Triangulations of balanced subdivisions of convex polygons

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THE PROBLEM

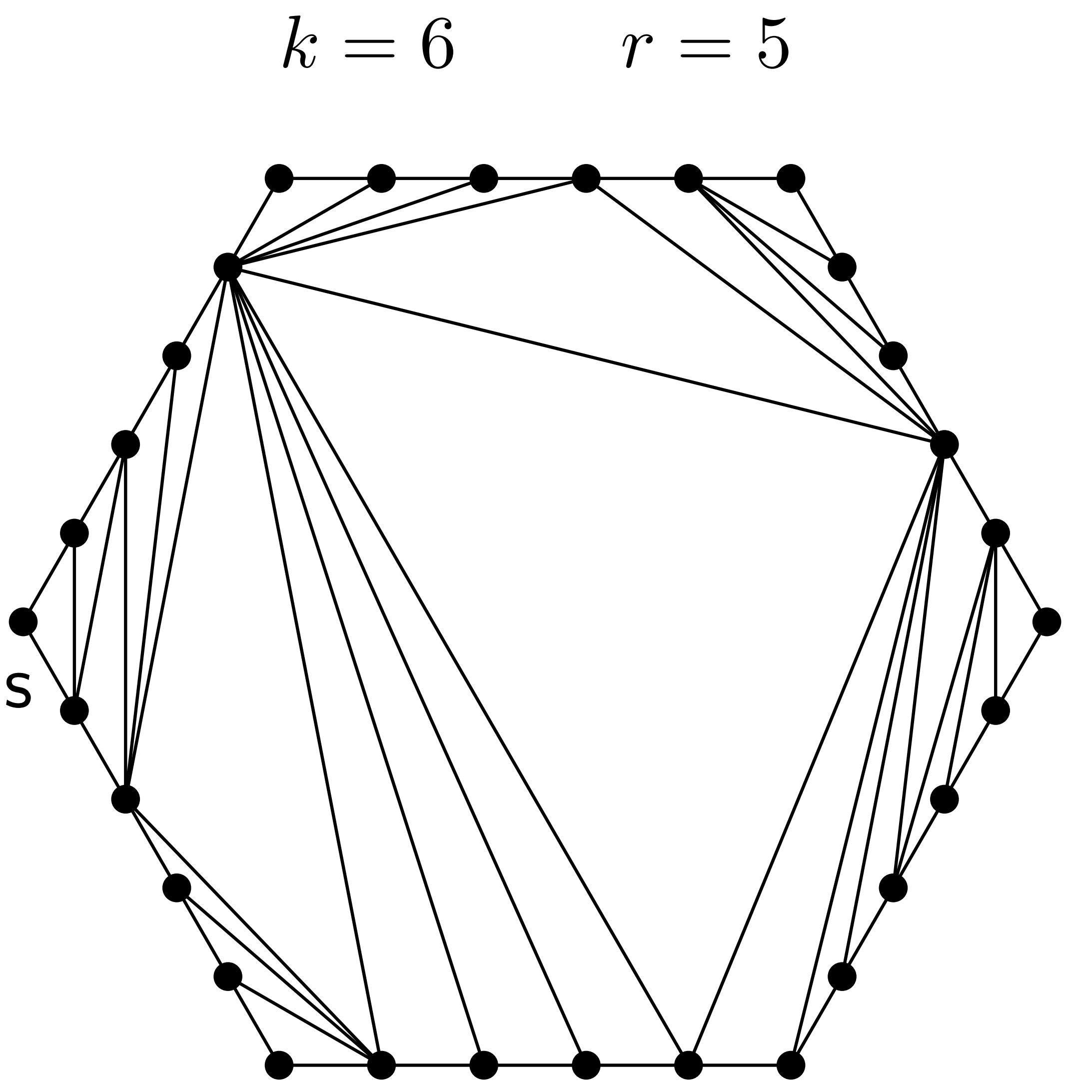
A convex k-gon

Each side is subdivided by $r-1$ points

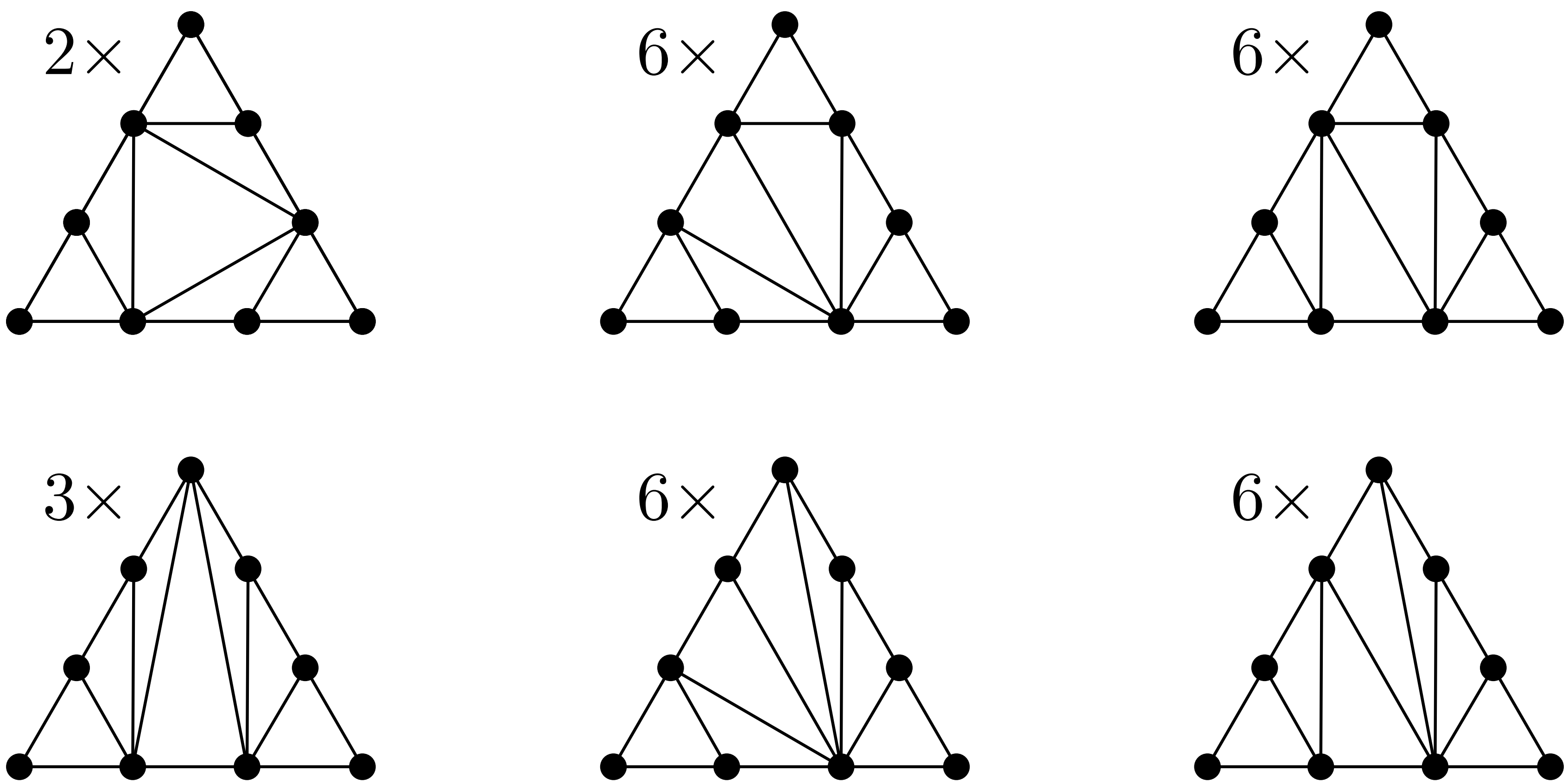
$n = kr$

How many triangulations has this configuration?

$\mathrm{tr}(k, r) = ?$



$\mathsf{tr}(3,3) = 29$



	$r=1$	2	3	4	5	6
$k=3$	1	4	29	229	1847	14974
4	2	30	604	12168	238848	4569624
5	5	250	13740	699310	33138675	1484701075
6	14	2236	332842	42660740	4872907670	510909185422
7	42	20979	8419334	2711857491	745727424435	182814912101920

