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LACUNARY LAGUERRE SERIES FROM A COMBINATORIAL PERSPECTIVE

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Abstract. In recent work, Babusci, Dattoli, Górska, and Penson have presented a number of lacunary generating functions for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, i.e., series of the type $\sum_{n\geq 0} c_n L_{2n}^{(\alpha)}(x)\, t^n$, by a method closely related to umbral calculus. This work is complemented here, deriving many of their results by interpreting Laguerre polynomials combinatorially as enumerators for discrete structures (injective partial functions). This combinatorial view pays in that it suggests natural extensions and gives a deeper insight into the known formulas.

1. Introduction

1.1. **History and motivation.** The following citation is taken from the Introduction of an article by Adriano Garsia and Jeff Remmel [19], published in 1980 in the very first volume of the *European Journal of Combinatorics*. It has been reproduced (in French) in a slightly shortened version by Dominique Foata in the introduction to the written version [17] of his talk on the combinatorics of orthogonal polynomials, presented at the International Congress of Mathematicians in Warsaw, 1983.

> *It has become increasingly apparent since the work of Foata–Schutzenberger and recent works of the Lotharingian school of combinatorics [...] that the special functions and identities of classical mathematics are gravid with combinatorial information. This information can be expressed in the form of correspondences or more precisely encodings of objects into words of certain languages and natural bijections between different classes of languages. The classical identities appear then as relations between enumerators of these words by suitable statistics. At present a systematic study is taking place to mine this information out of the classical literature. This increasingly rich inventory of correspondences has led to new identities as well as more revealing proofs of the old ones.*

In this context, Foata mentions various schools of combinatorics (écoles *bostonienne*, *californienne*, *lotharingienne*, *québecoise*, *viennoise*) inspired by these ideas, and the number of references, witnessing the enormous activities in the spirit expressed in this citation, has drastically increased compared to the original text.

The above quotation very well describes the atmosphere of departure around 1980. I will not try to give an overview, least of all a comprehensive account of what has been achieved in those days. But in relation to the present article I have to mention some particular work that I have personally been involved in. Foata's work on the

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combinatorics of Hermite polynomials, see [11], and also [12] and [13], with a simple proof of the bilinear generating function, known as Mehler's formula, and its multilinear generalizations, constitutes an early striking example for the elegance and efficiency of combinatorial concepts and methods in this domain. The combinatorial model itself, involutions alias matchings of complete graphs, belongs to combinatorial folklore.

In 1981 I have been invited by Dominique Foata to jointly work on similar items for the Laguerre polynomials, i.e., their bilinear generating function that goes under the name Hille–Hardy identity, which will appear in this article towards the end as Eq. (12.2). We have set up a combinatorial model, the very same one that will be introduced here in Section 4.2, and we have given a proof of the Hille–Hardy identity based on this model. The model then guided us to invent and to prove a multilinear extension, see [14] and [15].

Another striking event in those days was the combinatorial proof of the generating function for the Jacobi polynomials, achieved by Foata and Leroux in [16]. The model deals with complementary pairs of Laguerre configurations, which naturally produces tree-like objects and therefore relates to Lagrange inversion. For me it has been a challenge to study further properties of the Jacobi polynomials, including Bailey's bilinear generating function, by looking through combinatorial glasses. Some of this work has been published in articles, but I will not give the references here, because it has materialized, together with lots of otherwise unpublished work, in my habilitation thesis [36].

After having completed [36], my interests shifted, well under the influence of the spectacular success of Zeilberger's ideas and implementations, towards the algorithmic side of hypergeometric series, binomial identities, and symbolic summation. See, e.g., my article [37] which deals with both, combinatorial and algorithmic aspects. The chapter on the combinatorics of special function identities appeared to be closed for me.

It came as a surprise when Karol Penson contacted me in 2012, sending me the list [1] (not containing any proofs) of lacunary generating functions for the Laguerre polynomials and asking, what a combinatorialist could say about these new results. My first reaction was reluctant, because the formulas looked pretty complicated, and I was not sure that one could produce combinatorial proofs that would be more enlightening than tour-de-force work. My mind changed when I received a first version of what is now the article [2]. I started to look closer and it appeared to me that indeed something interesting could be said when interpreting these identities in terms of the good old Laguerre configurations. The present article is the result of my investigations. I have taken some of the major results of [2] and cast them into combinatorics. But it would not have been too exciting for me to present intricate combinatorial proofs for results that have been obtained (formally) analytically more easily. In my opinion the value of the combinatorial approach lies in that it guides on the way towards reasonable generalizations, specializations, and refinements in an intuitive, even "visible" way.

1.2. **Structure of the article.** Sections 2 to 5 prepare the ground for the combinatorial work with Hermite and Laguerre series. To start with, Section 2 is about notation and conventions relating to exponential generating functions for families of combinatorial structures. It will be assumed that the reader is familiar with these notions that have been formalized in essentially equivalent ways in many places. The classical titles (in chronological order) [33], [4], [10], [21], [39], [3], [9] constitute just a subjective selection from the literature.

The main actors of this article, Hermite and Laguerre configurations, are introduced in Section 3. These are families of combinatorial objects used to interpret the integer coefficients (in a suitable normalization) of the classical polynomials bearing the same name. In Section 3.3 some pointers are given to other work on Laguerre polynomials (or, more generally, classical orthogonal polynomials) and combinatorics, that is not touched upon in this article.

In the subsequent section, Section 4, some elementary examples are given that demonstrate the use of these combinatorial models. Some of these ideas are used later on. Another item of the same kind is the use of differential operators from a combinatorial perspective, which is presented in Section 5. This material has been included mainly because in the article [2] by Babusci et al. differential operators play an important role, but here this aspect is not treated any further.

The proper combinatorial contributions of this article are presented beginning with Section 6, where a very natural idea for obtaining lacunary generating function combinatorially is exploited: just copy the base sets where the combinatorial structures live, and use a basic decomposition property of Laguerre configuration. These copying procedures could be used to obtain k-lacunary series for $k = 2, 3, 4, \ldots$, but only the cases $k = 2$ and $k = 3$ are treated in detail, and for $k = 4$ the result is given. It should then be obvious what the result for arbitrary k looks like.

The approach to lacunary Laguerre series as presented in Section 7 is different, and, in my opinion, it is more interesting than the previous one. The basic idea is to consider superpositions of Laguerre configurations and perfect matchings (i.e., a special class of Hermite configurations) as combinatorial objects. This forces the ground set to be of *even* cardinality, which naturally leads to a lacunary generating function. As a result, we obtain a very nice, strikingly simple double lacunary generating function for the Laguerre polynomials, which is expressed in terms of Hermite polynomials. The interest of this combinatorial set-up lies in the fact, that it can be easily generalized in various ways, one of which is presented in Section 8. By considering the superposition of Laguerre and (general) Hermite configurations, i.e., involutions alias general matchings, we are led to a bilateral (not lacunary) generating function, that can be specialized to yield further lacunary Laguerre series. In my experience, it would have been difficult to imagine the general result without the underlying combinatorial model in mind.

Another extension of the basic identity in Section 7 is then tackled in Section 9. This generalizes from the ordinary Laguerre polynomials $L_{2n}^{(0)}$ to the generalized Laguerre polynomials $L_{2n}^{(k)}$. The cases $k = 1$ and $k = 2$ have been treated experimentally by Babusci et al. in [2], with rather strange (at first sight, at least) coefficient polynomials showing up which were not explained any further. The combinatorial approach, however, yields a very nice and simple explanation for these "wild" polynomials, that are counting certain balls-into-boxes configurations. This leads to a very satisfactory result for general k through a combinatorial proof.

Section 10 can be seen as an addendum to the foregoing: first, in Section 10.1 the same situation as in Section 9 is treated, but now with an additional cycle counting parameter. Secondly, in Section 10.2 it is indicated that also triple lacunary Laguerre series can be obtained this way. Not all details are given, but they could be worked out, if desired.

A result from [2], featuring the polynomials $L_{2n}^{(\alpha-2n)}(x)$, which is already typographically different from all the others, is the object of Section 11. Indeed, these polynomials are not proper Laguerre polynomials, as their parameter $\alpha - 2n$ depends on n. These polynomials are essentially Charlier polynomials, which are generally easier to treat than Laguerre polynomials, as the combinatorial objects counted by them are just colored permutation. Anyway, the daunting analytical expression has a very simple combinatorial explanation.

Finally, in Section 12 another interesting hypergeometric series from [2] is considered, which is actually equivalent to one treated here already in Section 6. By relating this series to the classical Hille–Hardy bilinear generating function for the Laguerre polynomials, for which a combinatorial proof is known since long, see [14] and [15], one obtains intriguing hypergeometric and binomial identities. The core result is expressed as a nice umbral substitution identity, which relates Laguerre and Legendre polynomials. This looks as if a direct combinatorial proof should be manageable, but I have not been able to construct one — so I leave this as a combinatorial puzzle.

2. Notation and conventions

If $\mathfrak{E}$ is a family of combinatorial structures (in the sense of the *combinatorial species* or *species of structure* of [3], but we will use only very basic facts and conventions), and A is any finite set, then $\mathfrak{E}[A]$ will denote the set of $\mathfrak{E}$-structures living on A. If $A = [n] = \{1, 2, \ldots, n\}$, then $\mathfrak{E}[n]$ will be written in place of $\mathfrak{E}[[n]]$. For any set A of cardinality n, which is denoted by $\sharp A = n$, the sets $\mathfrak{E}[A]$ and $\mathfrak{E}[n]$ are equivalent in a functorial way. This is made precise in species theory, but needs not to be detailed here. As mentioned at the beginning of Section 1.2, there are many places, where a similar formalism for the combinatorics of exponential generating is described.

Generally, we are dealing here with structures which are both *labeled* (by the elements of the respective ground set A), and *weighted*, in the sense that a mapping (*weight function*), $w : \mathfrak{E}[A] \to R[x]$ is specified that is compatible with transport of structures via bijections. Here R is some suitable ring, $R[x]$ is its polynomial ring, in the variable x. Typically, the weight $w(e)$ for $e \in \mathfrak{E}[A]$ will be of the kind $w(e) = \tilde{w}(e)\, x^{\sharp A}$, where $\tilde{w}(e) \in R$ encodes some structural information about e and the term $x^{\sharp A}$ keeps track of the size (also called *order*) of the underlying vertex set.

The adequate tools for enumeration are then exponential generating functions:

$$\mathcal{E}(x) = \sum_{n\geq 0} \frac{1}{n!} \sum \big\{ w(e)\, ;\, e \in \mathfrak{E}[n] \big\} = \sum_{n\geq 0} \frac{x^n}{n!} \sum \big\{ \tilde{w}(e)\, ;\, e \in \mathfrak{E}[n] \big\}.$$

It will be convenient to use the set-up of *multisorted* structures, which then gives rise to exponential generating functions in more than one variable. The case of two-sorted structures is typical: the family of structures e belonging to $\mathfrak{E}$ and living on a pair (A, B) of disjoint finite sets will be denoted as $\mathfrak{E}[A, B]$, and these structures are weighted by $w(e) = \tilde{w}(e)\, x^{\sharp A} y^{\sharp B}$. The exponential generating function would then be written as

$$\mathcal{E}(x, y) = \sum_{n\geq 0} \sum_{X \uplus Y = [n]} \frac{1}{\sharp X!} \frac{1}{\sharp Y!} \sum \big\{ w(e)\, ;\, e \in \mathfrak{E}[X, Y] \big\}$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{k+\ell=n} \binom{n}{k,\ell} x^k\, y^\ell \sum \big\{ w(e)\,;\, e \in \mathfrak{E}[k,\ell] \big\}$$
$$= \sum_{n\geq 0} \frac{1}{n!}\, \mathcal{E}_n(x,y),$$

where $\mathfrak{E}[k,\ell] = \mathfrak{E}[A,B]$ for $A = [k]$ and $B = [k+\ell] \setminus [k]$, say. In this notation the enumerating polynomials $\mathcal{E}_n(x,y)$ for $\mathfrak{E}$-structures of order n are *homogeneous* polynomials in x and y of total degree n. There is no need for an extra variable t, with the term t^n indicating the order of $\mathfrak{E}$-structures, but it can be made visible by making the substitution $x \to x\,t$, $y \to y\,t$, if desired.

3. The combinatorial vocabulary: Hermite and Laguerre configurations

3.1. **Hermite configurations.** For any finite set A, let $\mathfrak{I}[A]$, respectively $\mathfrak{M}[A]$, denote the set of all involutions, respectively fixed point free involutions on A (alias *matchings* respectively *perfect matchings* of the complete graph with vertex set A). For $\mu \in \mathfrak{I}[A]$ the set of its fixed points will be denoted by $\mathit{fix}(\mu)$, and $\mathsf{fix}(\mu) = \sharp\mathit{fix}(\mu)$ is its cardinality. Similarly, we write $\mathit{trans}(\mu)$ for the set of transpositions of μ, and $\mathsf{trans}(\mu) = \sharp\mathit{trans}(\mu)$ for its cardinality. Each involution μ will be given the valuation

$$\nu_{x,y}(\mu) = x^{\mathsf{fix}(\mu)} y^{2\,\mathsf{trans}(\mu)},$$

and the homogeneous generating polynomial for involutions of order n is

$$\mathcal{H}_n(x,y) = \sum \big\{\, \nu_{x,y}(\mu)\,;\, \mu \in \mathfrak{I}[n] \,\big\},$$

our combinatorial version of the classical Hermite polynomials. Specifically, for any pair (A,B) of disjoint finite sets the set $\mathfrak{H}[A,B]$ of *Hermite configurations* is the set

$$\mathfrak{H}[A,B] = \big\{\, \mu \in \mathfrak{I}[A \uplus B]\,;\, \mathit{fix}(\mu) = A \,\big\},$$

so $\mu \in \mathfrak{H}[A,B]$ is essentially given by A (as a set) and a fixed-point free involution on B. For $\mathfrak{M}[B]$ to be non-empty, the cardinality of B must be even, and for a set B of cardinality $2k$ one has

$$m_k = \sharp\mathfrak{M}[B] = \begin{cases} 1\cdot 3\cdot 5 \cdots (2m+1) = \dfrac{(2m)!}{2^m\, m!}, & \text{if } k = 2m, \\ 0, & \text{if } k \text{ is odd.} \end{cases} \tag{3.1}$$

Thus

$$\mathcal{H}_n(x,y) = \sum_{A\uplus B=[n]} \sum \big\{\, \nu_{x,y}(\mu)\,;\, \mu \in \mathfrak{H}[A,B] \,\big\}$$
$$= \sum_{0\leq k\leq n} \binom{n}{k} m_k\, x^{n-k}\, y^k$$
$$= \sum_{0\leq 2k\leq n} \binom{n}{2k} m_{2k}\, x^{n-2k}\, y^{2k}$$
$$= \sum_{0\leq 2k\leq n} \frac{n!}{k!\,(n-2k)!\,2^k}\, x^{n-2k}\, y^{2k}.$$

In hypergeometric terms,

$$\mathcal{H}_n(x,y) = x^n \cdot {}_2\mathsf{F}_0\left[\begin{matrix}-n/2\,,\,1-n/2\\ -\end{matrix}\,;\,\frac{2y^2}{x^2}\right],$$

and it is easy to obtain the exponential generating function

$$\sum_{n\geq 0} \frac{1}{n!}\,\mathcal{H}_n(x,y) = e^{x+y^2/2}.$$

As a special case, for $x=0$ one has

$$\sum_{n\geq 0} \frac{1}{n!}\mathcal{H}_n(0,y) = \sum_{n\geq 0} \frac{y^{2n}}{(2n)!}\,m_{2n} = e^{y^2/2}.$$

Referring to the standard notation $H_n(x)$ for the Hermite polynomials, as given for instance in the NIST Handbook [29], one observes

$$H_n(t) = \mathcal{H}_n(2\,t, i\sqrt{2}), \qquad \mathcal{H}_n(x,y) = (\frac{-i\,y}{\sqrt{2}})^n H_n\left(\frac{i\,x}{\sqrt{2}\,y}\right).$$

3.2. **Laguerre configurations.** As for the combinatorial version of the Laguerre polynomials, it is also preferable to use a homogeneous version in two variables.

