymbol{i}+\boldsymbol{e}_1} a_{\boldsymbol{i}-\boldsymbol{e}_2} = a_{\boldsymbol{i}} a_{\boldsymbol{i}+\boldsymbol{e}_1-\boldsymbol{e}_2} - 1 = a_{\boldsymbol{i}}^2 - 1,$$

where we used Lemma 8 in the last equality. But this implies $a_{\boldsymbol{i}+\boldsymbol{e}_1} < a_{\boldsymbol{i}}$ or $a_{\boldsymbol{i}-\boldsymbol{e}_2} < a_{\boldsymbol{i}}$, contradicting minimality. □

Corollary 11. *For $n = 3$, there are precisely 4 signature matrices $\boldsymbol{\epsilon}$ for which there exists an $\boldsymbol{\epsilon}$-SL_2-tiling. For such $\boldsymbol{\epsilon}$, this $\boldsymbol{\epsilon}$-SL_2-tiling is unique (up to translation). More precisely, an $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^3$ exists if and only if $\epsilon_{12}\epsilon_{13}\epsilon_{23} = -1$.*

Proof. The claim follows immediately from the observation that any signature matrix for $n = 3$ is either one of the two satisfying $\epsilon_{12} = \epsilon_{13} = \epsilon_{23}$ or is obtained from one of these with a single application of Lemma 6. □

We are finally ready to classify all $\boldsymbol{\epsilon}$-SL_2-tilings for any $n \geq 3$.

Theorem 12. *For $n \geq 3$, there are precisely 2^{n-1} signature matrices $\boldsymbol{\epsilon}$ for which there exists an $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^n$. They are precisely the signature matrices obtainable from the anti-SL_2-signature matrix by repeated application of Lemma 6. Whenever an $\boldsymbol{\epsilon}$-SL_2-tiling exists, it is unique up to translation.*

Proof. Let $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^n}$ be an $\boldsymbol{\epsilon}$-SL_2-tiling of $\mathbb{Z}^n$. Fixing all but any three distinct entries of $\boldsymbol{i}$ gives an $\boldsymbol{\epsilon}'$-tiling of $\mathbb{Z}^3$ whose signature matrix $\boldsymbol{\epsilon}'$ is the submatrix of $\boldsymbol{\epsilon}$ with the corresponding rows and columns. Therefore, it follows from Corollary 11 that we have an inclusion $E \subset E'$, where E is the set of $n \times n$ signature matrices $\boldsymbol{\epsilon}$ which admit an $\boldsymbol{\epsilon}$-SL_2-tiling, and E' is the set of $n \times n$ signature matrices $\boldsymbol{\epsilon}$ satisfying $\epsilon_{jk}\epsilon_{k\ell}\epsilon_{j\ell} = -1$ for any triple of distinct indices j, k, ℓ.

Any row (or equivalently any column) of a matrix $\boldsymbol{\epsilon}$ in E' uniquely determines all the remaining entries of $\boldsymbol{\epsilon}$, moreover all possible choices of entries in this fixed row (or equivalently column) are allowed. Indeed, assume for the sake of clarity that the matrix $\boldsymbol{\epsilon}$ has been computed using row 1, then

$$\epsilon_{jk}\epsilon_{k\ell}\epsilon_{j\ell} = (-\epsilon_{1k}\epsilon_{1j})(-\epsilon_{1k}\epsilon_{1\ell})(-\epsilon_{1j}\epsilon_{1\ell}) = -1.$$

Therefore E' is in bijection with $\{\pm 1\}^{n-1}$ and $\#E' = 2^{n-1}$.

Using Lemma 6, there is an action of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ on E given by $\boldsymbol{\epsilon} \mapsto \boldsymbol{\epsilon}^{(r)}$ for $1 \leq r \leq n-1$. This action is free; indeed the only element of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ leaving invariant the last column of any given matrix of E is the identity.

Due to Theorem 9, E is not empty, and so we compute $\#E \geq 2^{n-1} = \#E' \geq \#E$ and deduce that $E = E'$.

The uniqueness claim also follows immediately from Corollary 11 by fixing all but any three distinct entries of $\boldsymbol{i}$. □

Remark 13. *It is now clear why we choose the diagonal entries of $\boldsymbol{\epsilon}$ to be equal to -1: any $\boldsymbol{\epsilon}$-SL_2-tiling consists of odd Fibonacci numbers and (3) is satisfied also for $k = \ell$.*

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