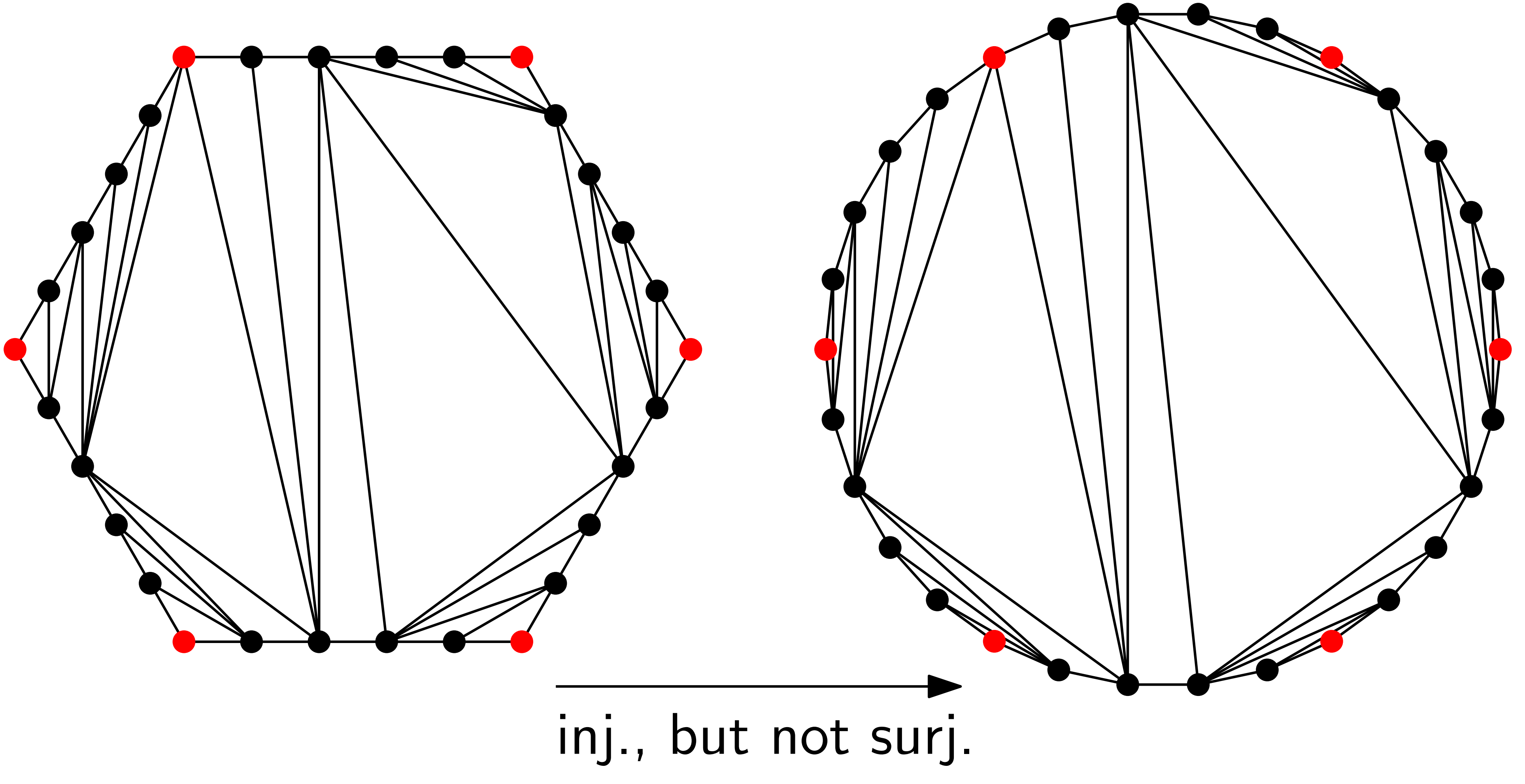
$$\mathsf{tr}(k,1) = C_{k-2}$$

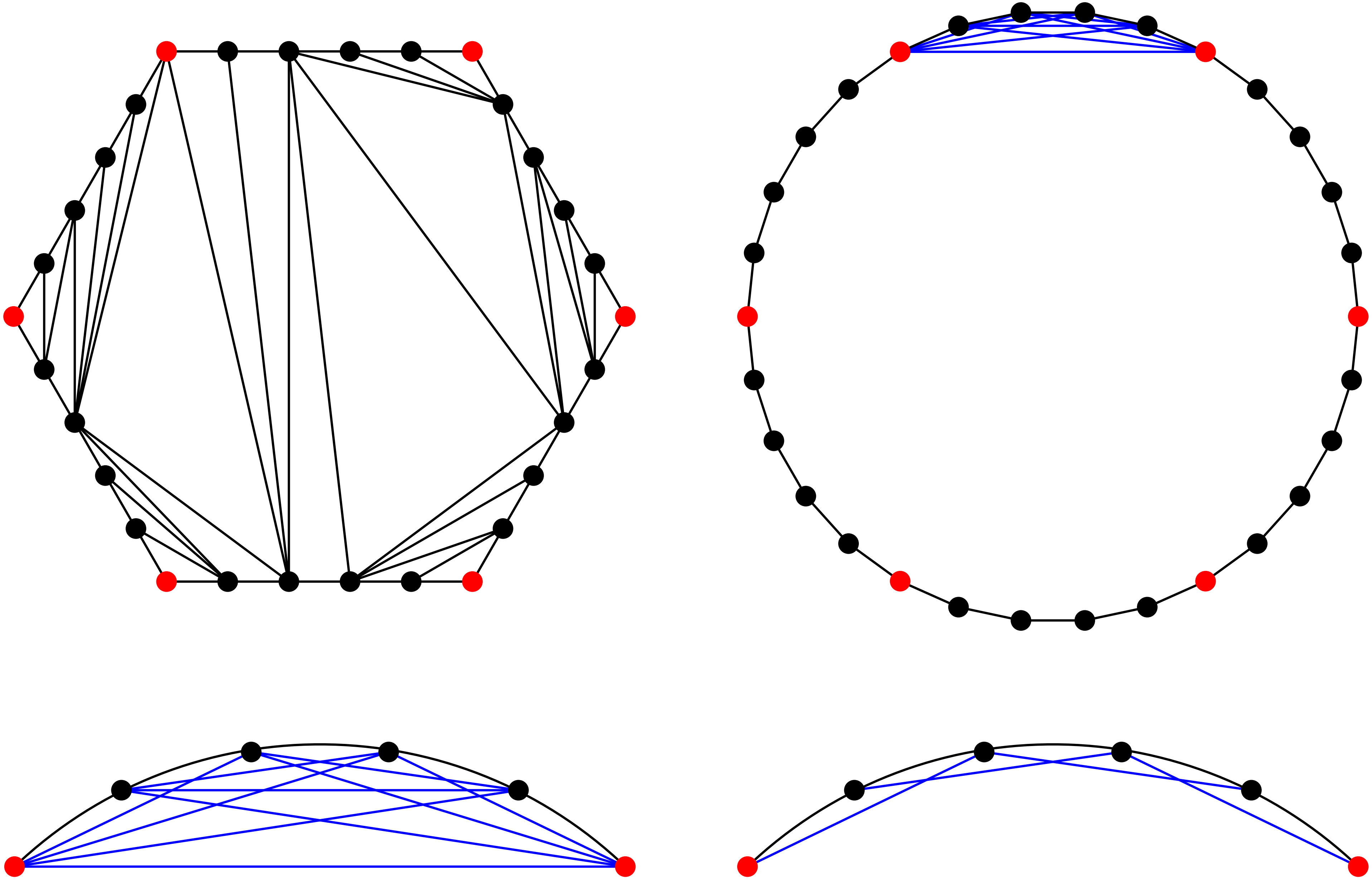
Some results on this topic:

F. Hurtado and M. Noy (1997).
Counting triangulations of almost-convex polygons.

R. Bacher and F. Mouton (2003-2010).
Triangulations of nearly convex polygons.

FORMULAS


inj., but not surj.



forbidden essentially forbidden

$$\mathsf{tr}(k,r) = \sum_{m=0}^{k\lfloor r\rfloor} \left([x^m] \left(\sum_{\ell\geq 0} \binom{r-\ell}{\ell} (-x)^\ell \right)^k \cdot C_{kr-m-2} \right)$$

$$= \sum_{m=0}^{k\lfloor r\rfloor} \left([x^m] \left(x^{r/2}\, U_r\left(\frac{1}{2\sqrt{x}}\right)\right)^k \cdot C_{kr-m-2} \right)$$

$U_r(x)$ is the Chebyshev polynomial of the second kind

$$\mathsf{tr}(k,r) = \sum_{m=0}^{k\lfloor r\rfloor} \left([x^m] \left(\sum_{\ell\geq 0} \binom{r-\ell}{\ell} (-x)^\ell \right)^k \cdot C_{kr-m-2} \right)$$

$$= \sum_{m=0}^{k\lfloor r\rfloor} \left([x^m] \left(x^{r/2}\, U_r \left(\frac{1}{2\sqrt{x}} \right) \right)^k \cdot C_{kr-m-2} \right)$$

$$= [x^{rk-2}] \left(\left(x^{r/2} U_r \left(\frac{1}{2\sqrt{x}} \right) \right)^k C(x) \right)$$

$$= [x^{rk-2}] \left(\frac{\left(\left(1+\sqrt{1-4x}\right)^{r+1} - \left(1-\sqrt{1-4x}\right)^{r+1} \right)^k}{2^{(r+1)k}(1-4x)^{k/2}} \cdot \frac{1-\sqrt{1-4x}}{2x} \right)$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\left(\left(1+\sqrt{1-4x}\right)^{r+1} - \left(1-\sqrt{1-4x}\right)^{r+1} \right)^{k}}{2^{(r+1)k+1} x^{rk} (1-4x)^{k/2}} \ \left(1-\sqrt{1-4x}\right) \ dx$$

$$\boxed{x = t(1-t)}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{((1-t)^{r+1} - t^{r+1})^{k}}{t^{rk-1}(1-t)^{rk}(1-2t)^{k-1}} \, dt$$

$$= -\frac{1}{4\pi i} \int_{\mathcal{C}} \frac{((1-t)^{r+1} - t^{r+1})^{k}}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}} \, dt$$

$$= [t^{rk-2}] \frac{((1-t)^{r+1} - t^{r+1})^{k}}{(1-t)^{rk}(1-2t)^{k-1}}$$

$$= \sum_{j=0}^{k} \sum_{\ell=0}^{rk-(r+1)j-1} (-1)^{j+1} \, 2^{\ell-1} \binom{k}{j} \binom{k-3+\ell}{\ell} \binom{(r-1)k-\ell-2}{rk-(r+1)j-\ell-1}.$$

GENERATING FUNCTIONS

"Vertical" generating functions (r is fixed):

$$\mathsf{tr}(k,r) = -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{((1-t)^{r+1}-t^{r+1})^k}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}}\ dt$$

$$\sum_{k\geq 1}\mathsf{tr}(k,r)x^k = -\frac{1}{4\pi i}\int_{\mathcal{C}}\sum_{k\geq 1}\frac{((1-t)^{r+1}-t^{r+1})^k x^k(1-2t)^2}{t^{rk}(1-t)^{rk}(1-2t)^k}\ dt$$

$$= -\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{(1-2t)^2\ dt}{1-x((1-t)^{r+1}-t^{r+1})t^{-r}(1-t)^{-r}(1-2t)^{-1}}$$

$$= -\frac{1}{4\pi i}\int_{\mathcal{C}}\frac{t^r(1-t)^r(1-2t)^2\ dt}{t^r(1-t)^r-x((1-t)^{r+1}-t^{r+1})(1-2t)^{-1}}$$

$$\sum_{k\geq 1} \mathsf{tr}(k,r)x^k = -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{t^r(1-t)^r(1-2t)^2\ dt}{t^r(1-t)^r - x((1-t)^{r+1}-t^{r+1})(1-2t)^{-1}}$$

$$= -\frac{1}{2}\sum_{i=1}^{r} \mathrm{Res}_{t=t_i(x)} \frac{t^r(1-t)^r(1-2t)^2}{P(x,t)}$$

$$= -\frac{1}{2}\sum_{i=1}^{r} \frac{t_i(x)^r(1-t_i(x))^r(1-2t_i(x))^2}{(\frac{d}{dt}P)(x,t_i(x))}$$

$P(x,t)$ is the denominator of the integrand.

$t_i(x)$ $(i=1,\ldots,r)$ are the “small roots” of $P(x,t)$.

That is, $\lim_{x\to 0} t_i(x) = 0$.

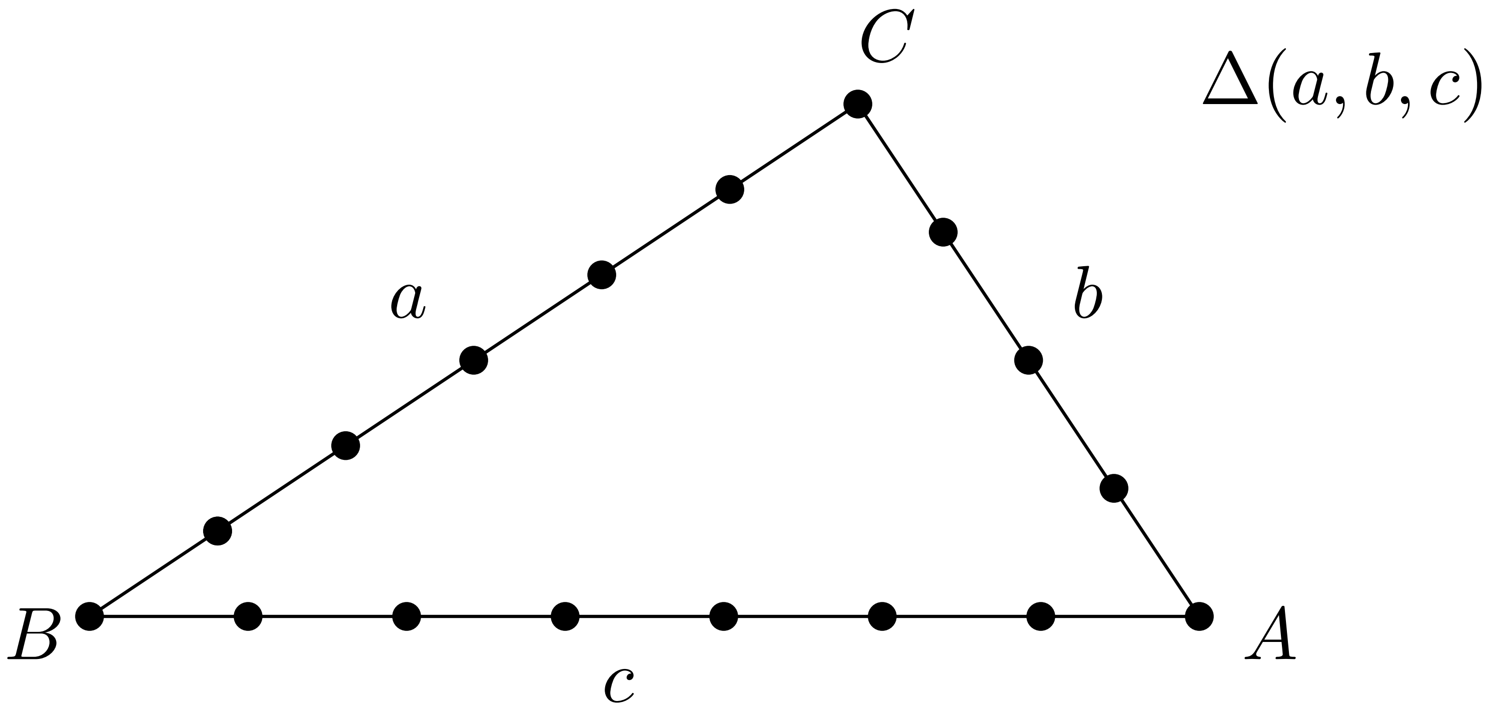
It follows: for fixed r, the “vertical” generating function $\sum_{k\geq 0} \mathsf{tr}(k,r)$ is algebraic.