For disjoint finite sets X, Y let $\mathfrak{L}[X,Y]$ denote the set of all *injective* functions $\lambda : X \to X \uplus Y$, henceforth called *Laguerre configurations*. Also put $\mathfrak{L}[N] = \bigcup_{X\uplus Y=N} \mathfrak{L}[X,Y]$, which can be seen as the set of *injective partial functions* on N.

The connected components of such a Laguerre configuration λ are of two types:

– λ-*cycles*, contained in X, of some length $\ell \geq 1$,

$$a_0 \xrightarrow{\lambda} a_1 \xrightarrow{\lambda} a_2 \xrightarrow{\lambda} \dots \xrightarrow{\lambda} a_{\ell-1} \xrightarrow{\lambda} a_0,$$

– λ-*chains* beginning in X and ending in Y, of some length $\ell \geq 0$,

$$a_0 \xrightarrow{\lambda} a_1 \xrightarrow{\lambda} a_2 \xrightarrow{\lambda} \dots \xrightarrow{\lambda} a_{\ell-1} \xrightarrow{\lambda} b,$$

with $a_0, \dots, a_{\ell-1} \in X$ and $b \in Y$. Here the case $\ell = 0$ means nothing but $b \notin \lambda(X)$.

Figure 1 shows a typical Laguerre configuration λ with three cycles and four chains, one of which is simply a vertex of Y which is not an endpoint of a proper chain. Note the following special cases:

– $\mathfrak{L}[X,\emptyset]$ is the family of permutations of X,
– $\mathfrak{L}[\emptyset,Y]$ is $\{Y\}$, i.e. the set Y taken as a singleton set of structures.

For Laguerre configurations the following valuation will be employed:

$$\omega^{\alpha}_{x,y}(\lambda) = (1+\alpha)^{\mathsf{cyc}(\lambda)} x^{\sharp X} y^{\sharp Y} \in \mathbb{Z}[\alpha,x,y],$$

where $\mathsf{cyc}(\lambda)$ is the number of λ-cycles[1], and it is then easily checked that

$$\sum \left\{\,\omega^{\alpha}_{x,y}(\lambda)\,;\,\lambda\in\mathfrak{L}[X,Y]\,\right\} = (1+\alpha+\sharp Y)_{\sharp X},$$

[1]Taking $1+\alpha$ instead of α as a counter for cycles is a matter of convenience, because it conforms with the traditional definition of generalized Laguerre polynomials. Specializing $\alpha \to 0$ means "cycles allowed, but no bookkeeping is done for the number of cycles" and $\alpha \to -1$ means "cycles are forbidden".

where $(\gamma)_n = \gamma \cdot (\gamma+1) \cdots (\gamma+n-1)$ is the rising factorial or *Pochhammer symbol*.

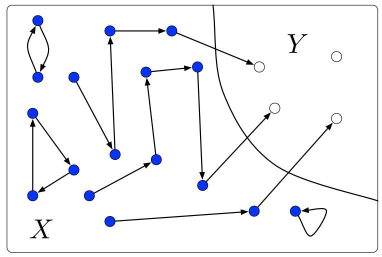

FIGURE 1. A typical Laguerre configuration λ with weight $\omega^{\alpha}_{x,y}(\lambda) = (1+\alpha)^3\, x^{17} y^4$

Now the combinatorial (generalized) Laguerre polynomials are, for any set N of cardinality n,

$$\begin{aligned}
\mathcal{L}^{(\alpha)}_n(x,y) &= \sum \left\{ \omega^{\alpha}_{x,y}(\lambda)\,;\, \lambda \in \mathfrak{L}[N] \right\} \\
&= \sum_{X \uplus Y = [N]} \sum \left\{ \omega^{\alpha}_{x,y}(\lambda)\,;\, \lambda \in \mathfrak{L}[X,Y] \right\} \\
&= \sum_{0 \le k \le n} \binom{n}{k} (1+\alpha+k)_{n-k}\, x^{n-k}\, y^k \\
&= x^n \cdot {}_1\mathsf{F}_1 \left[\begin{matrix} -n \\ 1+\alpha \end{matrix} ; -\frac{y}{x} \right].
\end{aligned}$$

This relates to the usual conventions for the Laguerre polynomials simply by

$$L^{(\alpha)}_n(y) = \frac{1}{n!}\, \mathcal{L}^{(\alpha)}_n(1,-y), \qquad \mathcal{L}^{(\alpha)}_n(x,y) = n!\, x^n\, L^{(\alpha)}_n(-y/x).$$

In the case $\alpha = 0$ the standard notation $L_n(y)$ for the traditional Laguerre polynomials will be used instead of $L^{(0)}_n(y)$, and consequently also $\mathcal{L}_n(x,y)$ is used in place of $\mathcal{L}^{(0)}_n(x,y)$.

A standard combinatorial reasoning, taking the two types of connected components into account, immediately shows that the exponential generating function for Laguerre configurations is

$$\sum_{n \ge 0} \frac{1}{n!}\, \mathcal{L}^{(\alpha)}_n(x,y) = (1-x)^{-(1+\alpha)} e^{y/(1-x)},$$

where the first factor on the right takes care of the cycles (thus counting permutations), and the second one represents the set of chains of elements of X ending in an element of Y.

A slight extension of this combinatorial model that is sometimes handy and that will be used in Section 6, is as follows: consider three mutually disjoint finite sets X, Y, Z, and let

$$\mathfrak{L}[X, Y; Z] = \mathfrak{L}[X, Y \uplus Z],$$

taking as valuation for these objects

$$\omega^{\alpha}_{x,y,z}(\lambda) = (1+\alpha)^{\mathsf{cyc}(\lambda)} x^{\sharp X} y^{\sharp Y} z^{\sharp Z}.$$

Then the generating polynomial is

$$\sum \left\{ \omega^{\alpha}_{x,y,z}(\lambda) \,;\, \lambda \in \mathfrak{L}[X,Y;Z] \right\} = (1+\alpha+\sharp Z+\sharp Y)_{\sharp X}\, x^{\sharp X} y^{\sharp Y} z^{\sharp Z},$$

and for any set N of cardinality n, keeping the "extra" set Z fixed,

$$\sum_{X \uplus Y = N} \sum \left\{ \omega^{\alpha}_{x,y,z}(\lambda) \,;\, \lambda \in \mathfrak{L}[X,Y;Z] \right\} = \mathcal{L}^{(\alpha+\sharp Z)}_{n}(x,y)\, z^{\sharp Z}.$$

A familiar special case occurs when the set Y is copy of X, say $Y = X'$. Then the Laguerre configurations $\lambda \in \mathfrak{L}[X, X'; Z]$ can be interpreted as matchings of the complete bipartite graph $K_{X,X' \uplus Z}$ with vertex sets X and $X' \uplus Z$, where the pairs $(a, \lambda(a)) \in X \times (X' \uplus Z)$ are the selected edges. Let $\sharp X = n$ and $\sharp Z = m$. The polynomial

$$\mathcal{L}^{(m)}_{n}(x,1) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+m}{k} x^k = {}_2\mathsf{F}_1\left[\begin{matrix} -n, -n-m \\ 1 \end{matrix} ; x\right]$$

is (essentially) what is known as the *matching polynomial* ([27]) of the complete bipartite graph $K_{n,n+m}$, also known as *rook polynomial* ([33]) of a rectangle of size $n \times (n+m)$. See Figure 2 for an illustration of how the coefficients of

$$\mathcal{L}^{(2)}_{4}(x,y) = y^4 + 24xy^3 + 180x^2y^2 + 480x^3y + 360x^4$$

can be seen as counters for matchings in $K_{4,6}$.

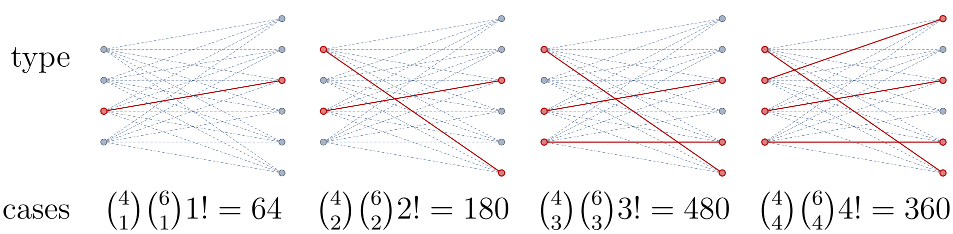

FIGURE 2. The coefficients of $\mathcal{L}^{(2)}_{4}(x,y)$ by counting matchings in $K_{4,6}$

3.3. **Other work.** A lot more could be said about Laguerre polynomials (or, more generally, classical orthogonal polynomials) and combinatorics. There are various models, like derangements, words, lattice paths, rook configurations, matchings, etc., that have been used for exploring properties of these families of polynomials. Since these developments, interesting and important as they are, will not be used in the remainder of this article, I will confine myself to shortly mention three areas of activity from the the period between the 1970's and the early 1990's.

– The general notion of a *reluctant function* was introduced by Mullin and Rota in [28] in the context of binomial enumeration, see also [25] for a recent reference. The particular subfamily of *injective* reluctant function is relevant for us: these are precisely the functions enumerated by the polynomials $\mathcal{L}^{(-1)}_{n}(x,y)$, i.e., Laguerre configurations in the present sense, but with $\alpha = -1$, which means that no cycles, only chains are allowed[2]. The chains also appear in other models, e.g.,

[2]Note that the Rota school uses a different convention for the generalized Laguerre polynomials: what Rota denotes $L^{(\alpha)}_{n}(x)$ is $L^{(\alpha-1)}_{n}(x)$ according to standard terminology.

as *tubes* in [19]. Laguerre polynomials are featuring as a prominent example in Rota's *finite operator calculus*, respectively *umbral calculus*, see, e.g., [34] and [35].

– Starting with articles [6] by Even and Gillis and [23] by Jackson, the exploration of the combinatorial aspects of the *linearization coefficients* for classical orthogonal polynomials has been a "hot topic" for quite some time, with Laguerre polynomials being of particular interest, see, e.g., the work [18] by Foata and Zeilberger. Finally, Gessel [20] and Zeng [40], besides substantial results, offer a systematic view on these achievements.
– In the field of algorithmics and algorithm analysis orthogonal polynomials made their appearance when studying the dynamic behavior of simple data structures (like lists, queues, dictionaries) under insertions, searches, and deletions, see [8] as an authoritative paper in this direction. The deeper reason for this application of orthogonal polynomials to discrete structures is the modeling via lattice paths and their relation to continued fractions, as put forward, e.g., by Flajolet in [7] . A combinatorial theory of orthogonal polynomials based on this lattice path model (where the *moment sequences* of these polynomials are of central importance) has been worked out by Viennot. It has been presented at a colloquium which was held in honor of Edmond Nicolas Laguerre at the occasion of his 150th birthday in his birthplace Bar-le-Duc (France), see [38].

4. Examples for the use of the combinatorial models

The purpose of this section is to illustrate the combinatorial reasoning with Hermite and Laguerre configurations by discussing typical examples.

4.1. **Hermite configurations.** Let N denote a set of cardinality $2n$. Here we will examine pairs (μ, S) where $\mu \in \mathfrak{M}[N]$ and where $S \subseteq N$. For convenience, put $R = N \setminus S$, and let $r = \sharp R$, $s = \sharp S = 2n - r$.

For any $\mu \in \mathfrak{M}[N]$ define

– μ_R : the set of all μ-transpositions contained in R,
  A : the subset of R on which μ_R acts, with $\sharp A = 2a$;
– μ_S : the set of all μ-transpositions contained in S,
  C : the subset of S on which μ_S acts, with $\sharp C = 2c$;
– β : the remaining transpositions of μ, which can be considered as an element of $\mathfrak{B}[B', B'']$, the set of bijections between $B' = R \setminus A$ and $B'' = S \setminus C$, both being sets of cardinality $b = n - a - c$.

Figure 3 shows the objects μ and S separately, in Figure 4 the decomposition of μ using S is displayed.