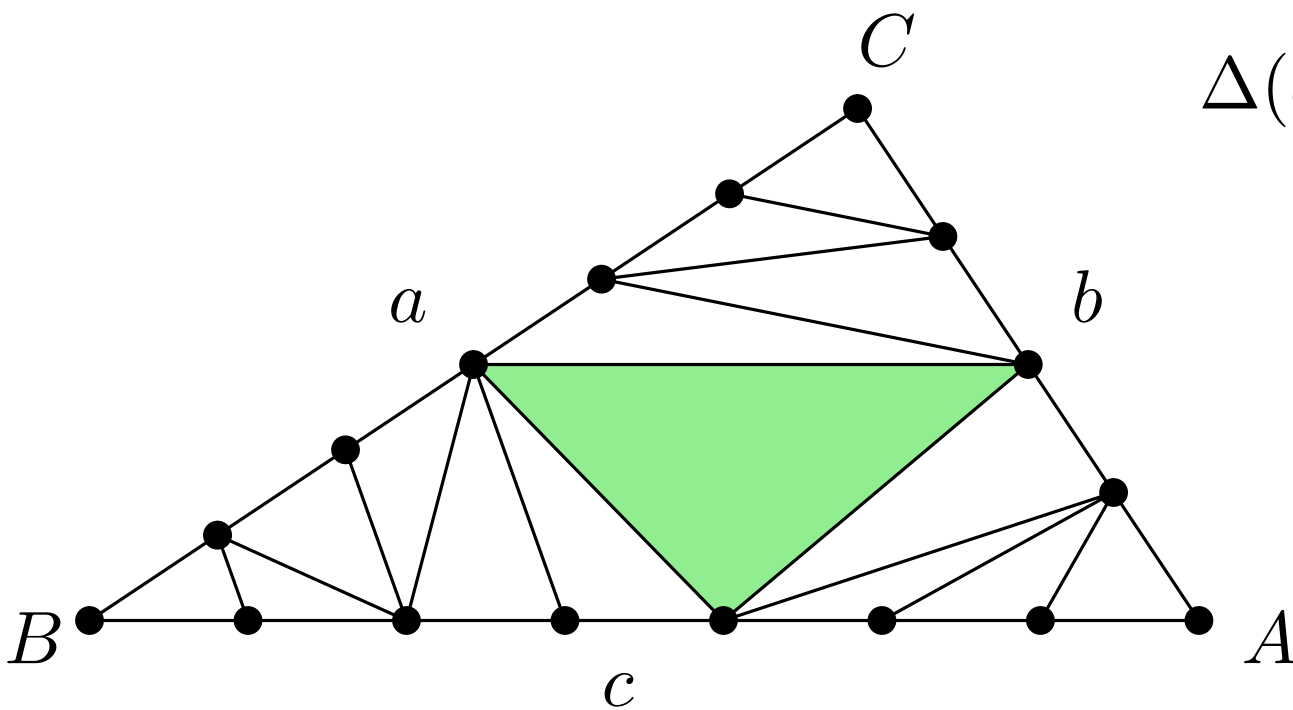
One can prove similarly that for fixed k, the “horizontal” generating function $\sum_{r\geq 0} \mathsf{tr}(k,r)$ is algebraic.

Example: For $r = 2$ we have

$$\sum_{k\geq 1} \mathsf{tr}(k,2)x^k = \frac{1}{8}\sqrt{\frac{x}{x+4}} \times$$
$$\times \left(\sqrt{1+2x+2\sqrt{x(x+4)}}\left(\sqrt{x}+\sqrt{x+4}\right)^2 - \right.$$
$$\left. -\sqrt{1+2x-2\sqrt{x(x+4)}}\left(\sqrt{x}-\sqrt{x+4}\right)^2 \right)$$

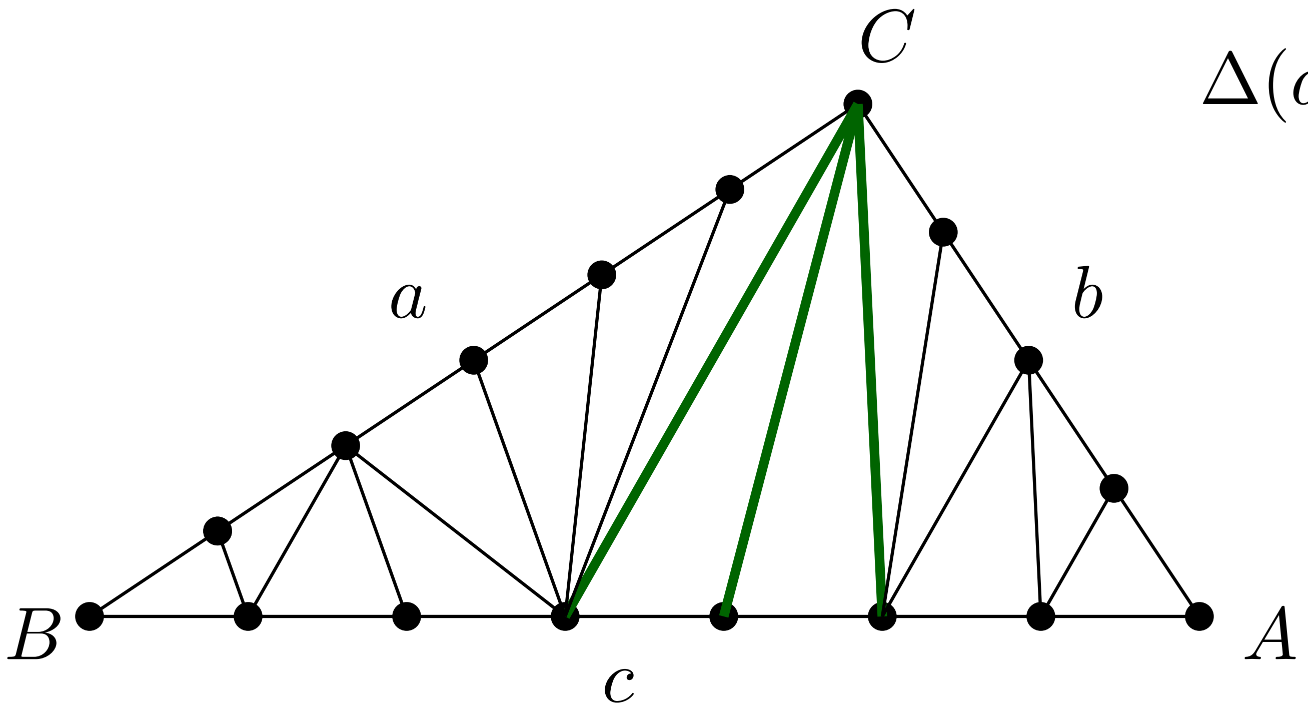
THE CASE $k = 3$ (NON-BALANCED)


C
a
b
B
A
c
$\Delta(a, b, c)$



$\Delta(a, b, c)$

T-trianguations

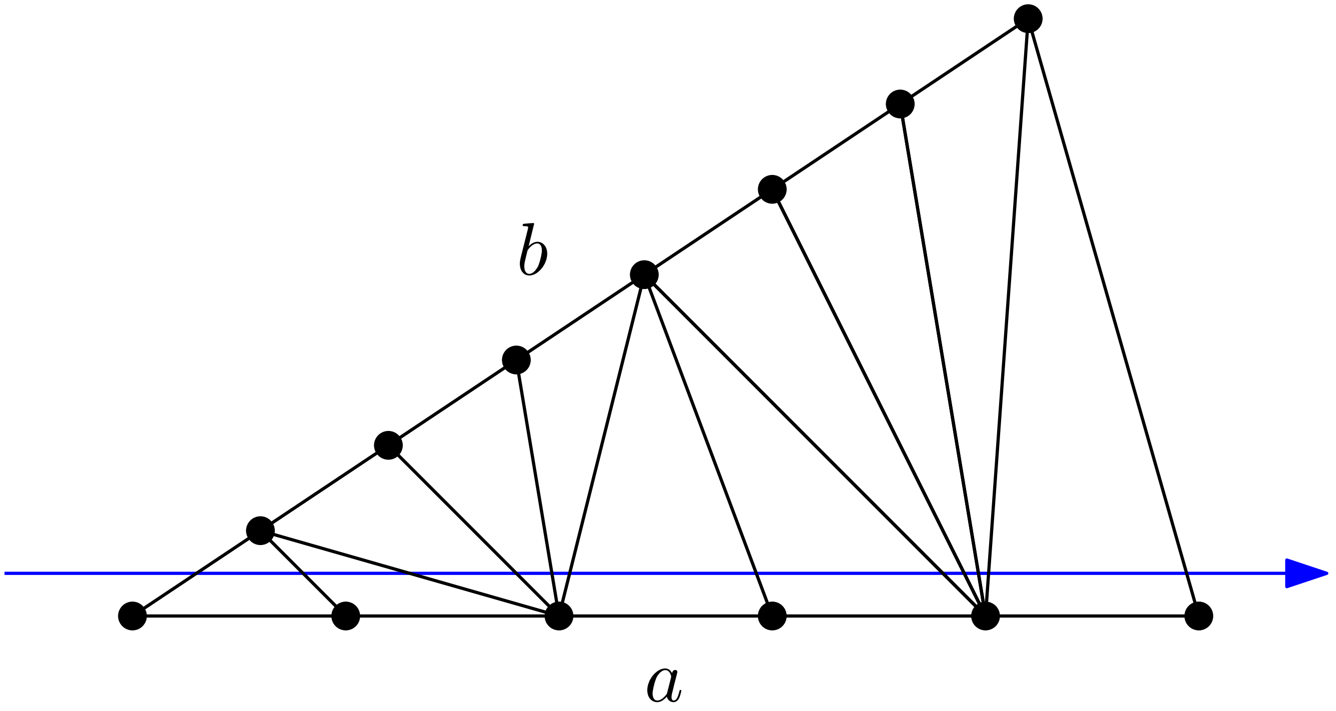


$\Delta(a, b, c)$

D-triangulations

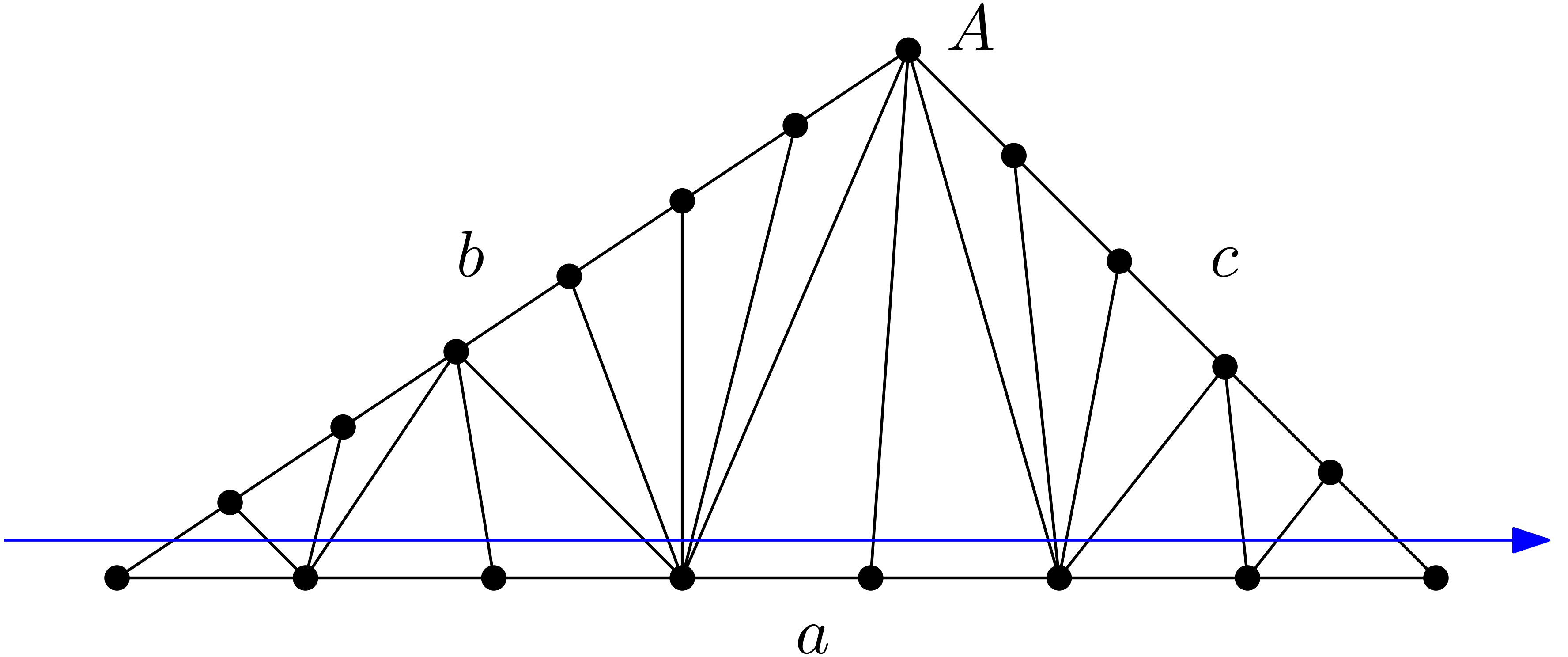
D-trianguations

$\mathsf{tr}(\Delta(a,b,0)) = \binom{a+b}{a}$



D-trianguations

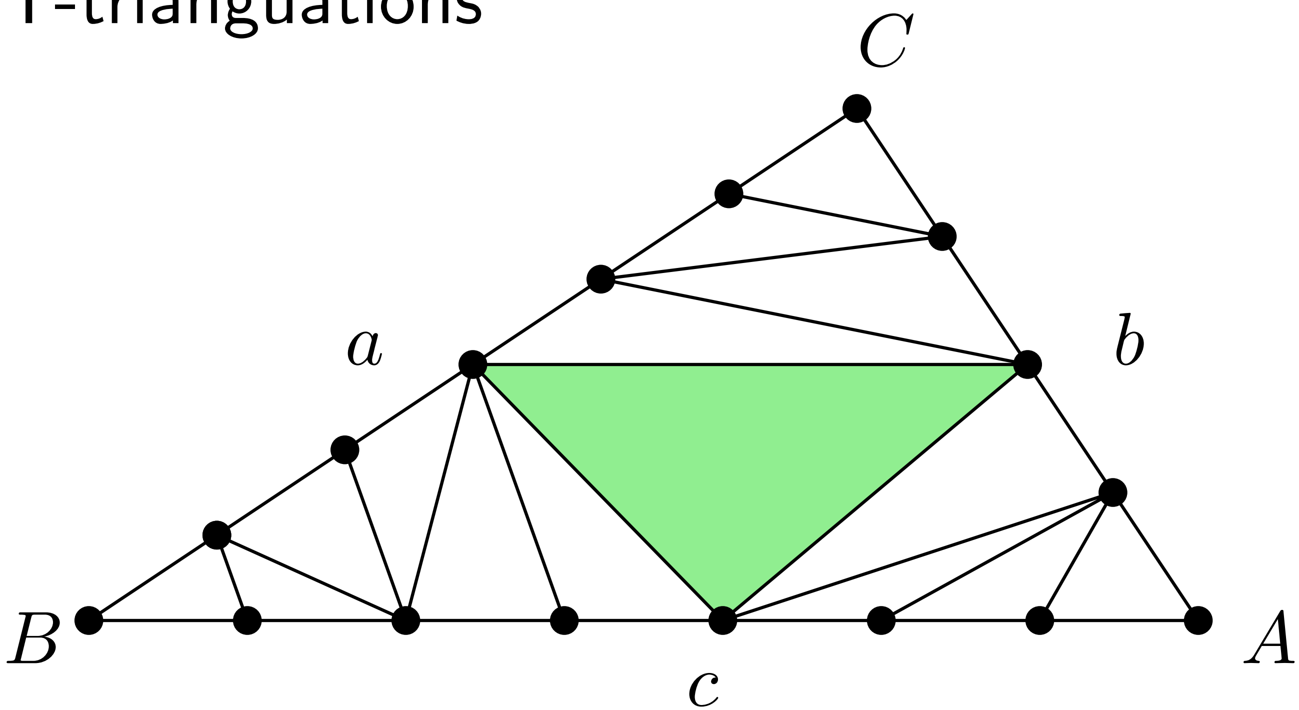
$\mathsf{tr}_{D,A}(\Delta(a,b,c)) = \binom{a+b+c-1}{a-1}$



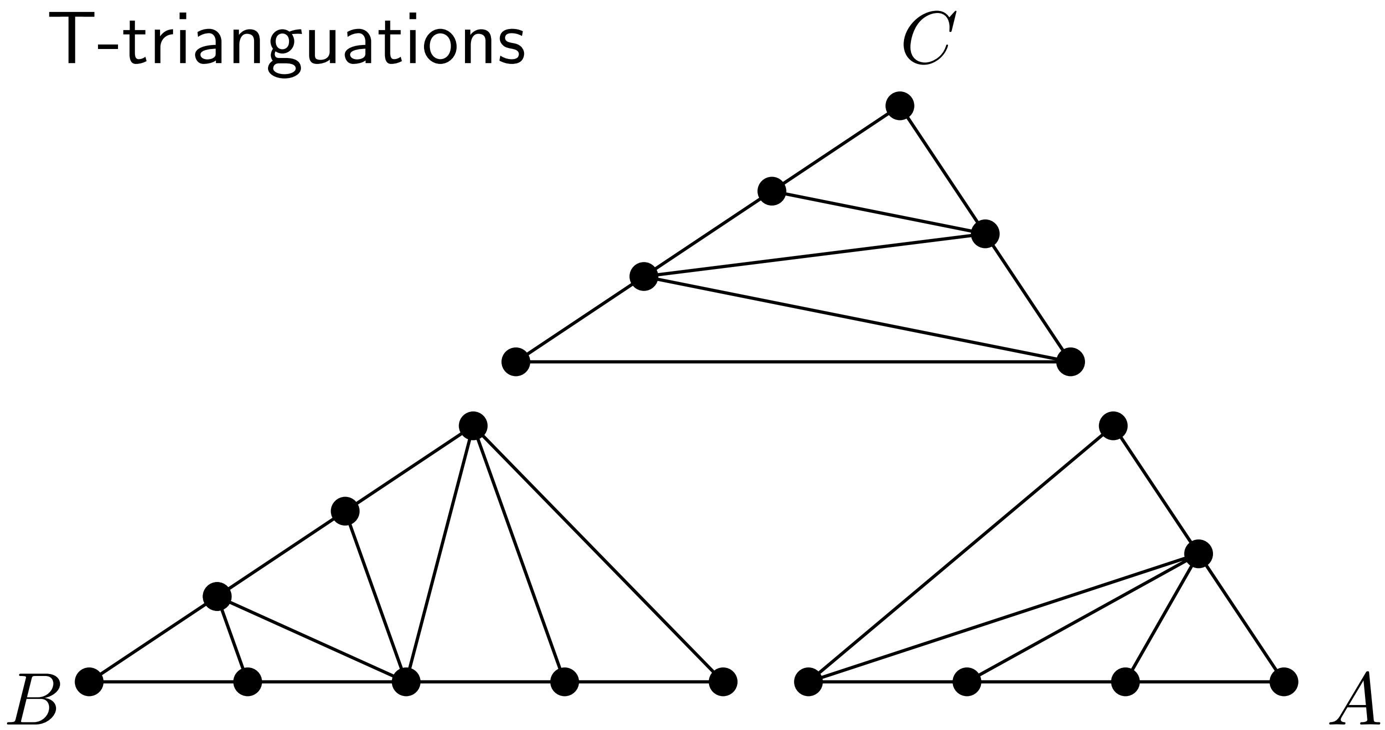
D-trianguations

$$\mathsf{tr}_D(\Delta(a,b,c)) = \binom{a+b+c-1}{a-1} + \binom{a+b+c-1}{b-1} + \binom{a+b+c-1}{c-1}$$

T-trianguations



T-triangulations



$$\binom{a+b}{a} = [x^a y^b]\ \frac{1}{1-x-y}$$

$$\mathsf{tr}_T(\Delta(a,b,c)) = [x^a y^b z^c] \frac{xyz}{(1-x-y)(1-y-z)(1-z-x)}$$

$$\mathsf{tr}_T(\Delta(a,b,c)) = [x^a y^b z^c] \frac{xyz}{(1-x-y)(1-y-z)(1-z-x)}$$

$$= 2^s - \sum_{\ell=0}^{a-1} \binom{s}{\ell} - \sum_{\ell=0}^{b-1} \binom{s}{\ell} - \sum_{\ell=0}^{c-1} \binom{s}{\ell}$$

$$s = a + b + c - 1$$

$$\mathsf{tr}(\Delta(a,b,c)) = 2^s - \sum_{\ell=0}^{a-2} \binom{s}{\ell} - \sum_{\ell=0}^{b-2} \binom{s}{\ell} - \sum_{\ell=0}^{c-2} \binom{s}{\ell}$$

$$\mathrm{tr}(\Delta(p,p,p)) = 2^{3p-1} - 3\sum_{\ell=0}^{p-2}\binom{3p-1}{\ell}$$