We have then the bijective mapping

$$\mathfrak{M}[N] \to \bigcup_{\substack{A \uplus B' = R \\ B'' \uplus C = S}} \mathfrak{M}[A] \times \mathfrak{B}[B', B''] \times \mathfrak{M}[C] \;:\; \mu \mapsto (\mu_R, \beta, \mu_S),$$

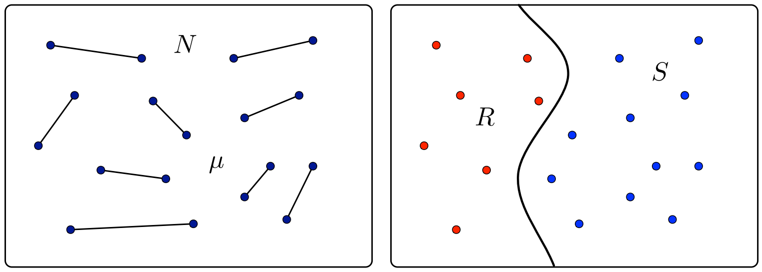

FIGURE 3. A perfect matching $\mu \in \mathfrak{M}[18]$ and a bipartition $R \uplus S = [18]$

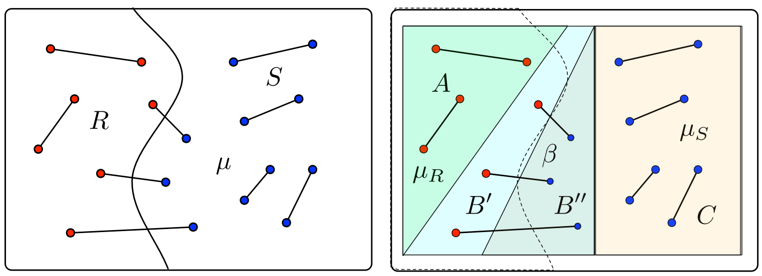

FIGURE 4. Decomposing the perfect matching μ using the bipartition $R \uplus S$

a quantitative version of which is given by

$$\begin{aligned} m_{2n} &= \sum_{2a+b=r} \binom{r}{2a} \sum_{2c+b=s} \binom{s}{2c} m_{2a}\, b!\, m_{2c} \\ &= \sum_{2a+b=r} \sum_{2c+b=s} \binom{r}{b} \binom{s}{b} m_{2a}\, b!\, m_{2c}. \end{aligned}$$

If we let run S over all subsets of N of cardinality s, we get

$$m_{2n} \binom{2n}{s} = (2n)! \sum_{2a+b=r} \sum_{2c+b=s} \frac{1}{(2a)!\, b!\, (2c)!} m_{2a}\, m_{2c},$$

which can be easily turned into

$$m_{2n} \binom{2n}{s} = \sum_{a+d=n} \binom{2n}{2a, 2d} m_{2a}\, m_{2d}\, 2^{2d-s} \binom{d}{s-d}$$

by simple manipulations. But this identity also has a simple combinatorial meaning, now starting reading from the right side: for any $\mu \in \mathfrak{H}[N]$

- we partition the transpositions of μ into two subsets, thus obtaining perfect matchings μ_A on a subset $A \subseteq N$ of cardinality $2a$, and μ_D on the set $D = N \setminus A$ of cardinality $2d = 2n - 2a$;
- we partition the d transpositions of μ_D into two subsets, thus obtaining perfect matchings μ_C on a set C of cardinality $2c$ with $c = s - d$ respectively μ_B on a set B of cardinality $2b$ with $b = 2d - s$, containing $s - d$ respectively $2d - s$ transpositions;

– from each transposition of μ_B we select one of the points: this defines $B' \subset B$ and $B'' = B \setminus B'$, both of cardinality $b = 2d - s$, and the transpositions of μ_B establishes a bijection between B' and B''.

So we are back to the first description of $\mathfrak{M}$ given above, where now $S = B'' \uplus C$ is indeed of cardinality $2c + b = 2(s-d) + (2d-s) = s$.

This kind of reasoning will be used below in Section 8.

4.2. **Laguerre configurations.** As a first illustration for the use of Laguerre configurations, I will mention a very simple, but useful technique, that will be employed later on.

For three mutually disjoint finite sets X, Y, Z one has

$$\mathfrak{L}[X \uplus Y, Z] \simeq \mathfrak{L}[X, Y \uplus Z] \times \mathfrak{L}[Y, Z], \tag{4.1}$$

where "$\simeq$" means: "there is a cycle-count-preserving bijection" between the sets of combinatorial objects on both sides.

To be precise, take $g \in \mathfrak{L}(X, Y \uplus Z)$ and $h \in \mathfrak{L}(Y, Z)$. Then the union $g \cup h$ (seen as a relation) is not necessarily an injective mapping: it may happen that for some $z \in Y \uplus Z$ one has $z = g(x) = h(y)$ for some $x \in X$ and some $y \in Y$, but if this happens, then x and y are uniquely determined. In this situation, leave g as it is and map y to the starting point of the unique g-chain that ends in z. This defines an injective function $f = g \star h$ from $X \uplus Y$ into $X \uplus Y \uplus Z$, so $f \in \mathfrak{L}(X \uplus Y, Z)$, and this construction is reversible. Furthermore

$$\mathsf{cyc}(g \star h) = \mathsf{cyc}(g) + \mathsf{cyc}(h).$$

The quantitative counterpart of Eq. (4.1) is nothing but factoring

$$(1 + \alpha + m)_{k+\ell} = (1 + \alpha + \ell + m)_k \cdot (1 + \alpha + m)_\ell.$$

This construction, which can also be read inversely as a decomposition of $f \in \mathfrak{L}[X \uplus Y, Z]$ into $g \in \mathfrak{L}(X, Y \uplus Z)$ and $h \in \mathfrak{L}(Y, Z)$, is illustrated in Figures 5, 6 and 7.

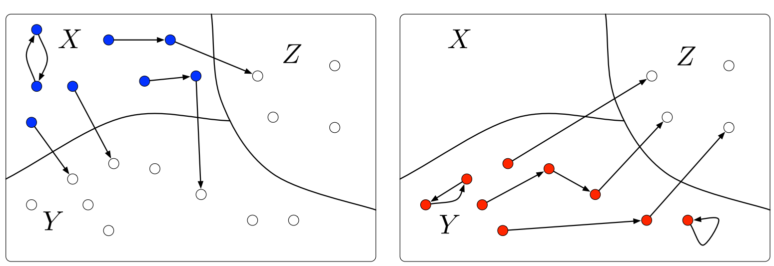

FIGURE 5. $g \in \mathfrak{L}[X, Y \uplus Z]$ (left) and $h \in \mathfrak{L}[Y, Z]$ (right)

In terms of the Laguerre polynomials the property just mentioned translates into

$$\mathcal{L}_n^{(\alpha)}(x + y, z) = \mathcal{L}_n^{(\alpha)}(x, \mathcal{L}^{(\alpha)}(y, z)), \tag{4.2}$$

where the expression on the right has to be read as an *umbral* substitution:

$$\mathcal{L}_n^{(\alpha)}(x, \mathcal{L}^{(\alpha)}(y, z)) = \mathsf{expand}(\mathcal{L}_n^{(\alpha)}(x, w))\big|_{w^k \mapsto \mathcal{L}_k^{(\alpha)}(y,z)}.$$

This umbral property of the Laguerre polynomials can also be neatly illustrated by using the model of matchings on complete bipartite graphs. Let M, N be disjoint finite sets with $\sharp M = m, \sharp N = n$, and let $\overline{N}$ be a copy of N. For the complete bipartite

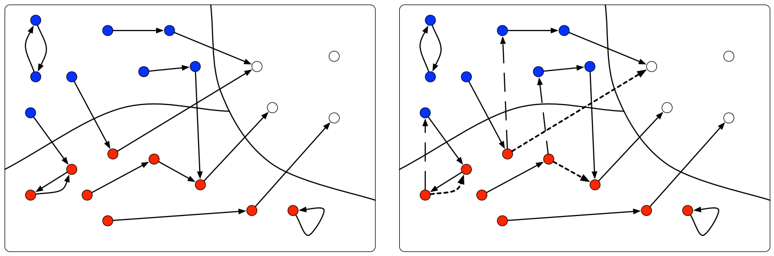

FIGURE 6. Superposition of g and h with collisions (left) and with resolved collisions (right)

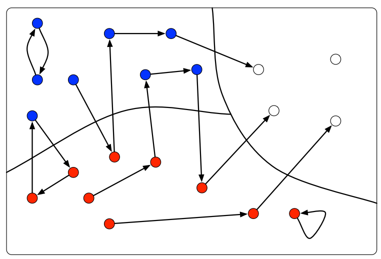

FIGURE 7. $f = g \star h \in \mathfrak{L}[X \uplus Y, Z]$

graph $K_{n,n+m} \simeq N \times (\overline{N} \uplus M)$ a 2-sorted matching is a pair (σ, τ) of matchings of $K_{n,n+m}$, considered as edge sets (with σ, respectively τ consisting of red, respectively blue edges), such that $\sigma \uplus \tau$ is again a matching of $K_{n,n+m}$. The weight of such a pair (σ, τ) is defined as

$$\phi_{x,y,z}(\sigma, \tau) = x^{\sharp\sigma} y^{\sharp\tau} z^{n-\sharp\sigma-\sharp\tau},$$

see Figure 8 for an illustration.

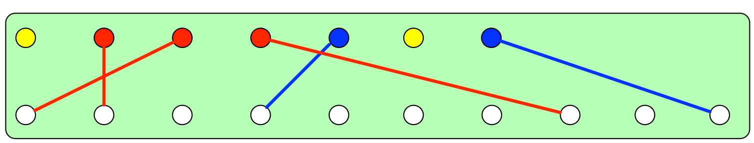

FIGURE 8. A 2-sorted matching (σ, τ) of $K_{7,10}$ of weight $x^3y^2z^2$ (σ in red, τ in blue, uncovered vertices of N in yellow)

Then $\mathcal{L}_n^{(m)}(x + y, z)$ is the generating polynomial for 2-sorted matchings (σ, τ) of $K_{n,n+m}$ w.r.t. $\phi_{x,y,z}$. But each pair (σ, τ) is specified by

- the number $s = \sharp\sigma$ with $0 \leq s \leq n$,
- subsets $S \subseteq N$ and $S' \subseteq \overline{N} \uplus M$ with $\sharp S = \sharp S' = s$,
- a bijection between S and S', which is the restriction of σ to $S \times S'$,
- a matching of $(N \setminus S) \times ((\overline{N} \uplus M) \setminus S')$, i.e., the restriction of ν to $N \setminus S$ and $(\overline{N} \uplus M) \setminus S'$.

For any s, there are $\binom{n}{s}\binom{n+m}{s}$ possibilities for choosing (S, S') and then $s!$ possibilities for the restriction of σ which thus contribute a total of $\mathcal{L}_s^{(0)}(x, 0) = s!x^s$. Finally, the possibilities for τ are the counted by $\mathcal{L}_{n-s}^{(m)}(y,z)$. As a result:

$$\mathcal{L}_n^{(m)}(x+y,z) = \sum_{0\le s\le n} \binom{n}{s}\binom{n+m}{s} \mathcal{L}_s^{(0)}(x,0)\,\mathcal{L}_{n-s}^{(m)}(y,z),$$

which is equivalent to Eq. (4.2). This will be used later in Section 12.

5. Differential operators for Laguerre configurations

It is interesting to see how familiar differential (or difference) formulas for the Laguerre polynomials can be interpreted (or deduced) from a combinatorial perspective.

For fixed $n \in \mathbb{N}$, let

$$\mathfrak{L}_n = \bigcup_{A\oplus B=[n]} \mathfrak{L}[A,B]$$

denote the set of all Laguerre configurations of order n, and for $0 \le k \le n$

$$\mathfrak{L}_{n,k} = \bigcup_{\substack{A\oplus B=[n]\\ \sharp A=k}} \mathfrak{L}[A,B].$$

Then, as mentioned before, $\mathfrak{L}_{n,0}$ is a singleton set, consisting of the function $\lambda_\emptyset$ with empty domain $\emptyset$, and $\mathfrak{L}_{n,n}$ is nothing but the set of all permutations of $[n]$. The corresponding generating polynomials are

$$\mathcal{L}_{n,0}^{(\alpha)}(x,y) = y^n, \qquad \mathcal{L}_{n,n}^{(\alpha)}(x,y) = (1+\alpha)_n\, x^n.$$

The set of all Laguerre configurations on $[n]$ can be constructed iteratively by starting with $\lambda_\emptyset$ on level $k=0$ by "adding n functional arrows", one at a time, thus passing from $\mathfrak{L}_{n,k}$ to $\mathfrak{L}_{n,k+1}$ for $0 \le k < n$, as follows:

- take $\lambda \in \mathfrak{L}[A,B]$, where $\sharp A = k$,
- pick one element $b \in B$ and one element $c \notin \lambda(A)$,
- add to λ the new arrow $b \to c$.

Then $\lambda' = \lambda \cup \{b \to c\} \in \mathfrak{L}[A',B']$, where $A' = A \cup \{b\}$ and $B' = B \setminus \{b\}$. Among all the $n-k$ possible choices for c (given one of the $n-k$ possible choices for b) there is precisely one which leads to the creation of a new cycle, namely when c is the start vertex of the (unique) λ-chain that ends in b. Note that for recovering λ from λ' one needs to know which of the $k+1$ arrows from λ' was added last.

Taking all possibilities into consideration, and expressing the choices by differential "pointing" operators, one sees that

$$(1+\alpha+y\,\partial_y)(x\,\partial_y)\,\mathcal{L}_{n,k}^{(\alpha)}(x,y) = (k+1)\,\mathcal{L}_{n,k+1}^{(\alpha)}(x,y) = (x\,\partial_x)\,\mathcal{L}_{n,k+1}^{(\alpha)}(x,y). \tag{5.1}$$

Note that $(x\partial_y)$ randomly selects in λ a point b from B (variable y) and turns it into a point of A' (variable x). Then c is chosen as the start vertex of a λ-chain, viz.,

- either of the unique λ-chain ending in b, thus forming a λ'-cycle (which gets valuation $1+\alpha$),
- or by selecting (by using $y\partial_y$) any of the remaining points from $B \setminus \{b\}$.

Thus one gets by induction

$$\frac{\left[(1+\alpha+y\,\partial_y)(x\,\partial_y)\right]^k}{k!}\,\mathcal{L}_{n,0}^{(\alpha)}(x,y) = \frac{\left[(1+\alpha+y\,\partial_y)(x\,\partial_y)\right]^k}{k!}\,y^n = \mathcal{L}_{n,k}^{(\alpha)}(x,y),$$

and, by summing over k,

$$\exp\left[(1+\alpha+y\,\partial_y)(x\,\partial_y)\right] y^n = \mathcal{L}_n^{(\alpha)}(x,y),$$

which should be compared to Eq. (1.5) in [2].

Summing identity (5.1) over k, we get

$$(1+\alpha+y\,\partial_y)(x\,\partial_y)\,\mathcal{L}_n^{(\alpha)}(x,y) = (x\,\partial_x)\,\mathcal{L}_n^{(\alpha)}(x,y),$$

which is nothing but the hypergeometric differential equation for the Laguerre polynomials in homogeneous form.

More examples of a combinatorially inspired use of differential operators in the context of Laguerre and Jacobi polynomials can be found in [26].

6. An natural approach to lacunary Laguerre series

The technique of Section 4.2 can be used for obtaining lacunary Laguerre series in a very natural setting. The idea is that the Laguerre configurations counted by $\mathcal{L}_{2n}(x,y)$ can be seen as configurations on a set $\overline{N} = N \uplus N'$, where N is a set of cardinality n, and where N' is a copy of N with a natural bijection (perfect matching) between N and N'.