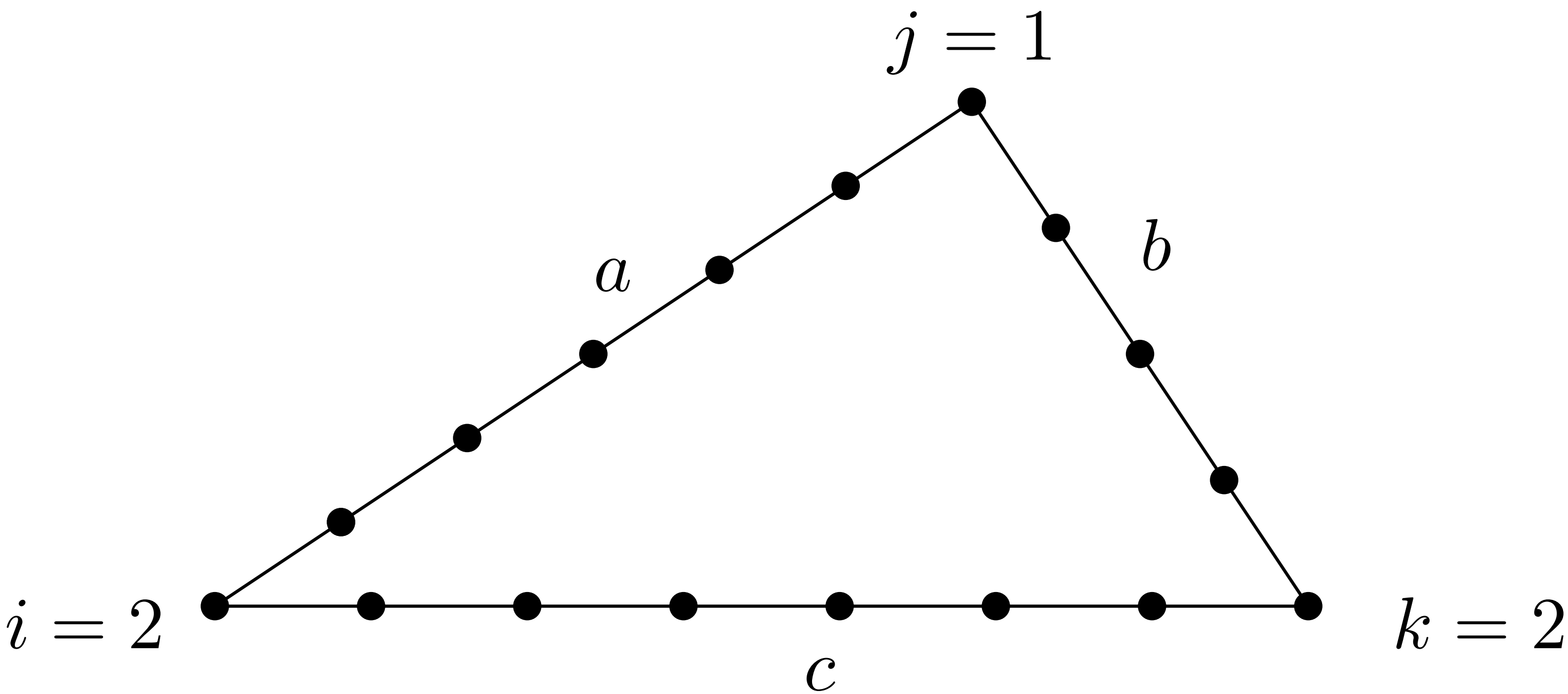
OEIS about $\mathrm{tr}(\Delta(p,p,p))$:

It seems that a(n)=sum_{i, j, k>=0}C(p, i+j)*C(p, j+k)*C(p, k+i)). - Benoit Cloitre, Oct 25 2004

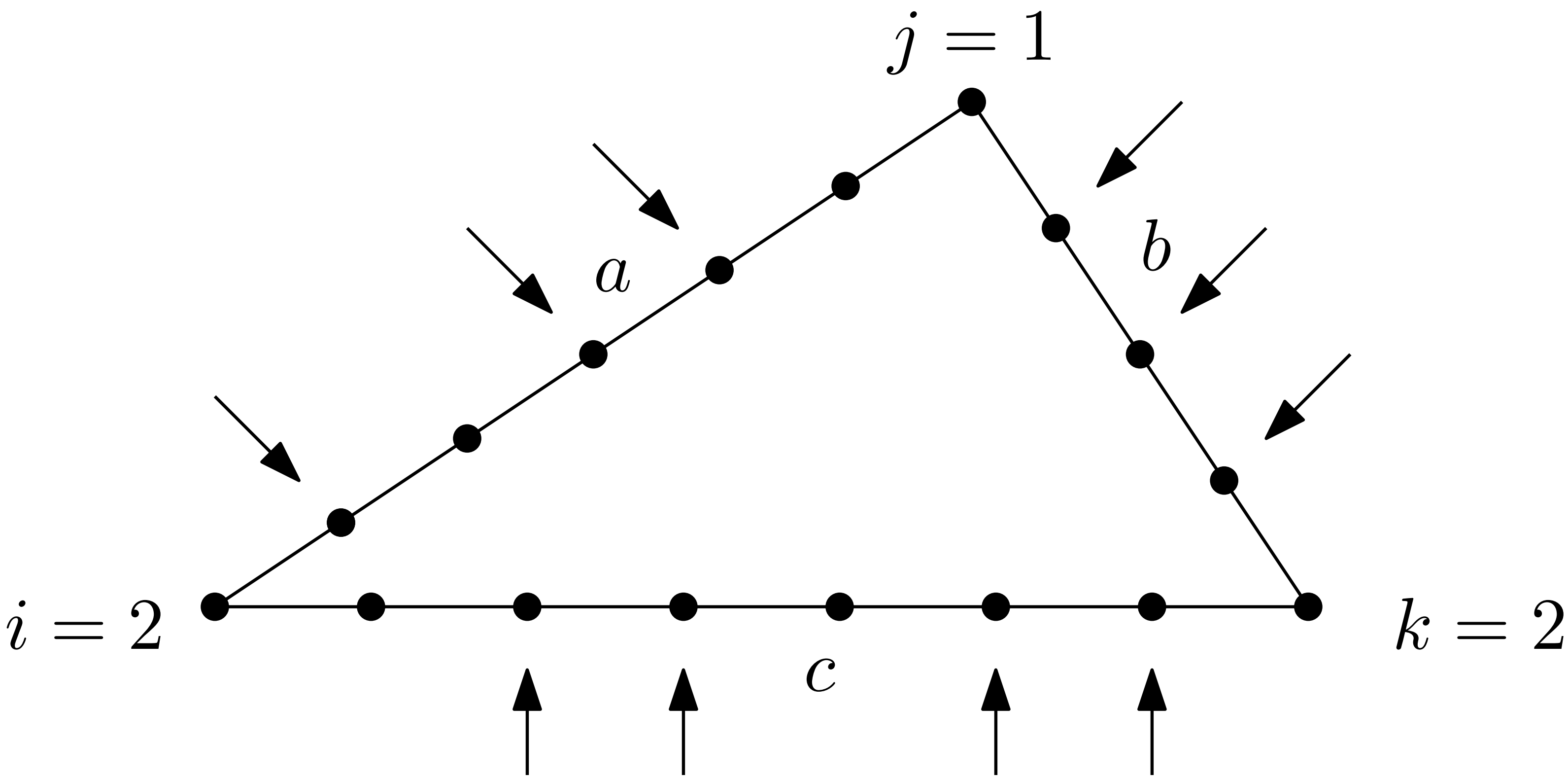
$$\mathrm{tr}(\Delta(p,p,p)) = \sum_{i,j,k\geq 0}\binom{p}{i+j}\binom{p}{j+k}\binom{p}{k+i}$$

$$\mathrm{tr}(\Delta(a,b,c)) = \sum_{i,j,k\geq 0}\binom{a}{i+j}\binom{b}{j+k}\binom{c}{k+i}$$

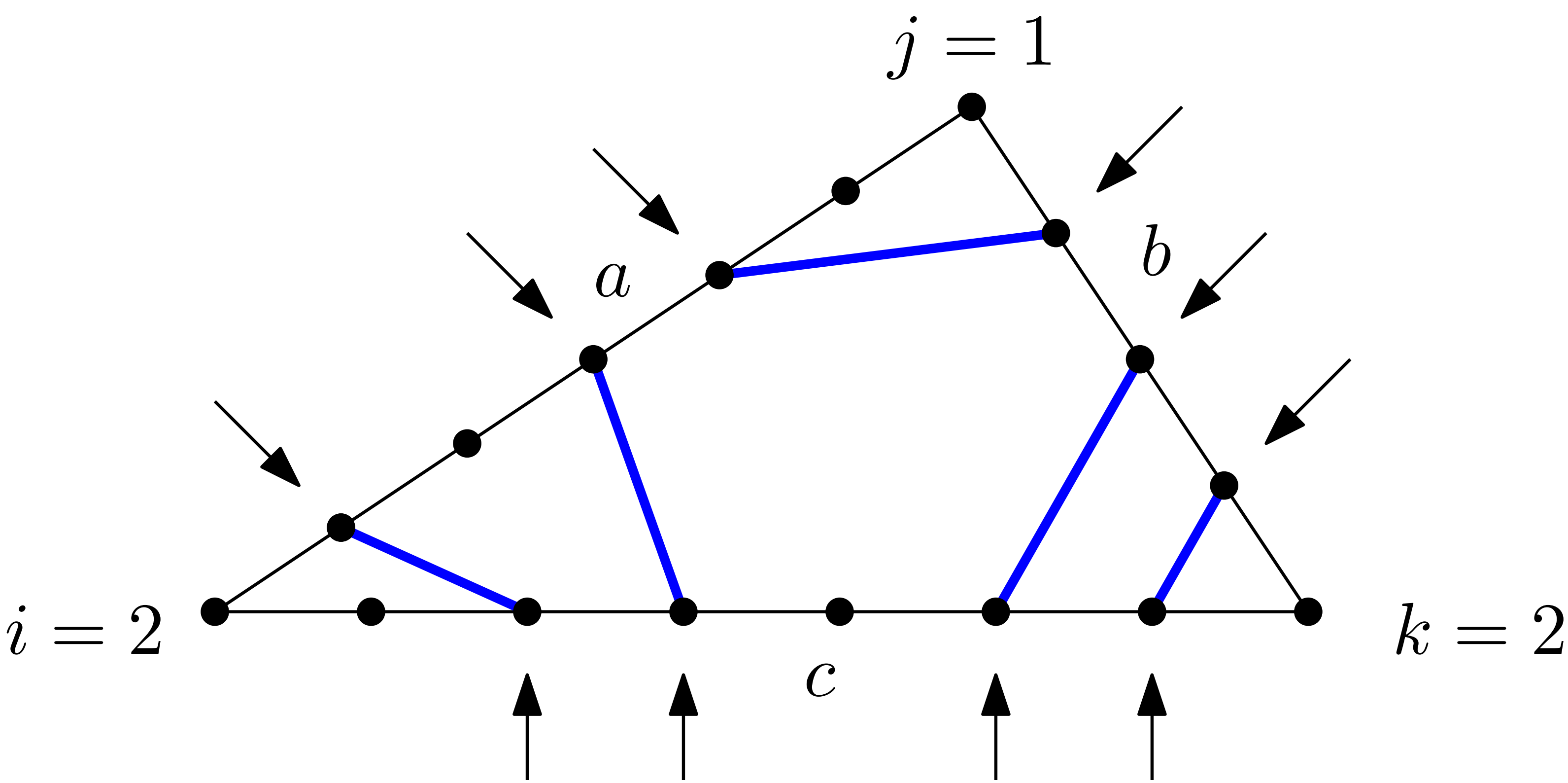
$$\mathsf{tr}(\Delta(a,b,c)) = \sum_{i,j,k\geq 0} \binom{a}{i+j}\binom{b}{j+k}\binom{c}{k+i}$$



$$\mathsf{tr}(\Delta(a,b,c)) = \sum_{i,j,k\geq 0} \binom{a}{i+j}\binom{b}{j+k}\binom{c}{k+i}$$



$$\mathsf{tr}(\Delta(a,b,c)) = \boxed{\sum_{i,j,k\geq 0} \binom{a}{i+j}\binom{b}{j+k}\binom{c}{k+i}}$$