The lacunary Laguerre series (3.11) in [2], viz.

$$\sum_{n\geq 0} t^n\,L_{2n}(x) = \frac{1}{1-t}\sum_{r\geq 0}\frac{L_r^{(r)}(x/2)}{(1/2)_r}\left[-\frac{t\,x}{2(1-t)}\right]^r, \tag{6.1}$$

is the special case for $\alpha \to 0$ of

$$\sum_{n\geq 0}\frac{(1/2)_n\,t^n}{((1+\alpha)/2)_n}\,L_{2n}^{(\alpha)}(x) = \frac{1}{(1-t)^{(1+\alpha/2)}}\sum_{r\geq 0}\frac{L_r^{(r+\alpha)}(x/2)}{((1+\alpha)/2)_r}\left[-\frac{t\,x}{2(1-t)}\right]^r. \tag{6.2}$$

By using the binomial series and comparison of coefficients for t^n, one gets

$$(2n)!\,L_{2n}^{(\alpha)}(x) = \sum_{0\leq r\leq n}\binom{n}{r}\,r!\,2^r\,L_r^{(r+\alpha)}(x/2)\,(1+2r+\alpha)_{2n-2r}(-x)^r, \tag{6.3}$$

or, in our notation,

$$\mathcal{L}_{2n}^{(\alpha)}(x,y) = \sum_{0\leq r\leq n}\binom{n}{r}\,(1+\alpha+2r)_{2n-2r}\,\mathcal{L}_r^{(\alpha+r)}(2x,y)\,x^{2n-2r}\,y^r. \tag{6.4}$$

Before delving into combinatorics, a short remark about these identities is in order. By using

$$\frac{((1+\alpha)/2)_n\,(1+\alpha/2)_n}{((1+\alpha)/2)_r\,(1+\alpha/2)_r} = (1+\alpha+2r)_{2n-2r}4^{r-n},$$

Eq. (6.3) is easily seen to be equivalent to

$$\frac{(1/2)_n}{((1+\alpha)/2)_n(1+\alpha/2)_n}L_{2n}^{(\alpha)}(x) = \sum_{0\leq r\leq n}\frac{(-x/2)^r}{((1+\alpha)/2)_r(1+\alpha/2)_r}L_r^{(r+\alpha)}(x/2)\frac{1}{(n-r)!}, \tag{6.5}$$

which leads to the generating function of Eq. (6.2). By exchanging the roles of $((1+\alpha)/2)_n$ and $(1+\alpha/2)_n$ when passing to the generating function one may also obtain

$$\sum_{n\geq 0}\frac{(1/2)_n\,t^n}{(1+\alpha/2)_n}L_{2n}^{(\alpha)}(x)=\frac{1}{(1-t)^{(1+\alpha)/2}}\sum_{r\geq 0}\frac{1}{(1+\alpha/2)_r}L_r^{(\alpha+r)}(x/2)\left[\frac{-x\,t}{2(1-t)}\right]^r. \quad (6.6)$$

Identities (6.2) and (6.6) look deceptively similar, yet they are not quite the same (although they may be considered equivalent in view of the preceding argument). From (6.2) on gets (6.1) by a simple specialization $\alpha\to 0$, but the same substitution leads from (6.6) to

$$\sum_{n\geq 0}\binom{2n}{n}\left(\frac{t}{4}\right)^n L_{2n}(x)=\frac{1}{(1-t)^{1/2}}\sum_{r\geq 0}\frac{1}{r!}L_r^{(r)}(x/2)\left[\frac{-x\,t}{2(1-t)}\right]^r. \quad (6.7)$$

Remark that the identity in Eq. (6.6) occurs as Eq. (4.6) of [2], see also my remarks reproduced in Appendix A of that article. Here this result will show up again in Section 12.

We now proceed towards a combinatorial proof of (6.4). Let N be a set of cardinality n and let N_1,N_2 be two copies of N, and $\overline{N}=N_1\uplus N_2$. There is a natural bijection $\rho=N_1\leftrightarrow N_2$, for $\xi\in\overline{N}$ we write $\xi'=\rho(\xi)$. For a subset $A\subseteq\overline{N}$ put $A'=\rho(A)$ and $A_i=A\cap N_i$ $(i\in\{1,2\})$.

From the bipartition $\overline{N}=N_1\uplus N_2$ one naturally gets, for any bipartition $\overline{N}=X\uplus Y$, a 4-partition $\overline{N}=A\uplus B\uplus C\uplus D$ as follows:

$$\text{for } \xi\in\overline{N}\ :\ \xi\in\begin{cases}A, & \text{if } (\xi,\xi')\in X\times X,\\ B, & \text{if } (\xi,\xi')\in X\times Y,\\ C, & \text{if } (\xi,\xi')\in Y\times X,\\ D, & \text{if } (\xi,\xi')\in Y\times Y.\end{cases}$$

Then $X=A\uplus B, Y=C\uplus D$ and the partition (A,B,C,D) satisfies

$$A=A', B_1=C_2', B_2=C_1', D=D'. \quad (6.8)$$

Schematically (see Figure 9, with bold contour lines indicating the regions where arrows are defined)

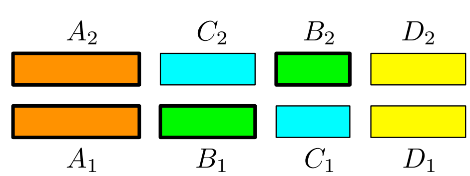

FIGURE 9. The scheme for $\mathfrak{L}[X,Y]$ before decomposition

Looking at the corresponding Laguerre configurations, one gets

$$\begin{aligned}\mathfrak{L}[X,Y]&=\mathfrak{L}[A\uplus B,C\uplus D]\\ &\simeq\mathfrak{L}[A,B\uplus C\uplus D]\times\mathfrak{L}[B,C\uplus D]\\ &\simeq\mathfrak{L}[A,B\uplus C\uplus D]\times\mathfrak{L}[B,D_1\,;\,C_1\uplus C_2\uplus D_2]\\ &=\mathfrak{L}[A,N\setminus A]\times\mathfrak{L}[B,D_1\,;\,(N\setminus A)_2],\end{aligned}$$

where the decomposition property, as discussed in Section 4.2, see Eq. (4.1), has been used. The situation for the two Laguerre-factors (omitting arrows) is now as depicted in Figure 10:

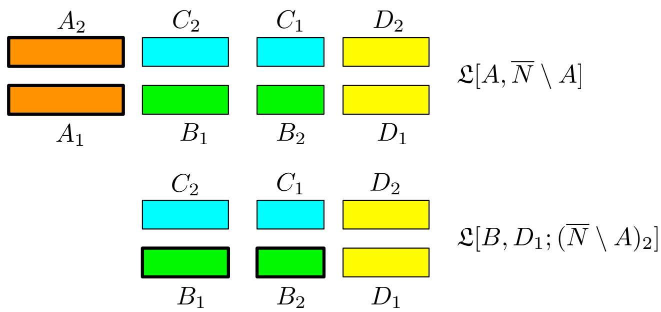

FIGURE 10. The scheme for $\mathfrak{L}[X, Y]$ after decomposition

We obtain

$$\begin{aligned}\mathcal{L}_{2n}^{(\alpha)}(x, y) &= \textstyle\sum_{X \uplus Y = \overline{N}} \sum \left\{ \omega_{x,y}^{(\alpha)}(f)\,;\ f \in \mathfrak{L}[X, Y] \right\} \\ &= \textstyle\sum'_{A \uplus B \uplus C \uplus D = \overline{N}} \sum \left\{ \omega_{x,1}(g)\,;\ g \in \mathfrak{L}[A, \overline{N} \setminus A] \right\} \times \\ &\qquad\qquad \times \textstyle\sum \left\{ \omega_{x,y,y}(h)\,;\ h \in \mathfrak{L}[B_1 \uplus C_1, D_1\,;\ (N \setminus A)_2] \right\},\end{aligned}$$

where $\sum'$ means that one sums over partitions (A, B, C, D) of $\overline{N}$ that satisfy Eq. (6.8). Then one realizes:

- For each $0 \le a \le n$ there are $\binom{n}{a}$ choices for the A-part.
- For each A-part the contribution from the $g \in \mathfrak{L}[A, \overline{N} \setminus A]$ is $(1 + \alpha + 2b)_{2a} x^{2a}$, where $b = n - a$.
- For each A, the contribution coming from all the $h \in \mathfrak{L}[B, D_1\,;\ (N \setminus A)_2]$ looks if it should be $\mathcal{L}_b^{\alpha+b}(x, y)\, y^b$, because $\sharp B + \sharp D_1 = \sharp C + \sharp D_2 = n - a = b$, but in addition the multiplicity from partitioning B into B_1 and B_2 in all possible ways (which all lead to the same contribution) has to be taken into account. This multiplicity is $2^{\sharp B}$, which is accommodated by simply replacing the variable x by $2x$.

If we put all this together this gives us Eq. (6.4), which we now write as

$$\mathcal{L}_{2n}^{(\alpha)}(x, y) = \sum_{a+b=n} \binom{n}{a, b} (1 + \alpha + 2b)_{2a}\, x^{2a} \cdot \mathcal{L}_b^{(\alpha+b)}(2x, y)\, y^b. \tag{6.9}$$

A surplus of the proof just given is that it immediately tells you what happens for a triple lacunary series, see identity (3.12) of [2], by looking at the combinatorial picture by stacking three copies of a base set N on top of each other, and adapting the definitions of A, B, C and D, see Figure 11 for a schematic illustration.

The result is

$$\begin{aligned}&\mathcal{L}_{3n}^{(\alpha)}(x, y) \\ &\quad = \sum_{a+b+c=n} \binom{n}{a, b, c} (1 + \alpha + b + 3c)_{3a+2b} \binom{3}{1}^b x^{3a+2b} \cdot \mathcal{L}_c^{(\alpha+b+2c)}(3x, y)\, y^{b+2c}.\end{aligned}$$

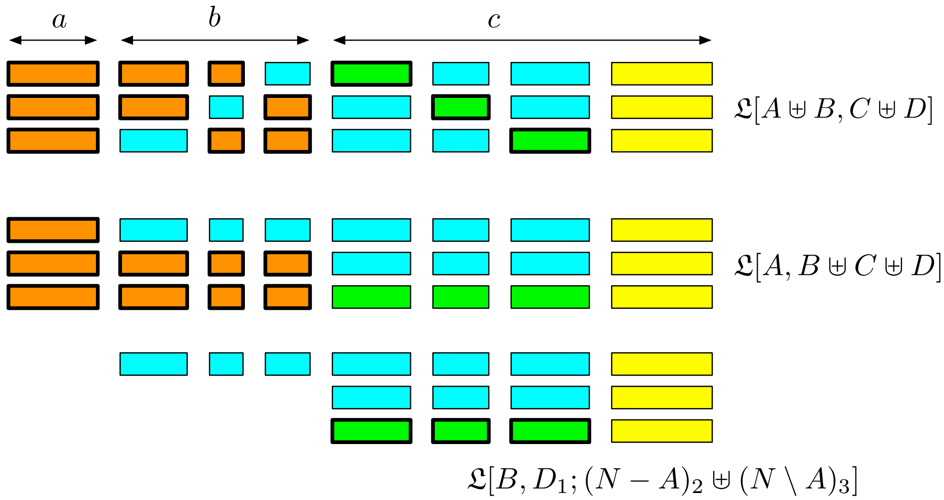

FIGURE 11. The scheme for $\mathfrak{L}[X, Y]$ in the triple lacunary case, again with A in red, B in green, C in cyan, D in yellow.

A completely analogous approach for a quadruple lacunary series gives

$$\mathcal{L}_{4n}^{(\alpha)}(x,y) = \sum_{a+b+c+d=n} \binom{n}{a,b,c,d} (1+\alpha+b+2c+4d)_{4a+3b+2c} \binom{4}{1}^b \binom{4}{2}^c \times \\ \times x^{4a+3b+2c} \mathcal{L}_d^{(1+\alpha+b+2c+3d)}(4x,y)\, y^{b+2c+3d}.$$

The general pattern should now be clear.

7. The simplest lacunary Laguerre series

In this section a different approach to lacunary Laguerre series will be employed, namely correlating Laguerre configurations with arbitrary perfect matchings on a ground set.

Identity (3.5) from [2] reads

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}(x) = e^t \cdot \sum_{r\geq 0} \frac{(ix\sqrt{t})^r}{r!^2} H_r(i\sqrt{t}). \tag{7.1}$$

The particular feature of this identity is that the n-th term on the l.h.s. comes with the Laguerre polynomial $L_{2n}(x)$, but with $n!$ (instead of $(2n)!$), as one would expect) in the denominator. If we force $(2n)!$ to appear in the denominator and make the innocuous substitution $t \to t^2/2$, we get an interesting factor:

$$\sum_{n\geq 0} \frac{t^{2n}}{(2n)!} \frac{(2n)!}{n!\,2^n} L_{2n}(x) = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!}\, m_{2n}\, L_{2n}(x) = \sum_{n\geq 0} \frac{t^n}{n!}\, m_n\, L_n(x).$$

The matching numbers m_n from Section 3.1 appear! This suggests a combinatorial meaning: consider it as an exponential generating function for pairs $(\mu, \lambda) \in \mathfrak{M}[A] \times \mathfrak{L}[A]$, i.e., μ is a perfect matching and λ is a Laguerre configuration on the same vertex set A, where A must have even cardinality by the perfectness of μ.

Rewriting Eq. (7.1) with $t \to t^2/2$ and $x \to -y$ in terms of our "combinatorial" Laguerre and Hermite polynomials, we are led to

$$\sum_{n\geq 0} \frac{t^{2n}}{(2n)!} \frac{(2n)!}{n!2^n} \frac{1}{(2n)!} \mathcal{L}_{2n}^{(0)}(1,y) = e^{t^2/2} \cdot \sum_{n\geq 0} \frac{(yt)^n}{n!^2} \mathcal{H}_n(t,1),$$

and then the substitutions $yt \to y$ and $t \to x$ give us

$$\sum_{n\geq 0} \frac{m_{2n}}{(2n)!} \frac{1}{(2n)!} \mathcal{L}_{2n}^{(0)}(x,y) = e^{x^2/2} \cdot \sum_{n\geq 0} \frac{1}{n!^2} \mathcal{H}_n(x\,y,y). \tag{7.2}$$

This will now be proven by a combinatorial counting argument. First note that all terms on both sides of (7.2) indeed have an even total degree in x, y. Therefore it suffices to consider the contributions for a fixed total degree $2n$ in x, y on both sides by writing

$$m_{2n} \cdot \mathcal{L}_{2n}^{(0)}(x,y) = (2n)!^2 \left[e^{x^2/2} \cdot \sum_{s\geq 0} \frac{1}{s!^2} \mathcal{H}_s(x\,y,y) \right]^{\langle 2n \rangle}, \tag{7.3}$$

where $[\ldots]^{\langle 2n \rangle}$ denotes the sum of the terms of total degree $2n$ in $[\ldots]$.

Let N denote a set of cardinality $2n$. The left-hand side of (7.3) enumerates pairs $(\mu, \lambda) \in \mathfrak{M}(N) \times \mathfrak{L}(N)$. This can be evaluated explicitly and shown to be equal to the right-hand side.

Fix a partition $R \uplus S = N$ of the base set N, let $r = \sharp R$ and $s = \sharp S = 2n - r$. The following argument gives the contribution coming from the (μ, λ) with $\lambda \in \mathfrak{L}[R, S]$. Actually, this contribution depends only on the cardinalities of R and S, not on the functional structure of λ !