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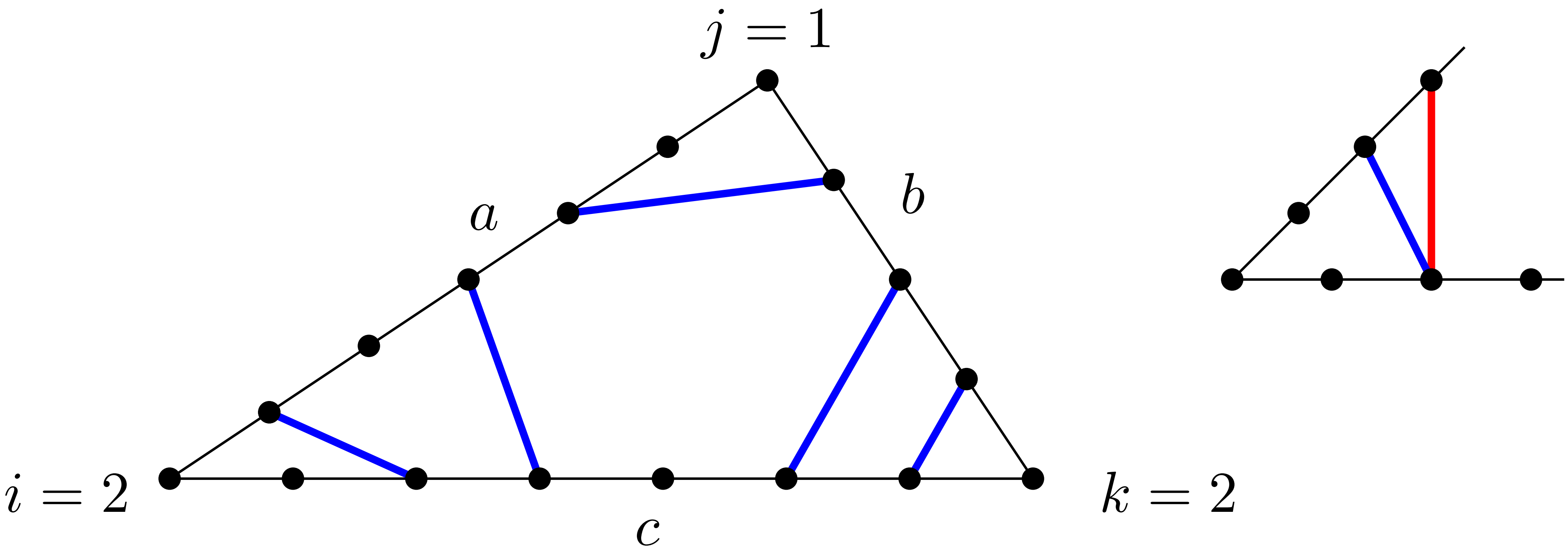

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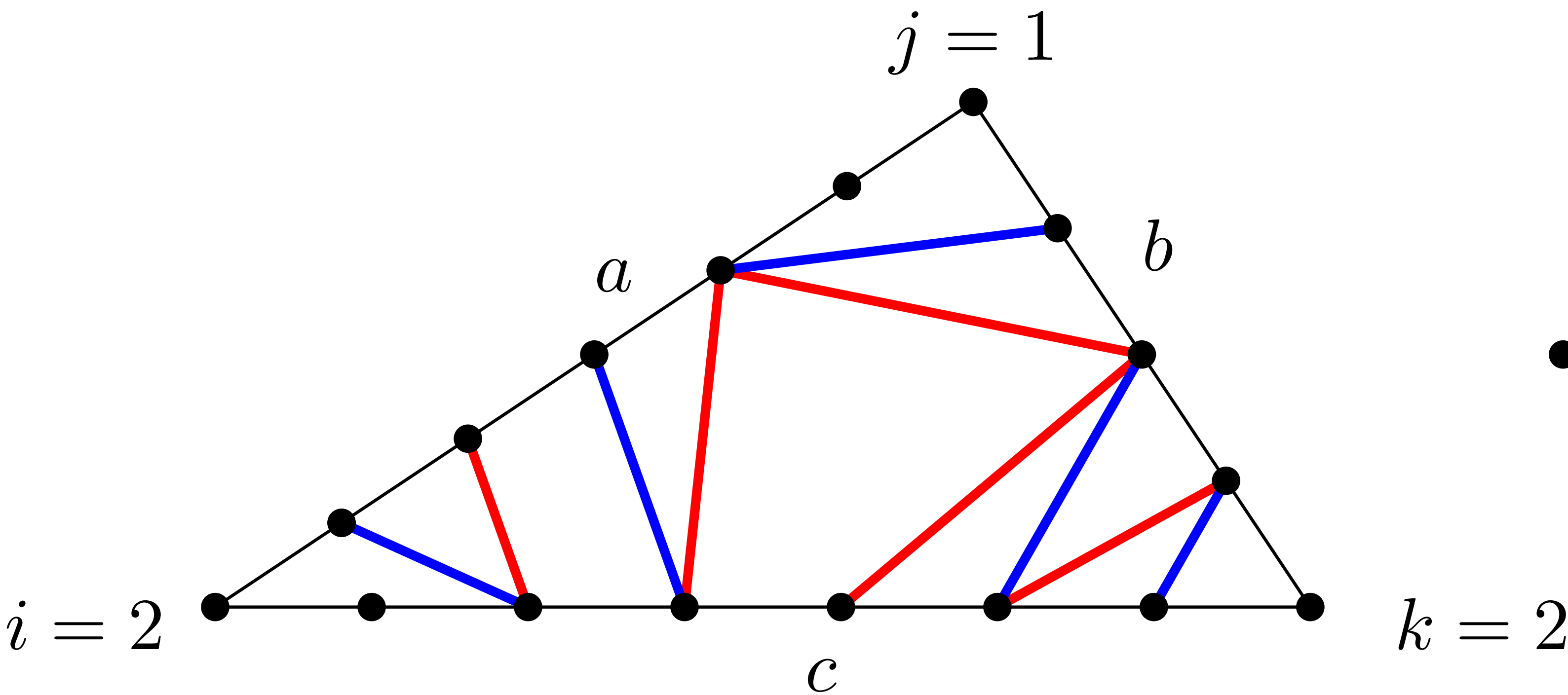

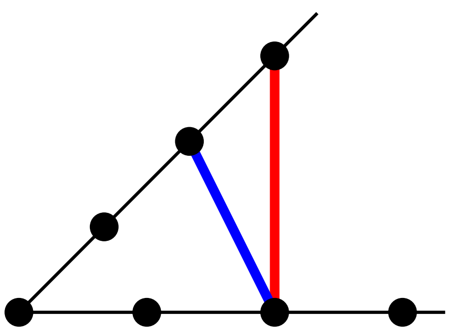

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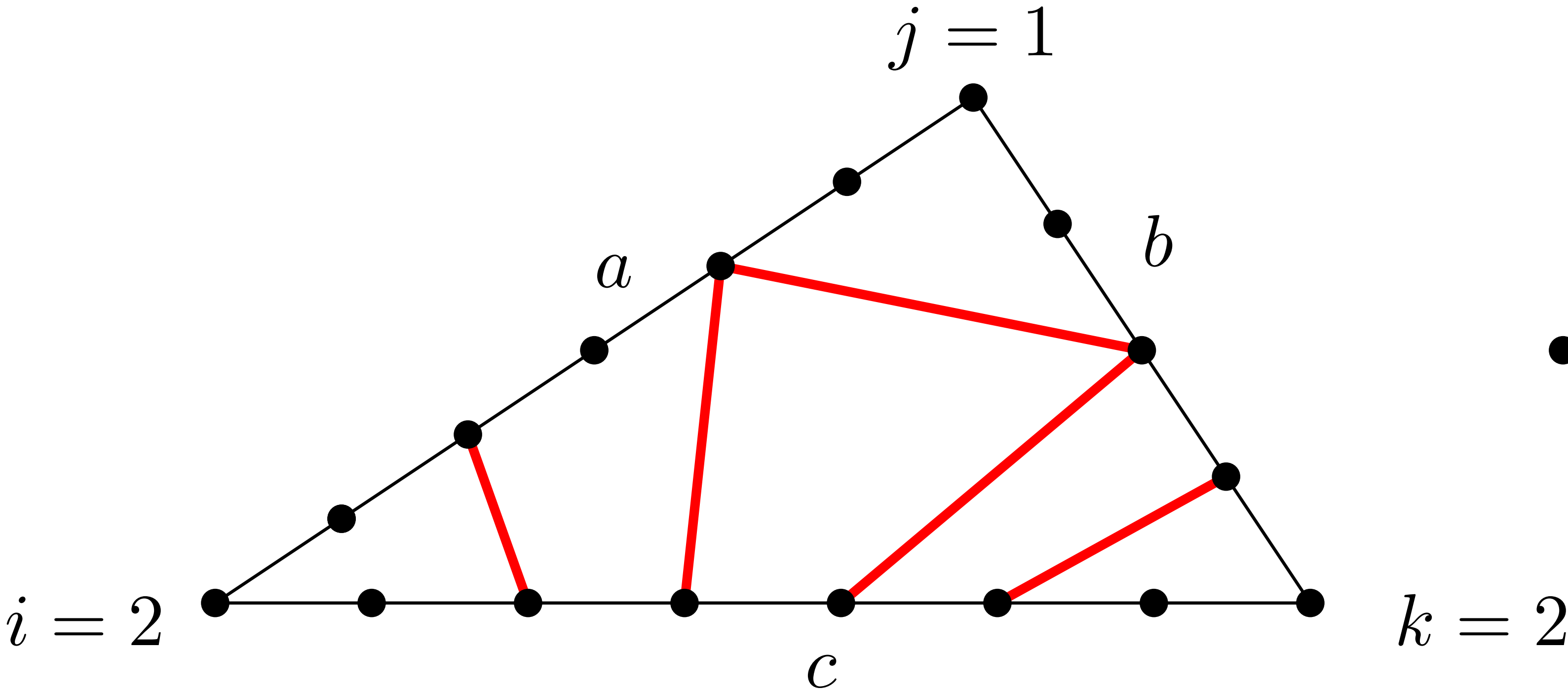

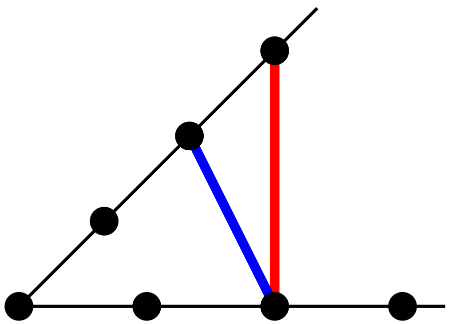

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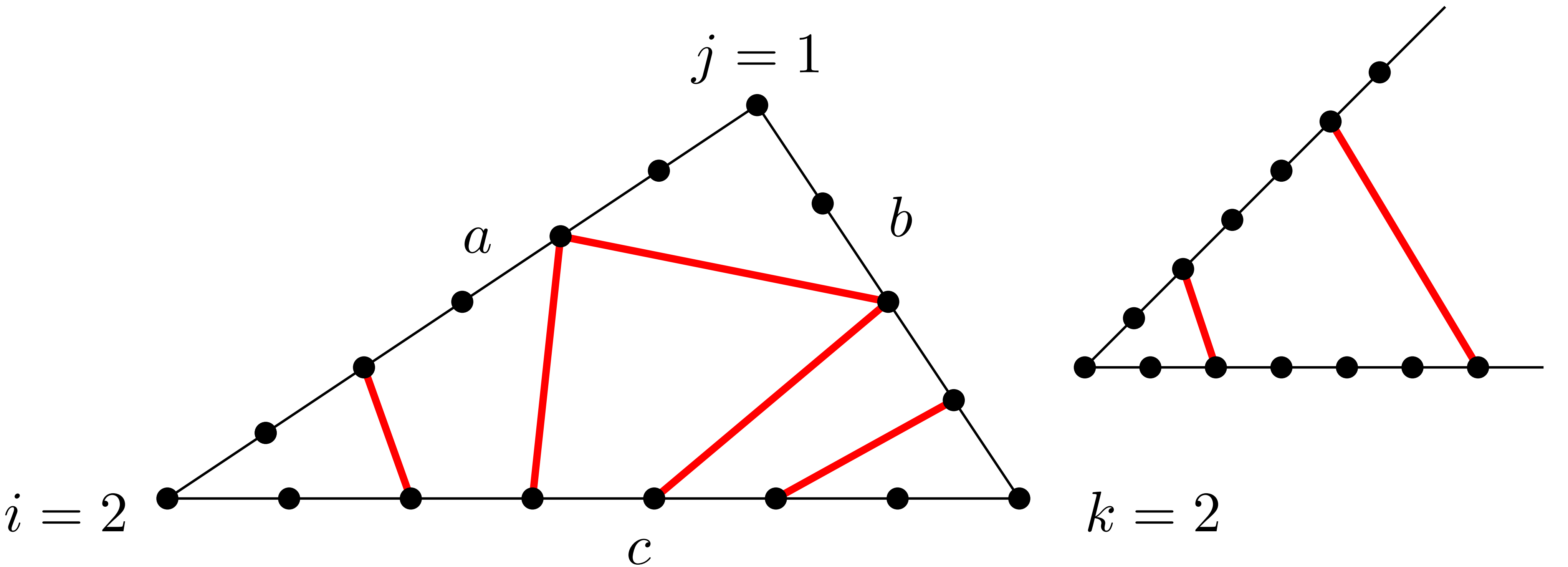

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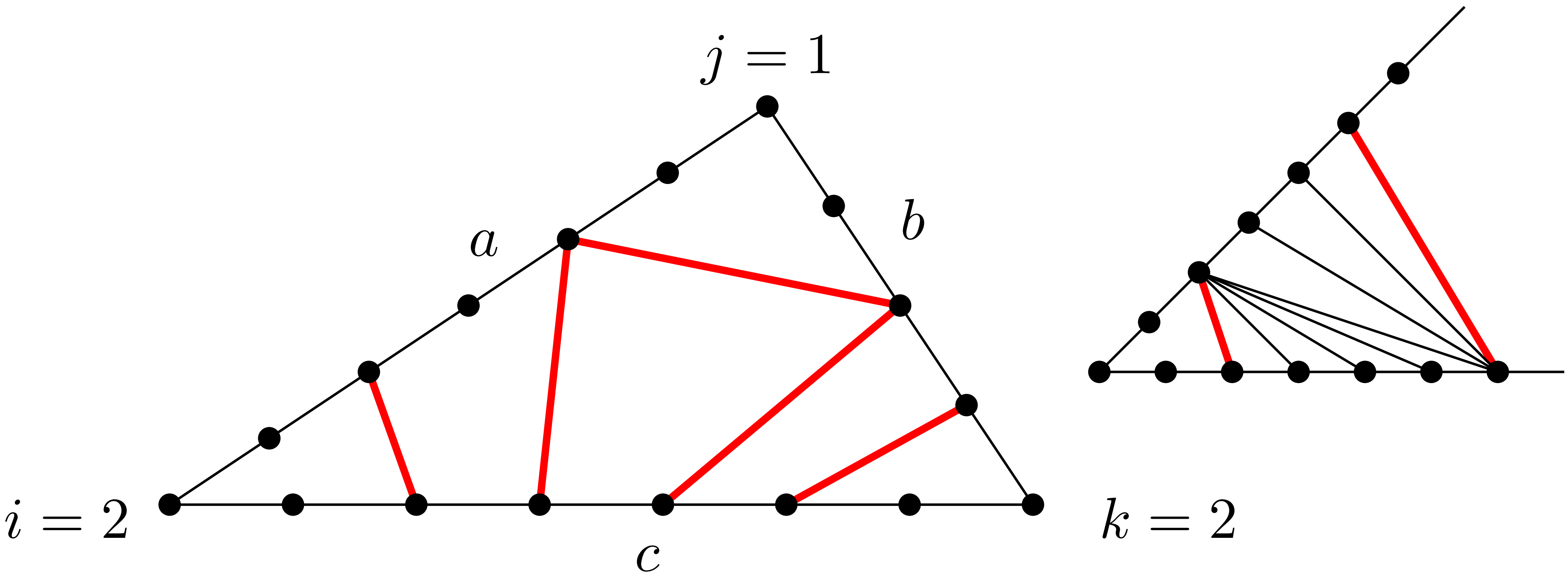

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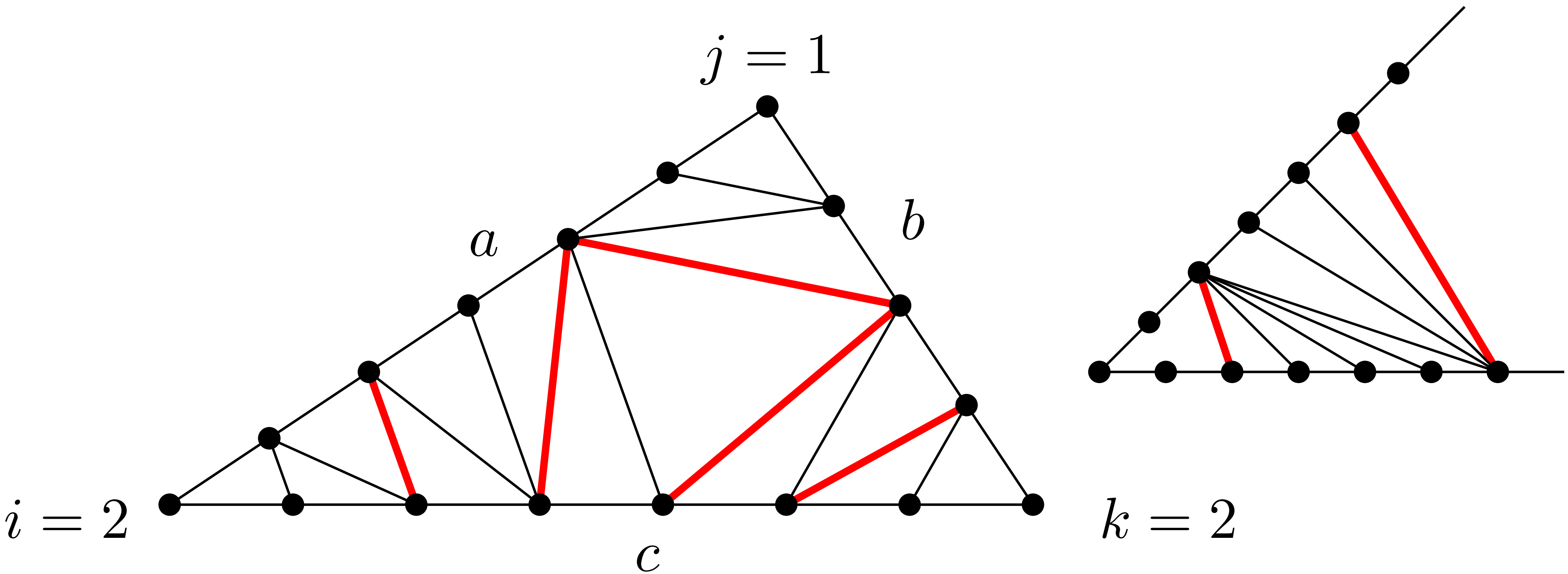
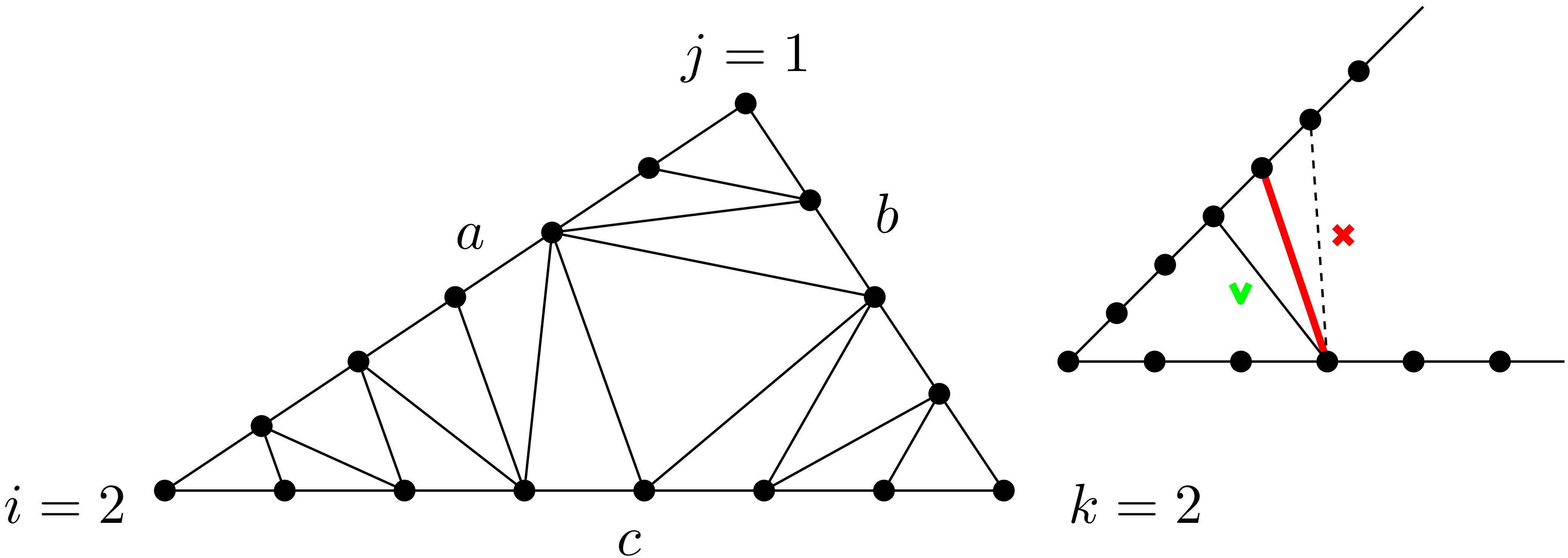
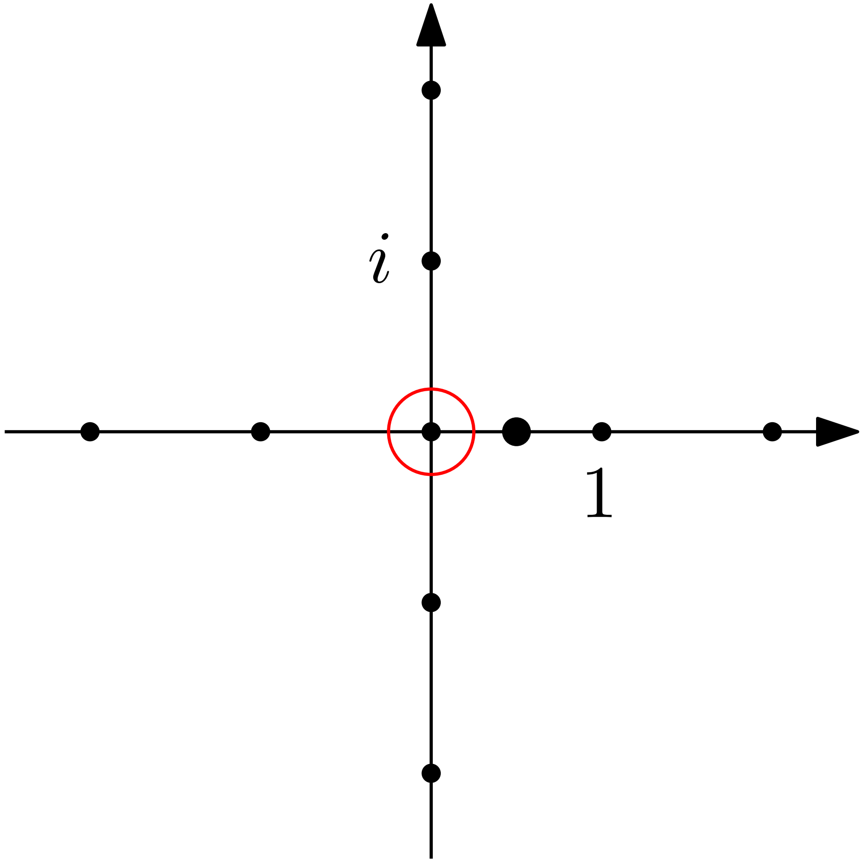