Now take the bijection considered in Section 4.1,

$$\mathfrak{M}[N] \to \bigcup_{\substack{A \uplus B' = R \\ B'' \uplus C = S}} \mathfrak{M}[A] \times \mathfrak{B}[B', B''] \times \mathfrak{M}[C] : \mu \mapsto (\mu_R, \beta, \mu_S),$$

with $r = \sharp R = 2a + b$ and $s = \sharp S = 2c + b$. Furthermore note that μ_S can be considered as an element of $\mathfrak{H}[B'', C]$. Now let us count:

- For fixed (A, B', B'', C) the number of relevant (μ_R, β, μ_S) is, by construction, $m_{2a} \cdot b! \cdot m_{2c}$.
- The number of choices for (A, B', B'', C) for fixed (a, b, c) is $\binom{r}{2a}\binom{s}{2c}$.
- For fixed (R, S) with cardinalities (r, s) there are $\sharp\mathfrak{L}[R, S] = (1 + s)_r = (2n)!/s!$ Laguerre configurations.
- There are $\binom{2n}{s}$ choices of the ordered pair (R, S) for given (r, s), which leads to a valuation $\omega_{x,y}^0(\lambda) = x^r y^s$ of the Laguerre configurations, cycles are (not yet) counted.

To summarize, the l.h.s. of Eq. (7.3) can be written as

$$\begin{aligned}
&\sum_{r+s=2n} \binom{2n}{s} \frac{(2n)!}{s!} x^r y^s \sum_{2a+b=r} \binom{r}{2a} \sum_{2c+b=s} \binom{s}{2c} m_{2a}\, b!\, m_{2c} \\
&\quad = \sum_{r+s=2n} \frac{(2n)!}{r!s!} \frac{(2n)!}{s!} \sum_{2a+b=r} \frac{r!}{(2a)!b!} m_{2a}\, x^{2a} \sum_{2c+b=s} \binom{s}{2c} b!\, m_{2c}\, (x\,y)^b y^{2c} \\
&\quad = (2n)!^2 \sum_{r+s=2n} \sum_{2a+b=r} \frac{m_{2a}}{(2a)!}\, x^{2a} \frac{1}{s!^2} \mathcal{H}_s(x\,y, y) \\
&\quad = (2n)!^2 \left[\sum_{a\geq 0} m_{2a} \frac{x^{2a}}{(2a)!} \sum_{s\geq 0} \frac{1}{s!^2}\, \mathcal{H}_s(x\,y, y) \right]^{\langle 2n\rangle} \\
&\quad = (2n)!^2 \left[e^{x^2/2} \sum_{s\geq 0} \frac{1}{s!^2}\, \mathcal{H}_s(x\,y, y) \right]^{\langle 2n\rangle} .
\end{aligned}$$

The combinatorial proof of Eq. (7.3) and hence of Eq. (7.1) is now complete. One could easily add cycle counting with a parameter α to this proof, thus obtaining a combinatorial proof of identity (3.6) of [2]. But much more can be done by extending the combinatorial perspective. Instead of pairing Laguerre configurations with *perfect* matchings, one can pair them with *arbitrary* matchings on the same ground set. This will be done in the next section, and the α for cycle counting will also be included in this generalization.

8. An extension motivated by combinatorics

The combinatorial picture presented in the foregoing Section 7 leads immediately to the following idea for an extension: play the same game, but now with arbitrary involutions (Hermite configurations) instead of perfect matchings! Note that the use of perfect matchings forced terms occurring on the left of (7.2) to have an even total degree — hence the lacunarity of the series expansion! If we replace $\mathfrak{M} \times \mathfrak{L}$ by $\mathfrak{I} \times \mathfrak{L}$, then lacunarity gets lost, but otherwise one can hope that a similar counting argument would go through. This is is indeed the case and, interestingly, now the functional structure of the Laguerre configurations plays a role, whereas in the previous proof only the underlying pair of sets (X, Y) was relevant. In addition, in this enlarged set-up also the cycle counting for the Laguerre configurations will be tracked by the parameter α.

The generating function identity to be proven in this section, written in terms of the combinatorial Hermite and Laguerre polynomials, is

$$\sum_{n\geq 0} \frac{\mathcal{H}_n(u,v)}{n!} \frac{\mathcal{L}_n^{(\alpha)}(x,y)}{(1+\alpha)_n} = \mathcal{M}(x\,v) \sum_{s,p\geq 0} \frac{\mathcal{H}_s(x\,y\,v^2, y\,v)}{s!} \frac{\mathcal{L}_p^{(\alpha+s)}(x\,u, y\,u)}{p!\,(1+\alpha)_{s+p}}. \tag{8.1}$$

Before delving into the details, I would like to stress that — at least from my personal perspective — it would be difficult to imagine such an identity without being guided by combinatorial imagination.

By considering the contributions of total degree n on both sides, one finds that identity (8.1) is equivalent to

$$\begin{aligned}\mathcal{H}_n(u,v)\cdot\mathcal{L}_n^{(\alpha)}(x,y)\\ =\sum_{r+s+p=n}\binom{n}{r,s,p}r!\,(1+\alpha+s+p)_r\cdot\Big[e^{(x\,v)^2/2}\cdot\mathcal{H}_s(x\,y\,v^2,y\,v)\cdot\mathcal{L}_p^{(\alpha+s)}(x\,u,y\,u)\Big]^{\langle n\rangle},\end{aligned}\tag{8.2}$$

where again $[\ldots]^{\langle n\rangle}$ means that the homogeneous part of degree n (in one of the variable pairs $\{x,y\}$ or $\{u,v\}$, for the other pair this follows automatically) has to be taken.

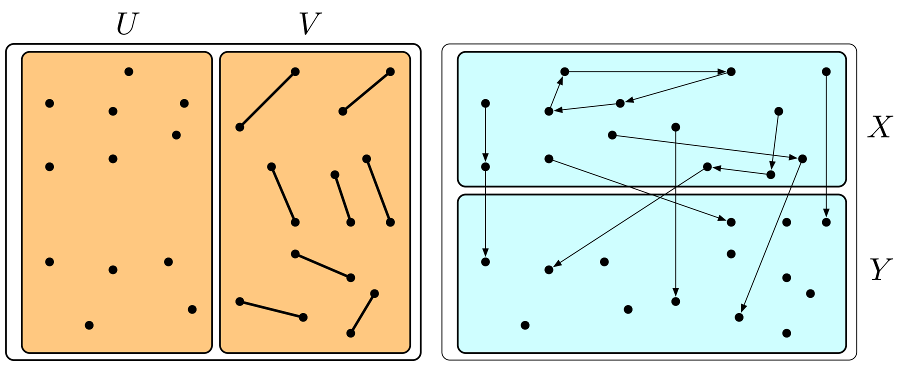

FIGURE 12. $\sigma\in\mathcal{H}[U,V]$ and $\lambda\in\mathfrak{L}[X,Y]$ separately

The combinatorial argument used here directly generalizes the approach taken in Section 7. For a set N of cardinality n take any $(\sigma,\lambda)\in\mathfrak{H}[N]\times\mathfrak{L}[N]$, more precisely, $\sigma\in\mathfrak{H}[U,V]$ and $\lambda\in\mathfrak{L}[X,Y]$, so that $N=U\uplus V$ and $N=X\uplus Y$ are two bipartitions of N. To shorten the notation somewhat, we will use $AB\equiv A\cap B$ for any two sets A,B in the sequel.

Let $\sharp U=p$, $\sharp V=q=n-p$, $\sharp X=i$, $\sharp Y=j=n-i$. The two bipartitions of N define

- a bipartition $UX\uplus UY$ of U, with $\sharp UX=k$, $\sharp UY=\ell$,
- a bipartition $VX\uplus VY$ of V, with $\sharp VX=r,\sharp VY=s$.

Now $\sigma\big|_V$ is a perfect matching of V, so that the ideas of Section 4.1 can be applied w.r.t. the bipartition of $V=VX\uplus VY$. This defines a set $W\subseteq V$ which contains precisely the transpositions of $\sigma\big|_V$ which "cross the border" between VX and VY. This gives a 4-partition

$$V=(VX\setminus WX)\uplus WX\uplus WY\uplus(VY\setminus WY)$$

with $\sharp(VX\setminus WX)=2a,\sharp WX=\sharp WY=b,\sharp(VY\setminus WY)=2c$.

Then

- $\sigma_{VX}=\sigma|_{VX\setminus WX}\in\mathfrak{M}[VX\setminus WX]$
- $\sigma_{VY}\in\mathfrak{H}[WY,VY\setminus WY]$ with $\sigma_{VY}\big|_{VY\setminus WY}\in\mathfrak{M}[VY\setminus WY]$ and $\mathsf{fix}(\sigma_{VY})=WY$,
- $\beta=\sigma\big|_{WX\uplus WY}\in\mathfrak{B}[WX,WY]$.

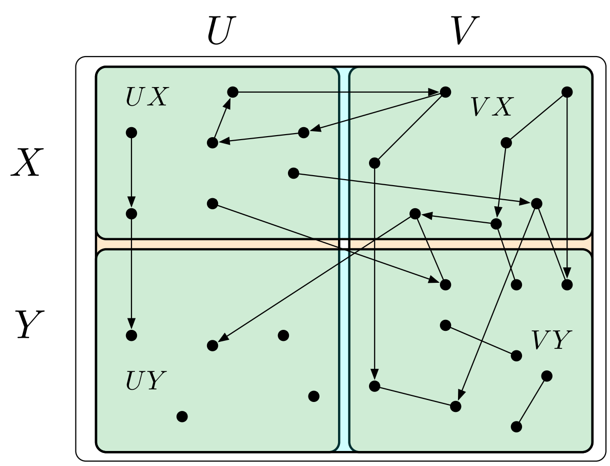

FIGURE 13. Superposition $(\sigma, \lambda) \in \mathfrak{H}[U, V] \times \mathfrak{L}[X, Y]$

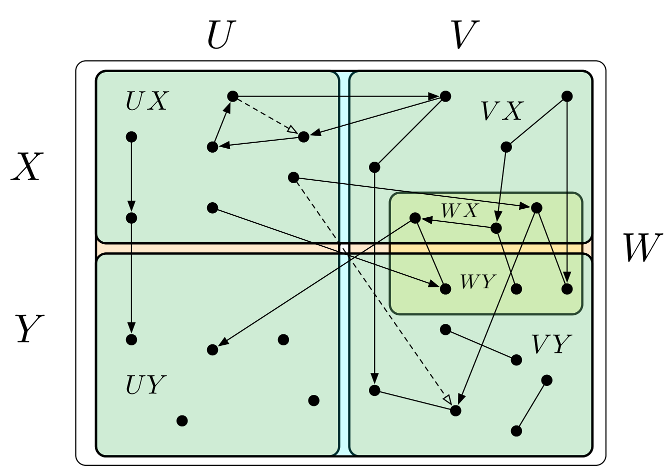

FIGURE 14. Superposition $(\sigma, \lambda) \in \mathfrak{H}[U, V] \times \mathfrak{L}[X, Y]$ with matching set W and matching between WX and WY.

We have $\sharp VX = r$ and $\sharp VY = s$, so that in particular

$$k + \ell = p, \ r + s = 2(a + b + c) = q, \ k + r = i, \ s + \ell = j.$$

As for $\lambda \in \mathfrak{L}[X, Y]$, due to $X = UX \uplus VX$, and using the decomposition of Laguerre configurations as in (4.1), we have the equivalence

$$\mathfrak{L}[UX \uplus VX, Y] \ni \lambda \leftrightarrow (\lambda_{VX}, \lambda_{UX}) \in \mathfrak{L}[VX, UX \uplus Y] \times \mathfrak{L}[UX, Y],$$

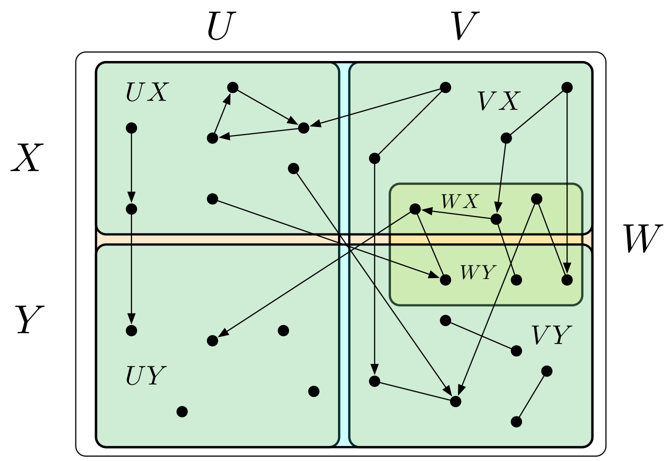

FIGURE 15. Final configuration after changing the critical arrows: $\sigma_{VX} \in \mathfrak{H}[\emptyset, VX \setminus WX]$, $\beta \in \mathcal{B}[WY, WX]$, $\sigma_{VY} \in \mathfrak{H}[WY, VY \setminus WY]$, $\lambda_{VX} \in \mathfrak{L}[VX, UX \uplus Y]$, $\lambda_{UX} \in \mathfrak{L}[UX, Y]$

and, furthermore,

$$\mathsf{cyc}(\lambda) = \mathsf{cyc}(\lambda_{UX}) + \mathsf{cyc}(\lambda_{VX}).$$

For fixed U, V, X, Y and hence for fixed UX, UY, VX, VY the transformation

$$(\sigma, \lambda) \mapsto (\sigma_{VX}, \beta, \sigma_{VY}, \lambda_{VX}, \lambda_{UX})$$

is a bijection which preserves cycle counting between

$$\mathfrak{H}[U, V] \times \mathfrak{L}[X, Y]$$

and the union of

$$\mathfrak{M}[VX \setminus WX] \times \mathfrak{B}[WX, WY] \times \mathfrak{H}[WY, VY \setminus WY] \times \mathfrak{L}[VX, UX \uplus Y] \times \mathfrak{L}[UX, Y],$$

running over all bipartitions $(VX \setminus WX) \uplus WX = VX$ and $(VY \setminus WY) \uplus WY = VY$. The correspondence is indeed literally:

$$\begin{array}{ccccc} \sigma_{VX} & \beta & \sigma_{VY} & \lambda_{VX} & \lambda_{UX} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathfrak{M}[VX \setminus WX] & \mathfrak{B}[WX, WY] & \mathfrak{H}[WY, VY \setminus WY] & \mathfrak{L}[VX, UX \uplus Y] & \mathfrak{L}[UX, Y] \end{array}$$

Note that the cycle counting part from λ splits into the corresponding parts of λ' and λ'':

$$(1 + \alpha + j)_i = (1 + \alpha + j)_k \cdot (1 + \alpha + j + k)_r = (1 + \alpha + j)_k \cdot (1 + \alpha + s + p)_r$$

Summing now over all possible partitions

$$(VX \setminus WX) \uplus WX \uplus WY \uplus (VY \setminus WY) \uplus UX \uplus UY$$

of N, and using the weight function

$$\nu_{u,v}(\sigma) \cdot \omega^{\alpha}_{x,y}(\lambda),$$

we obtain the expression

$$\sum_{r+s+p=n} \binom{n}{r,s,p} r!\,(1+\alpha+s+p)_r \cdot \left[e^{(x\,v)^2/2} \cdot \mathcal{H}_s(x\,y\,v^2, y\,v) \cdot \mathcal{L}_p^{(\alpha+s)}(x\,u, y\,u)\right]^{\langle n\rangle},$$

where the contribution

- from $\sigma_{VX} \in \mathfrak{M}[VX \setminus WX]$ goes into $\mathcal{M}(x\,v) = e^{(x\,v)^2/2}$,
- from $(\beta, \sigma_{VY}) \in \mathfrak{B}[WX, WY] \times \mathfrak{H}[WY, VY \setminus WY]$ goes into $\mathcal{H}_s(x\,y\,v^2, y\,v)$,
- from $\lambda_{VX} \in \mathfrak{L}[VX, UX \uplus Y]$ goes into $\mathcal{L}_p^{(\alpha+s)}(x\,u, y\,u)$,
- from $\lambda_{UX} \in \mathfrak{L}[UX, Y]$ goes into the term $(1+\alpha+s+p)_r$.

This proves Eq. (8.2) and hence Eq. (8.1).

Specializing the result (8.1) by setting $v = 1$ (for convenience) and extracting the coefficient of u^k (for fixed k) on both sides, we obtain a whole range of lacunary Laguerre series:

$$\sum_{m\geq 0} \frac{\mathcal{L}_{2m+k}^{(\alpha)}(x,y)}{2^m\, m!\,(1+\alpha)_{2m+k}} = \mathcal{M}(x) \sum_{s\geq 0} \frac{\mathcal{H}_s(x\,y, y)}{s!} \frac{\mathcal{L}_k^{(\alpha+s)}(x,y)}{(1+\alpha)_{s+k}}.$$

For the proof, just note that for having a term u^k to appear in $\mathcal{H}_n(u,1)$ on the left-hand side of the main result one must have that $n-k$ is even, so put $n = 2m+k$. In this case the coefficient is

$$h_{n,m} = \frac{n!}{k!\,m!\,2^m}.$$

On the other hand, on the right-hand side of the main result the variable u appears only in the term $\mathcal{L}_p^{(\alpha+s)}(xu, yu)$, which is homogeneous of degree p, so that it can be written as $\mathcal{L}_p^{(\alpha+s)}(x,y)\,u^p$. This shows that the summation over p collapses into the single contribution for $p = k$.

9. An extension featuring balls-into-boxes combinatorics

Recall that in Section 7 we showed combinatorially that

$$\sum_{n\geq 0} \frac{t^n}{n!} L_{2n}(x) = e^t \cdot \sum_{r\geq 0} \frac{(i\,x\sqrt{t})^r}{r!^2} H_r(i\sqrt{t}). \tag{9.1}$$

This is the case $k = 0$ of

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(k)}(x) = e^t \cdot \sum_{r=0}^{\infty} \frac{p_{2k}(r; 1, -x, t)}{r!(r+3k)!} (i\,x\sqrt{t})^r H_r(i\sqrt{t}), \tag{9.2}$$

where $p_{2k}(r; x, y; t)$ is a homogeneous polynomial of degree $2k$ in $\{x, y\}$ and also of degree $2k$ in r, with integer coefficients. It can be argued algebraically that such polynomials must exists, and one can use computer algebra to determine the first cases. Indeed, the authors of [2] have done this for the cases $k = 1$ and $k = 2$, see their identities (3.7), (3.8) and (3.9). Here is what their results are:

$$p_2(r;x,y,t) = (1+2tx^2)r^2 + (5+4xyt+10tx^2)r + (6+12tx^2+12xyt+2ty^2), \tag{9.3}$$

$$\begin{aligned} p_4(r;x,y,t) = {} & (2+10tx^2+4t^2x^4)r^4 \\ & + [36+(180x^2+20yx)t+(72x^4+16x^3y)t^2]r^3 \\ & + [238+(10y^2+1190x^2+300yx)t \\ & \quad + (240x^3y+24x^2y^2+476x^4)t^2]r^2 \\ & + [684+(110y^2+1480yx+3420x^2)t \\ & \quad + (1184x^3y+264y^2x^2+16xy^3+1368x^4)t^2]r \\ & + [720+(2400yx+3600x^2+300y^2)t \\ & \quad + (1920x^3y+96xy^3+720y^2x^2+1440x^4+4y^4)t^2]. \end{aligned} \tag{9.4}$$

Presented as such this does not really look inspiring. I went further and did the computation for $k=3$, and again, at first glance the result, presented in Figure 16 does not look inviting: Indeed, this expression does not seem to simplify in any reasonable way. But change of the priority of the variables leads to a simplified expression that

$$\begin{aligned} p_6(r;x,1,t) = {} & \left(8\,t^3+48\,t^2+54\,t+6\right)r^6 \\ & + \left(48\,xt^3+312\,t^3+192\,xt^2+1872\,t^2+108\,xt+2106\,t+234\right)r^5 \\ & + \left(120\,x^2t^3+1680\,xt^3+288\,x^2t^2+5000\,t^3+6720\,xt^2+54\,x^2t \right. \\ & \quad \left. +30000\,t^2+3780\,xt+33750\,t+3750\right)r^4 \\ & + \left(160\,x^3t^3+3600\,x^2t^3+192\,x^3t^2+23280\,xt^3+8640\,x^2t^2+42120\,t^3 \right. \\ & \quad \left. +93120\,xt^2+1620\,x^2t+252720\,t^2+52380\,xt+284310\,t+31590\right)r^3 \\ & + \left(120\,x^4t^3+3840\,x^3t^3+48\,x^4t^2+40200\,x^2t^3+4608\,x^3t^2+159600\,xt^3 \right. \\ & \quad +96480\,x^2t^2+196592\,t^3+638400\,xt^2+18090\,x^2t \\ & \quad \left. +1179552\,t^2+359100\,xt+1326996\,t+147444\right)r^2 \\ & + \left(48\,x^5t^3+2040\,x^4t^3+30560\,x^3t^3+816\,x^4t^2+198000\,x^2t^3 \right. \\ & \quad +36672\,x^3t^2+541152\,xt^3+475200\,x^2t^2+481728\,t^3+2164608\,xt^2 \\ & \quad \left. +89100\,x^2t+2890368\,t^2+1217592\,xt+3251664\,t+361296\right)r \\ & \quad +8\,x^6t^3+432\,x^5t^3+8640\,x^4t^3+80640\,x^3t^3+3456\,x^4t^2+362880\,x^2t^3 \\ & \quad +96768\,x^3t^2+725760\,xt^3+870912\,x^2t^2+483840\,t^3+2903040\,xt^2 \\ & \quad +163296\,x^2t+2903040\,t^2+1632960\,xt+3265920\,t+362880 \end{aligned}$$

FIGURE 16. The polynomial $p_6(r;x,1,t)$

definitely looks attractive:

$$\begin{aligned}
p_6(r;x,1,t) = 8\,x^6t^3&\\
&+ 48\,x^5t^3\,(5\,t+2)\,(r+9)\\
&+ 24\,x^4t^2\,(r+9)\,(r+8)\\
&+ 32\,x^3t^2\,(5\,t+6)\,(r+9)\,(r+8)\,(r+7)\\
&+ 6\,x^2t\,\big(20\,t^2+48\,t+9\big)\,(r+9)\,(r+8)\,(r+7)\,(r+6)\\
&+ 12\,x\,t\,\big(4\,t^2+16\,t+9\big)\,(r+5)\,(r+9)\,(r+8)\,(r+7)\,(r+6)\\
&+ (8\,t^3+48\,t^2+54\,t+6)\,(r+9)\,(r+8)\,(r+7)\,(r+6)\,(r+5)\,(r+4)\,.
\end{aligned}$$

This encourages one to look back to the expressions given in the cases $k=1$,

$$\begin{aligned}
p_2(r;x,1,t) &= 2\,x^2t + 4\,xt\,(r+3) + (2\,t+1)\,(r+3)\,(r+2)\\
&= \sum_{i=0}^{2}\sum_{j=0}^{1} x^i\,t^j\binom{2j}{i}\frac{(r+3)!}{(r+1+i)!}a_{1,j}, \qquad \text{with } (a_{1,0},a_{1,1})=(1,2),
\end{aligned}$$

and $k=2$,

$$\begin{aligned}
p_4(r;x,1;t) = 4\,x^4t^2&\\
&+ 16\,x^3t^2\,(r+6)\\
&+ 2\,x^2\,t\,(12\,t+5)\,(r+6)\,(r+5)\\
&+ 4\,x\,t\,(4t+5)\,(r+6)\,(r+5)\,(r+4)\\
&+ (4\,t^2+10\,t+2)\,(r+6)\,(r+5)\,(r+4)\,(r+3)\\
= \sum_{i=0}^{4}\sum_{j=0}^{2} x^i\,t^j\binom{2j}{i}\frac{(r+6)!}{(r+2+i)!}a_{2,j},& \qquad \text{with } (a_{2,0},a_{2,1},a_{2,2})=(2,10,4).
\end{aligned}$$

This suggests that in general one should have

$$\sum_{n\ge0} L_{2n}^{(k)}(x)\frac{t^n}{n!} = e^t\sum_{r\ge0}\frac{\tilde p_{2k}(r;-x,t)}{r!}\,(i\,x\sqrt{t})^r\,H_r(i\sqrt{t}), \tag{9.5}$$

where

$$\tilde p_{2k}(r;x,t) = \sum_{0\le i\le 2k}\,\sum_{0\le j\le k} x^i\,t^j\binom{2j}{i}\frac{1}{(r+k+i)!}\,a_{k,j},$$

with (positive integer) coefficients $(a_{k,j})_{0\le j\le k}$. Note that in this notation the term $(r+3)!$ (for $k=1$) and $(r+6)!$ (for $k=2$) disappears from the identity. This change is indicated by writing $\tilde p_{2k}(r;x,t)$ instead of $p_{2k}(r;x,t)$. Now $\tilde p_{2k}(r;x,t)$ is no longer a polynomial in r, but the advantage is that the term $(r+\text{something})!$ cancels and we do not need to know what "something" is.

Computing the first values of the $a_{k,\ell}$, we get

$k\backslash j$	0	1	2	3	4	5	6
0	1						
1	1	2					
2	2	10	4				
3	6	54	48	8			
4	24	336	492	176	16		
5	120	2400	5100	2920	560	32	
6	720	19440	55800	45240	13680	1632	64

This clearly suggests a linear recurrence

$$a_{k,\ell} = 2a_{k-1,\ell-1} + (k+2\ell)\,a_{k-1,\ell},$$

which would show that the numbers $a_{k,\ell}/2^\ell$ are integers. Based on this conjectured recurrence one could derive various generating functions, but it seems more interesting to look at the numbers and generating polynomials

$$\overline{a}_{k,\ell} = a_{k,\ell}\,\frac{\ell!}{k!} \quad \text{and} \quad \overline{a}_k(t) = \sum_{0\le\ell\le k} \overline{a}_{k,\ell} t^\ell.$$

Then the recurrence reads

$$k\,\overline{a}_{k,\ell} = 2\,\ell\,\overline{a}_{k-1,\ell-1} + (k+2\ell)\,\overline{a}_{k-1,\ell},$$

and the $\overline{a}_k(t)$ $(k \ge 0)$ have a very simple rational generating function:

$$\sum_{k\ge0} \overline{a}_k(t)\,z^k = \frac{1-z}{1-2(1+t)z+(1+t)z^2}.$$

The vertical generating functions of the $\overline{a}_{k,\ell}$ are given by

$$\begin{aligned}\sum_{k\ge\ell} \overline{a}_{k,\ell}\,z^k &= \frac{1}{1-z}\left(\frac{z(2-z)}{(1-z)^2}\right)^\ell \\ &= \frac{1}{1-z}\left(\frac{1-(1-z)^2}{(1-z)^2}\right)^\ell \\ &= \frac{(-1)^\ell}{1-z}\cdot\left(1-\frac{1}{(1-z)^2}\right)^\ell.\end{aligned}$$

An exponential generating function for the $\overline{a}_k(t)$ $(k \ge 0)$ can be obtained, but it does not have a particularly nice form.

Here is a table of first values of the $\overline{a}_{k,\ell}$:

$k\backslash\ell$	0	1	2	3	4	5	6
0	1						
1	1	2					
2	1	5	4				
3	1	9	16	8			
4	1	14	41	44	16		
5	1	20	85	146	112	32	
6	1	27	155	377	456	272	64

These numbers are known to the *Online Encyclopedia of Integer Sequences* [30] as entry A056242. They were apparently first considered by Hwang and Mallows [22] in counting *order-consecutive partitions*, but in the present context another combinatorial interpretation of the balls-into-boxes type is better suited.

- For integers k, ℓ with $0 \leq \ell \leq k$ consider
  - k (ordered) singleton boxes, each of which can contain at most one ball,
  - ℓ (ordered) twin boxes, each of which can contain at most one ball in each of the two twin compartments.
- Then $\overline{a}_{k,\ell}$ is the number of distributions of balls into the boxes s.th.
  - each twin box contains at least one ball,
  - the total number of balls in singleton boxes equals the total number of balls in twin boxes.

In Figure 17 this model is illustrated for $k = 3$:

- The first row shows that without any twin box ($\ell = 0$) there is just one possible configuration, so $\overline{a}_{3,0} = 1$;
- The second and the third row show representatives for the case $\ell = 1$, the second row (one ball) giving 6 possibilities, the third row (two balls) giving 3 possibilities, so $\overline{a}_{3,1} = 9$,
- The fourth and the fifth row show representatives for the case $\ell = 2$, the fourth row (two balls) giving 12 possibilities, the third row (three balls) giving 4 possibilities, so $\overline{a}_{3,2} = 16$,
- The sixth row shows a representative for the case $\ell = 3$, where only three balls are necessary. This gives 8 possibilities, so $\overline{a}_{3,3} = 8$.

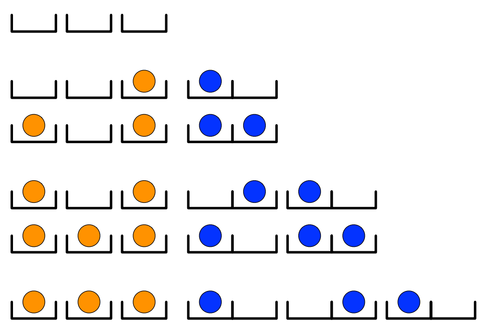

FIGURE 17. Balls-into-boxes for $k = 3$ and $0 \leq \ell \leq 3$

Expansion of the main identity (9.5) gives

$$m_{2n}\binom{2n}{s}\binom{2n+k}{k} = \sum_{2a+2b+i+j=2n} \binom{2n}{2a, 2b, i, j} m_{2a}\, \beta_{s-i,b}\, m_{i+j}\, \overline{a}_{k,(i+j)/2},$$

where

$$\beta_{r,b} = \binom{2b}{r}\binom{r}{b} b!\, 2^{b-r} = m_{2b}\binom{b}{r-b} 2^{2b-r},$$

so that

$$
\begin{aligned}
m_{2n}\binom{2n}{s}\binom{2n+k}{k}
&= \sum_{2a+2b+i+j=2n} \binom{2n}{2a,2b,i,j} m_{2a}\, m_{2b}\, m_{2c}\, 2^{2b-s+i}\binom{b}{s-i-b}\overline{a}_{k,(i+j)/2} \\
&= \sum_{a+b+c=n} \binom{2n}{2a,2b,2c} m_{2a}\, m_{2b}\, m_{2c}\, \overline{a}_{k,c} \sum_{i+j=2c}\binom{2c}{i,j}\binom{b}{2b-s+i} 2^{2b-s+i}. \qquad (9.6)
\end{aligned}
$$

There are two interesting special cases:

– Case $k = 0$:

$$
m_{2n}\binom{2n}{s} = \sum_{a+b=n}\binom{2n}{2a,2b} m_{2a}\, m_{2b}\, 2^{2b-s}\binom{b}{s-b}
$$

We know this already! This is the numerical consequence from the combinatorial proof given in Section 7.

– Case $s = 0$:

$$
m_{2n}\binom{2n+k}{k} = \sum_{a+c=n}\binom{2n}{2a,2c} m_{2a}\, m_{2c}\, \overline{a}_{k,c} \qquad (9.7)
$$

In this case the relevance of the balls-into-boxes argument becomes obvious:
  – Consider a set N of cardinality $2n$ and a set K of cardinality k, disjoint from N.
  – The left-hand side of (9.7) counts the pairs (μ, S), where $\mu \in \mathfrak{M}[N]$ and $S \subset N \uplus K$, with $\sharp S = k$.
  – The right-hand side of (9.7) counts pairs $(\mu_A, \mu_C) \in \mathfrak{M}[A] \times \mathfrak{M}[C]$ for bipartitions $N = A \uplus C$, together with a balls-into-boxes configuration on $K \times \mathit{trans}(\mu_C)$, the elements of K being the singleton boxes and the transpositions of μ_C being the "twin boxes".
    For $\mathfrak{M}[A] \times \mathfrak{M}[C]$ to be nonempty, the cardinalities of A and C must be even, so $\sharp A = 2a$ and $\sharp C = 2c$ with $a + c = n$.
  – The balls-into-boxes configuration configuration just mentioned contains ℓ balls, say, where $c \leq \ell \leq 2c$. The ℓ balls placed on $\mathit{trans}(\mu_B)$ together with the $k - \ell$ empty singleton boxes in K make a k-subset S of $N \uplus K$, μ_A and μ_B together give an element $\mu \in \mathfrak{M}[N]$.
  – The argument just outlined is perfectly reversible.

This situation is illustrated in Figure 18 in a situation where $n = 8$, $k = 10$, $a = c = 4$, $\ell = 6$.

The general case of Eq. (9.6) can be illustrated by amalgamating these combinatorial views for the special cases.

10. More extensions: balls-into-boxes with parameter α, and a triple lacunary Laguerre series

In this section two extension of the results given in Section 9 will be presented, but the details will only be sketched.

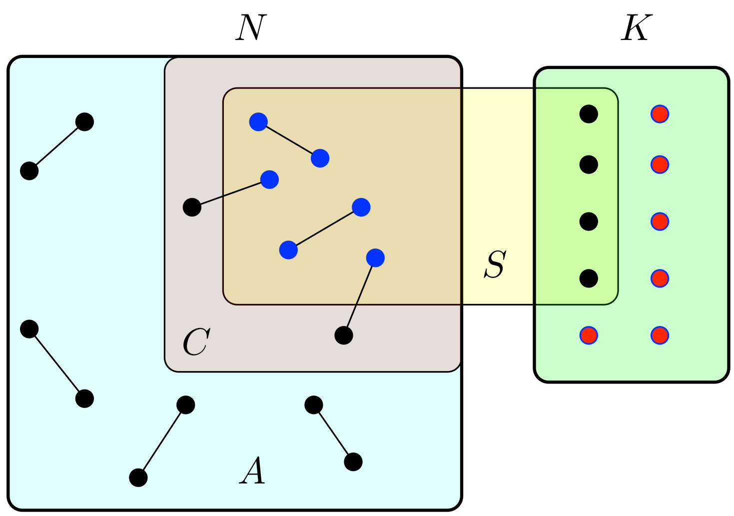

FIGURE 18. Illustration of the second special case, where balls-into-boxes appear (the balls are the red and blue vertices).

10.1. **The α-extension.** Identity (9.7) can be generalized into

$$m_{2n} \cdot (1+\alpha+2n)_k = \sum_{2a+2c=2n} \binom{2n}{2a,\, 2c} m_{2a}\, m_{2c}\, \overline{aa}^{(\alpha)}_{k,c},$$

where now the coefficients $\overline{aa}^{(\alpha)}_{k,c}$ are defined by

$$\overline{aa}^{(\alpha)}_{k,\ell} = 2\ell \cdot \overline{aa}^{(\alpha)}_{k-1,\ell-1} + (\alpha+k+2\ell) \cdot \overline{aa}^{(\alpha)}_{k-1,\ell},\ \ \overline{aa}^{(\alpha)}_{0,0} = 1.$$

This leads to

$$\begin{aligned} m_{2n} \binom{2n}{s} (1+\alpha+2n)_k \\ &= \sum_{a+b+c=n} \binom{2n}{2a,\, 2b,\, 2c} m_{2a}\, m_{2b}\, m_{2c}\, \overline{aa}^{(\alpha)}_{k,c} \sum_{i+j=2c} \binom{2c}{i,\, j} \binom{b}{2b-s+i} 2^{2b-s+i}, \end{aligned}$$

which is the same as

$$\begin{aligned} m_{2n}\, \mathcal{L}^{(\alpha+k)}_{2n}(1,x) &= (1+\alpha)_{2n} \sum_{0\le s\le 2n} \frac{x^s}{(1+\alpha)_{k+s}} \sum_{a+b+c=n} \binom{2n}{2a,\, 2b,\, 2c} m_{2a}\, m_{2b}\, m_{2c}\, \overline{aa}^{(\alpha)}_{k,c} \\ &\qquad \times \sum_{i+j=2c} \binom{2c}{i,\, j} \binom{b}{2b-s+i} 2^{2b-s+i} \\ &= (1+\alpha)_{2n} \sum_{a+b+c=n} \binom{2n}{2a,\, 2b,\, 2c} m_{2a}\, m_{2c}\, \overline{aa}^{(\alpha)}_{k,c} \\ &\qquad \times \sum_{0\le s\le 2n} \frac{x^s}{(1+\alpha)_{k+s}} \sum_{i+j=2c} \binom{2c}{i} \frac{(2b)!}{(s-i)!} \, [t^{2b}]\, \mathcal{H}_{s-i}(t^2,t). \end{aligned}$$

With
$$p_k^{(\alpha)}(r,x,t) = \sum_{0\le i\le 2k}\sum_{0\le j\le k} x^i t^j \binom{2j}{i} \frac{1}{(\alpha)_{r+k+i}}\, a_{k,j}^{(\alpha)},$$
where $a_{k,j}^{(\alpha)} = \overline{aa}_{k,j}^{(\alpha)}/j!$, one gets
$$m_{2n}\,\mathcal{L}_{2n}^{(\alpha+k)}(1,x) = (2n)!\,(\alpha)_{2n}\,[t^{2n}]\,e^{t^2/2}\sum_{r\ge 0}\frac{x^r}{r!}\,p_k^{(\alpha)}(r,x,t^2/2)\,\mathcal{H}_r(t^2,t),$$
and hence
$$\sum_{n\ge 0}\frac{t^{2n}}{2^n\,n!\,(\alpha)_{2n}}\mathcal{L}_{2n}^{(\alpha+k)}(1,x) = e^{t^2/2}\sum_{r\ge 0}\frac{x^r}{r!}\,p_k^{(\alpha)}(r,x,t^2/2)\,\mathcal{H}_r(t^2,t),$$
which in conventional notation reads
$$\sum_{n\ge 0}\frac{t^{2n}}{n!}\,\frac{(2n)!}{(1+\alpha)_{2n}}L_{2n}^{(\alpha+k)}(x) = e^t\sum_{r\ge 0}\frac{1}{r!}\,p_k^{(\alpha)}(r,-x,t)\,(i\,x\,\sqrt{t})^r\,H_r(i\,\sqrt{t}),$$
where the polynomials $p_k^{(\alpha)}(r,x,t)$ have a nice combinatorial meaning.

10.2. **A triple lacunary Laguerre series.** In analogy to Eq. (9.2) one may consider the case of triple lacunary Laguerre series of the form
$$\sum_{n\ge 0}\frac{t^n}{n!}L_{3n}^{(k)}(x) = e^t\cdot\sum_{r\ge 0}\frac{x^r\,q_{3k}(r;x,t)}{(r+4k!)}\sum_{0\le s\le\lfloor r/3\rfloor}\frac{(-t)^s\,(i\,\sqrt{3t})^{r-3s}}{(r-3s)!\,s!}H_{r-3s}(i\,\sqrt{3t}/2), \quad (10.1)$$
with polynomial factors $q_{3k}(r;-x,t)$. The case $k=1$ has been worked out in [2], see identity (4.4) and the following line, which reads
$$\begin{aligned} q_3(r;x,t) = (1+3\,t)\,r^3 + (9+27\,t-9t\,x)\,r^2 + (26+78\,t-63\,t\,x+9\,t\,x^2)\,r \\ + (24+72\,t-108t\,x+36\,t\,x^2-3\,t\,x^3), \end{aligned}$$
but without any hint what these integer coefficients might count. Indeed, we have the general form
$$q_{3k}(r;x,t) = \sum_{0\le i\le 3k}\sum_{0\le j\le k}(-x)^i\,t^j\binom{3j}{i}\frac{(r+4k)!}{(r+k+i)!}\,b_{k,j},$$
where the generating polynomials
$$b_k(t) = \sum_{0\le\ell\le k} b_{k,\ell}\,t^\ell$$
are defined via the linear recurrence
$$b_{k,\ell} = 3\,b_{k-1,\ell-1} + (k+3\ell)\,b_{k-1,\ell},$$
with exponential generating function
$$\sum_{k\ge 0}\sum_{0\le\ell\le k} b_{k,\ell}\,t^\ell\,\frac{z^k}{k!} = \frac{e^{t/(1-z)^2}}{(1-z)\,e^t}.$$

The first values for the $b_{k,\ell}$ are

$k\backslash\ell$	0	1	2	3	4	5
0	1					
1	1	3				
2	2	18	9			
3	6	114	135	27		
4	24	816	1692	756	81	
5	120	6600	21060	15660	3645	243

Again, the numbers $\overline{b}_{k,\ell} = b_{k,\ell} \cdot \ell!/k!$ are integers, which satisfy the recurrence

$$k\,\overline{b}_{k,\ell} = 3\,\ell\,\overline{b}_{k-1,\ell-1} + (k+3\ell)\,\overline{b}_{k-1,\ell}.$$

The first values (not yet in the OEIS) are

$k\backslash\ell$	0	1	2	3	4	4
0	1	0	0	0	0	0
1	1	3	0	0	0	0
2	1	9	9	0	0	0
3	1	19	45	27	0	0
4	1	34	141	189	81	0
5	1	55	351	783	729	243

The straightforward combinatorial interpretation for $\overline{b}_{k,\ell}$ is a balls-into-boxes situation as before, but now with k singleton boxes and ℓ triple boxes and a corresponding occupancy rule.

With this information at hand it is not difficult to obtain a combinatorial proof of Eq. (10.1) along the lines presented in Section 9.

11. A lacunary Charlier series

In this section we consider the lacunary Laguerre series identity

$$\sum_{n\geq 0} t^n L_{2n}^{(\alpha-2n)}(x,y) = (1-t\,x^2)^{\alpha/2}\cosh\left(\sqrt{t}\,y + i\,\alpha\arcsin\left(\frac{\sqrt{t}\,x}{\sqrt{t\,x^2-1}}\right)\right), \tag{11.1}$$

which is Eq. (3.16) in [2]. It is convenient to replace t by t^2 and α by $-\alpha$, so that in terms of the combinatorial version of the Laguerre polynomials with variables x, y the identity reads

$$\sum_{n\geq 0}\frac{t^{2n}}{(2n)!}\mathcal{L}_{2n}^{(-\alpha-2n)}(x,y) = (1-(t\,x)^2)^{-\alpha/2}\cosh\left(t\,y - i\,\alpha\arcsin\left(\frac{t\,x}{\sqrt{(t\,x)^2-1}}\right)\right). \tag{11.2}$$

As usual, the variable t can be dropped, so that after setting $t=-1$ we get

$$\sum_{n\geq 0}\frac{1}{(2n)!}\mathcal{L}_{2n}^{(-\alpha-2n)}(-x,y) = (1-x^2)^{-\alpha/2}\cosh\left(y + i\,\alpha\arcsin\left(\frac{x}{\sqrt{x^2-1}}\right)\right). \tag{11.3}$$

It should be stressed that the polynomials

$$\mathcal{L}_{2n}^{(-\alpha-2n)}(x,y)$$

are not really Laguerre polynomials (because their cycle-counting parameter $-\alpha-2n$ depends on n). Instead, they are essentially homogeneous versions of the Charlier polynomials, which can be seen from

$$\begin{aligned}\mathcal{L}_n^{(-\alpha-n)}(-x,y) &= \sum_{0\le k\le n}\binom{n}{k}(1-\alpha-n+n-k)_k\,(-x)^k\,y^{n-k}\\ &= \sum_{0\le k\le n}\binom{n}{k}(\alpha)_k\,x^k\,y^{n-k}\\ &= y^n\cdot{}_1\mathsf{F}_0\left[\begin{matrix}-n,\alpha\\ -\end{matrix};\frac{x}{y}\right],\end{aligned}$$

so that

$$\mathcal{L}_{2n}^{(-\alpha-2n)}(-x,y) = \sum_{0\le k\le 2n}\binom{2n}{k}(\alpha)_k\,x^k\,y^{2n-k}.$$

Obviously, these polynomials are the enumerating polynomials for those Laguerre configurations on $X\uplus Y=\{1,2,\ldots,2n\}$ which consist, apart from the singletons in Y, of permutations of X only, and with a weight α put on each cycle. Those structures will be called *Charlier configurations* for short.

As for the factor $(1-x^2)^{-\alpha/2}$ in Eq. (11.3), this is nothing but the exponential generating functions for permutations consisting of cycles of *even* length only, with weight α put on each cycle. Therefore it suffices to show that

$$\cosh\left(y+i\,\alpha\arcsin(\frac{x}{\sqrt{x^2-1}}\right) \tag{11.4}$$

is the exponential generating function for Charlier configurations of *even* order, where all cycles are of *odd* length.

For this purpose, consider the familiar series

$$\arcsin(\frac{x}{\sqrt{x^2-1}}) = -i\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right).$$

This shows that

$$i\,\alpha\arcsin(\frac{x}{\sqrt{x^2-1}}) = \alpha\left(x+2!\frac{x^3}{3!}+4!\frac{x^5}{5!}+\ldots\right)$$

is the exponential generating function for cyclic permutations of *odd* length only. By general principles,

$$\exp\left(y+i\,\alpha\arcsin(x/\sqrt{x^2-1})\right)$$

is the exponential generating function for structures which are composed of an arbitrary number of

– singletons with weight y,
– cycles of odd length $2\ell+1$ ($\ell\ge 1$) with weight $\alpha\,x^{2\ell+1}$ for each cycle.

Now we restrict attention to only those compound structures of *even* order, i.e., for which the total weight (product over all component weights) is even as a monomial in

x and y, which is achieved if and only if the number of components is even. But this means that instead of substituting into the exponential series

$$e^z = \sum_{n\ge 0} \frac{z^n}{n!},$$

we have to substitute into the cosh-series

$$\cosh(z) = \sum_{n\ge 0} \frac{z^{2n}}{(2n)!}.$$

Thus Eq. (11.4) is indeed the exponential generating function of Charlier configurations of *even* order composed of cycles of *odd* length (variable x) and singletons (variable y, cycle count α), as required.

12. Binomial identities, umbral substitution, and a combinatorial puzzle

In Section 4 of [2], Eqs. (4.6) ff., the authors derive an interesting lacunary Laguerre series in terms of Bessel functions by using their proper methods. We have already met this generating function here as Eq. (6.7). I have pointed out to the authors of [2] that this result can be deduced from a mixed bilateral generating function for products of Laguerre and Gegenbauer polynomials. My hint is reproduced in Appendix A of [2]. In this last section, I want to take a closer look on this result, with combinatorics in mind.

For convenience, I write Eq. (6.7) in the following form:

$$\sum_{n\ge 0} \frac{(1/2)_n}{(1+\alpha)_n} L_{2n}^{(2\alpha)}(2y)\, t^n = \frac{1}{(1-t)^{\alpha+1/2}} e^{-2y\,t/(1-t)} {}_0F_1\left[\begin{matrix} - \\ 1+\alpha \end{matrix}; \frac{y^2\,t}{(1-t)^2}\right]. \tag{12.1}$$

The right-hand side is reminiscent of the well-known Hille–Hardy identity, see [32, Theorem 69, p. 212], which has received a combinatorial proof, including a combinatorially motivated multivariate extension in [15], see also [14] for an analytic proof:

$$\sum_{n\ge 0} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)\, t^n = \frac{1}{(1-t)^{\alpha+1}} e^{-(x+y)t/(1-t)} {}_0F_1\left[\begin{matrix} - \\ 1+\alpha \end{matrix}; \frac{x\,y\,t}{(1-t)^2}\right]. \tag{12.2}$$

By setting $x = y$, one gets

$$\sum_{n\ge 0} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(y) L_n^{(\alpha)}(y)\, t^n = \frac{1}{(1-t)^{\alpha+1}} e^{-2y\,t/(1-t)} {}_0F_1\left[\begin{matrix} - \\ 1+\alpha \end{matrix}; \frac{y^2\,t}{(1-t)^2}\right] \tag{12.3}$$

and hence, by comparison of (12.1) and (12.3), one has

$$\sum_{n\ge 0} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(y)^2\, t^n = \frac{1}{(1-t)^{1/2}} \sum_{n\ge 0} \frac{(1/2)_n}{(1+\alpha)_n} L_{2n}^{(2\alpha)}(2y)\, t^n,$$

or, equivalently,

$$\frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(y)^2 = \sum_{0\le k\le n} \frac{(1/2)_k}{k!} \frac{(1/2)_{n-k}}{(1+\alpha)_{n-k}} L_{2n-2k}^{(2\alpha)}(2y).$$

Written in terms of the combinatorial Laguerre polynomials, this reads

$$\begin{aligned}\mathcal{L}_n^{(\alpha)}(2x,y)^2 &= \sum_{0\le k\le n}\binom{n}{k}\frac{(2k)!}{k!}\frac{(1+\alpha)_n}{(1+\alpha)_{n-k}}\,x^{2k}\mathcal{L}_{2n-2k}^{(2\alpha)}(x,y)\\ &= \sum_{0\le k\le n}\binom{n}{k}\binom{\alpha+n}{k}(2k)!\,x^{2k}\,\mathcal{L}_{2n-2k}^{(2\alpha)}(x,y).\end{aligned} \tag{12.4}$$

As a side remark: the Hille–Hardy identity itself can be written as

$$\mathcal{L}_n^{(\alpha)}(x,y)\,\mathcal{L}_n^{(\alpha)}(x,z) = \sum_{0\le k\le n}\binom{n}{k}(1+\alpha+k)_{n-k}(y\,z)^k\mathcal{L}_{n-k}^{(\alpha+2k)}(x^2,x\,y+x\,z),$$

(Bateman project [5, Sec. 10.12, Eq. (42)]). So we have both

$$\mathcal{L}_n^{(\alpha)}(x,y)^2 = \sum_{0\le k\le n}\binom{n}{k}\binom{\alpha+n}{n-k}(n-k)!\,x^{n-k}\,y^{2k}\,\mathcal{L}_{n-k}^{(\alpha+2k)}(x,2y),$$

$$\mathcal{L}_n^{(\alpha)}(x,y)^2 = \frac{1}{4^n}\sum_{0\le k\le n}\binom{n}{k}\binom{\alpha+n}{k}(2k)!\,x^{2k}\,\mathcal{L}_{2n-2k}^{(2\alpha)}(x,2y).$$

Back to the main identity Eq. (12.4). Written as a family of binomial identities, it can be stated for $0\le m\le 2n$ as

$$2^m\sum_k\frac{\binom{n}{k}\binom{n+\alpha}{k}\binom{n}{m-k}\binom{n+\alpha}{m-k}}{\binom{m}{k}} = \sum_k\frac{\binom{n}{k}\binom{n+\alpha}{k}\binom{2n-2k}{m-2k}\binom{2n-2k+2\alpha}{m-2k}}{\binom{m}{2k}}. \tag{12.5}$$

One has to be careful with the range of summation! The cases $0\le m\le n$ and $n\le m\le 2n$ should be considered separately! For $0\le m\le n$ the summation $\sum_k$ runs on the left-hand side effectively from $k=0$ to $k=m$; for $n\le m\le 2n$ the summation $\sum_k$ runs effectively from $k=m-n$ to $k=n$.

In the case $\alpha=0$, Eqs. (12.4) respectively (12.5) simplify to

$$\mathcal{L}_n(2x,y)^2 = \sum_{0\le k\le n}\binom{n}{k}^2(2k)!\,x^{2k}\,\mathcal{L}_{2n-2k}(x,y), \tag{12.6}$$

respectively

$$2^m\sum_{0\le k\le n}\frac{\binom{n}{k}^2\binom{n}{m-k}^2}{\binom{m}{k}} = \sum_{0\le k\le n}\frac{\binom{n}{k}^2\binom{2n-2k}{m-2k}^2}{\binom{m}{2k}}\ (0\le m\le 2n). \tag{12.7}$$

Two particular cases are worth mentioning:

– For $m=2n$, Eq. (12.7) reduces to the well-known

$$4^n = \sum_{0\le k\le n}\binom{2k}{k}\binom{2n-2k}{n-k},$$

for which several combinatorial proofs (e.g., counting lattice paths) are known.

– For $m = n$ one gets from (12.7):

$$\begin{aligned}
2^n \sum_{0\le k\le n} \binom{n}{k}^3 &= \sum_{0\le k\le n} \binom{n}{k}^2 \binom{2n-2k}{n-2k}^2 (2k)!(n-2k)! \\
&= \sum_{0\le k\le n} \binom{2k}{k}\binom{2n-2k}{n-k}\frac{(2n-2k)!}{(n-2k)!} \\
&= \sum_{0\le k\le n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{2k}{n}. \qquad (12.8)
\end{aligned}$$

This is an interesting identity, because it features the *Franel numbers* $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$, which have an intimate relation with the more famous *Apéry numbers* a_n, viz.,

$$a_n = \sum_{0\le k\le n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{0\le k\le n} \binom{n}{k}\binom{n+k}{k} f_n,$$

see [37] for comprehensive information about this intriguing identity, which surprisingly can be seen as a descendant of Bailey's bilinear generating function for the Jacobi polynomials, for which I have given a combinatorial proof a long time ago in my habilitation thesis [36]. Incidentally, there is another formula which looks deceptively similar to (12.8) — watch the last binomial coefficient of the right of both formulas! —, but which is not the same:

$$\sum_{0\le k\le n} \binom{n}{k}^3 = \sum_{0\le k\le n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{2k}. \qquad (12.9)$$

The equivalence of the two summations of Eq. (12.8) and Eq. (12.9) is not obvious! It amounts to show the hypergeometric identity

$$\frac{1}{2^n}\binom{2n}{n} {}_3\mathsf{F}_2\left[\begin{matrix}1/2, 1/2-n/2, -n/2 \\ 1/2-n, 1/2-n\end{matrix}; 1\right] = {}_3\mathsf{F}_2\left[\begin{matrix}-n, 1/2-n/2, -n/2 \\ 1, 1/2-n\end{matrix}; 1\right],$$

which does not seem to fall under one of the standard hypergeometric transformations — but see a remark on this below.

Identity (12.5), as well as the particular cases mentioned, can be verified algorithmically (independently from the derivation given here, which after all is based on two nontrivial results about generating functions) by

a) using Doron Zeilberger's method for determining recurrence operators for hypergeometric expressions, see e.g. [31], which I have executed myself, but I will not present here details about the operators and the proof certificates;
b) using Christian Krattenthaler's program HYP for verifying and/or discovering hypergeometric transformation identities — which Christian has done successfully in [24].

After this hypergeometric digression, we turn to the identity of Eq. (12.4) again. In the sequel it will be assumed for convenience that $\alpha = 0$, thus writing now $\mathcal{L}(x, y)$ in place of $\mathcal{L}^{(0)}(x,y)$. We then have

$$\mathcal{L}_n(2x,y)^2 = \sum_{0\le k\le n} \binom{n}{k}^2 (2k)!\, x^{2k}\, \mathcal{L}_{2n-2k}(x,y). \tag{12.10}$$

From the umbral property (4.2) we can write

$$\begin{aligned}\mathcal{L}_n(2x,y) &= \mathcal{L}_n(x,\mathcal{L}(x,y)) \\ &= \sum_{0\le k\le n} \binom{n}{k} (1+n-k)_k x^k \mathcal{L}_{n-k}(x,y) \\ &= \sum_{0\le k\le n} \binom{n}{k}^2 k!\, x^k \mathcal{L}_{n-k}(x,y) \\ &= \sum_{0\le k\le n} \binom{n}{k}^2 \mathcal{L}_k(x,0)\mathcal{L}_{n-k}(x,y).\end{aligned}$$

If we now introduce

$$\mathcal{P}_n(x,y) = \sum_{0\le k\le n} \binom{n}{k}^2 x^k y^{n-k}$$

as the *homogeneous Legendre polynomials*, then, using the umbral substitution

$$\Lambda : \begin{cases} x^k \to \mathcal{L}_k(x,0) & (k\ge 0), \\ y^\ell \to \mathcal{L}_\ell(x,y) & (\ell \ge 0), \end{cases}$$

and writing it in compositional form, we turn the preceding identity into

$$\mathcal{L}_n(2x,y) = \Lambda \circ \mathcal{P}(x,y).$$

On the other hand, the right-hand side of (12.10) is just

$$\begin{aligned}\sum_{0\le k\le n} \binom{n}{k}^2 (2k)!\, x^{2k}\, \mathcal{L}_{2n-2k}(x,y) &= \sum_{0\le k\le n} \binom{n}{k}^2 \mathcal{L}_{2k}(x,0)\, \mathcal{L}_{2n-2k}(x,y) \\ &= \Lambda \circ \mathcal{P}(x^2,y^2),\end{aligned}$$

so we end up with a very pleasing version of (12.4):

$$[\,\Lambda \circ \mathcal{P}(x,y)\,]^2 = \Lambda \circ \mathcal{P}(x^2,y^2).$$

One can even show that the combinatorial Legendre polynomials are the only (nontrivial) homogeneous polynomials which satisfy this umbral substitution property w.r.t. Λ.

Back to combinatorics. Write (12.4) as

$$\mathcal{L}_n(2x,1)^2 = \sum_{0\le k\le n} \binom{n}{k}^2 (2k)!\, x^{2k}\, \mathcal{L}_{2n-2k}(x,1). \tag{12.11}$$

Then, in view of the interpretation of the Laguerre polynomials $\mathcal{L}_n^{(0)}(x,y)$ as matching polynomials of the complete bipartite graph $K_{n,n}$, both sides of Eq. (12.11) count matchings (not necessarily perfect ones), the factor 2 on the left giving rise to 2-sorted matchings, as mentioned in Section 4.2.

– On the left: pairs of 2-sorted (red and blue edges) matchings of $K_{n,n}$, as shown in Figure 19 for $n = 5$, with two independent $K_{5,5}$'s (in green and magenta);

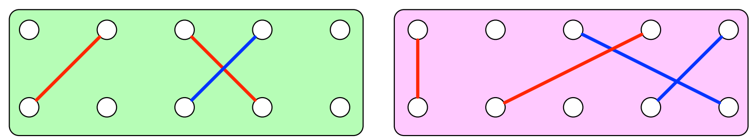

FIGURE 19. A configuration for $n = 5$, counted by $\mathcal{L}_5(2x + 1)^2$

– On the right: matchings of $K_{2n,2n}$ having a perfect kernel $K_{2k,2k}$ $K_{2k,2k}$ and an ordinary matching for the complementary $K_{2n-2k,2n-2k}$ (in green), as shown in Figure 20.

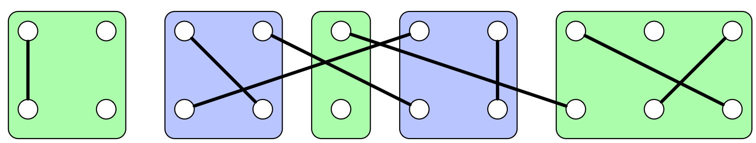

FIGURE 20. A configuration for $n = 5, k = 4$, counted by $\binom{5}{2}^2 4!\, x^4\, \mathcal{L}_6(x, 1)$

One would think that such an explicit description of two families of matchings should make a bijective proof feasible, but I have not been able to achieve this — so I leave it as a puzzle.

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