$$\boxed{\mathsf{tr}(\Delta(a,b,c))} = \sum_{i,j,k\geq 0} \binom{a}{i+j}\binom{b}{j+k}\binom{c}{k+i}$$



ASYMPTOTICS

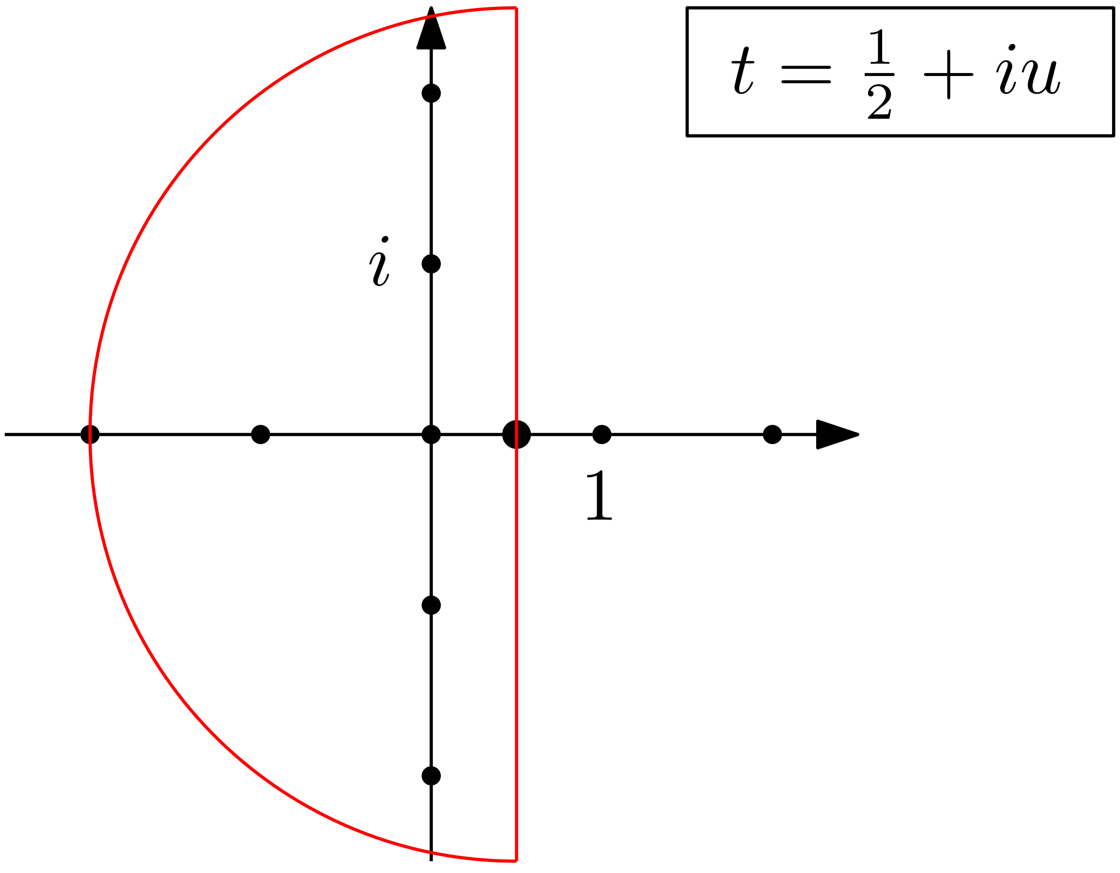
$$\mathsf{tr}(k,r) = -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{((1-t)^{r+1} - t^{r+1})^k \ dt}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}}$$



$$\mathsf{tr}(k,r) = -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{((1-t)^{r+1}-t^{r+1})^k \; dt}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}}$$

$\rho \to \infty \;\Rightarrow$

$\int$ over half-circle $\to 0$



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$$\boxed{t = \tfrac{1}{2} + iu}$$

$$= -\frac{2^{(r-2)k}}{\pi}\int_{-\infty}^{\infty} \frac{du}{(1+4u^2)^{rk}(iu)^{k-2}} \left(\left(1+2iu\right)^{r+1} - \left(1-2iu\right)^{r+1}\right)^k$$

k fixed, $r \to \infty$

$$\mathsf{tr}(k,r) = -\frac{2^{(r-2)k}}{\pi}\int_{-\infty}^{\infty}\frac{du}{(1+4u^2)^{rk}(iu)^{k-2}}\left(\Big(1+2iu\Big)^{r+1}-\Big(1-2iu\Big)^{r+1}\right)^{k}$$

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$$\boxed{w = ur}$$

$$= -\frac{2^{(r-2)k}\, r^{k-3}}{\pi}\int_{-\infty}^{\infty}\frac{dw}{\left(1+4\frac{w^2}{r^2}\right)^{rk}(iw)^{k-2}}\left(\left(1+\frac{2iw}{r}\right)^{r+1}-\left(1-\frac{2iw}{r}\right)^{r+1}\right)^{k}$$

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$$\mathrm{tr}(k,r) = -\frac{2^{(r-2)k}}{\pi} \int_{-\infty}^{\infty} \frac{du}{(1+4u^2)^{rk}(iu)^{k-2}} \left(\left(1+2iu\right)^{r+1} - \left(1-2iu\right)^{r+1} \right)^k$$

$$\boxed{w = ur}$$

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$$1$$

$$(e^{2iw} - e^{-2iw})^k$$

$$= (2i\,\sin(2w))^k$$

$$= \frac{2^{(r-1)k}\, r^{k-3}}{\pi} \left(\int_{-\infty}^{\infty} \frac{\sin^k(2w)}{w^{k-2}}\, dw \right) (1+o(1)).$$

$$k \to \infty$$

$$\mathsf{tr}(k,r) = -\frac{2^{(r-2)k}}{\pi} \int_{-\infty}^{\infty} \frac{du}{(1+4u^2)^{rk}(iu)^{k-2}} \left(\left(1+2iu\right)^{r+1} - \left(1-2iu\right)^{r+1} \right)^{k}$$

$$k \to \infty$$

$w = u(kR)^{1/2}$, where $R = r(r+5)/6$

$$\mathsf{tr}(k,r) = -\frac{2^{(r-2)k}}{\pi} \int_{-\infty}^{\infty} \frac{du}{(1+4u^2)^{rk}(iu)^{k-2}} \left(\left(1+2iu\right)^{r+1} - \left(1-2iu\right)^{r+1} \right)^k$$

$$= \frac{2^{(r-1)k}}{\pi\,(kR)^{3/2}} \int_{-\infty}^{\infty} \frac{w^2\,dw}{\left(1+\frac{4w^2}{kR}\right)^{rk} \left(\frac{2iw}{(kR)^{1/2}}\right)^k} \left(\left(1+\frac{2iw}{(kR)^{1/2}}\right)^{r+1} - \left(1-\frac{2iw}{(kR)^{1/2}}\right)^{r+1} \right)^k$$

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$\exp(4w^2 r/R)$

$$= \left(2(r+1)\frac{2iw}{(kR)^{1/2}} \;+\; 2\frac{(r+1)r(r-1)}{6}\,\frac{(2iw)^3}{(kR)^{3/2}} + \dots\right)^k$$

$$= 2^k(r+1)^k \left(\frac{2iw}{(kR)^{1/2}}\right)^k \left(1+\frac{r(r-1)}{6}\,\frac{(2iw)^2}{kR} + \dots\right)^k$$

$\exp\left(-4w^2\left(\frac{r}{R}+\frac{r(r-1)}{6R}\right)\right) = \exp(-4w^2)$

$\exp(-4w^2 r(r-1)/6R)$

$$= \left(\frac{2^{rk}(r+1)^k}{\pi\,(kR)^{3/2}} \int_{-\infty}^{\infty} w^2 \exp(-4w^2)\,dw\right)(1+o(1)) \;=\; \frac{2^{rk}(r+1)^k}{16\sqrt{\pi}\,(kR)^{3/2}}\,(1+o(1))$$

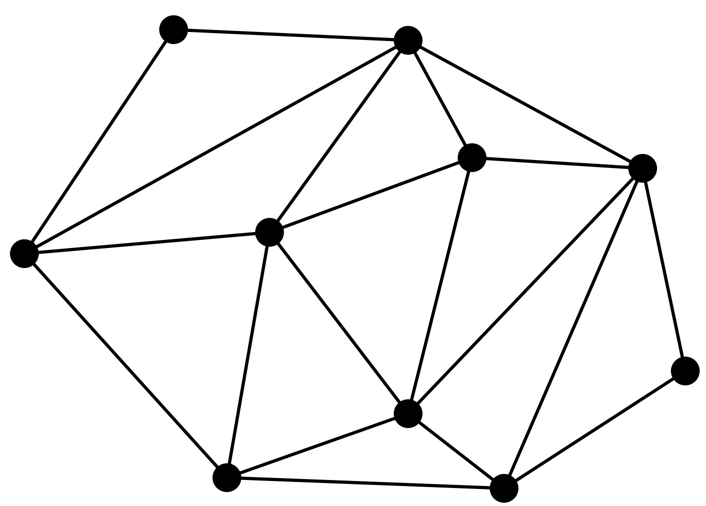
Motivation / starting point:

PLANAR SETS WITH FEW TRIANGULATIONS

What is the minimum / maximum number of triangulations that a planar point set of size n in general position can have?

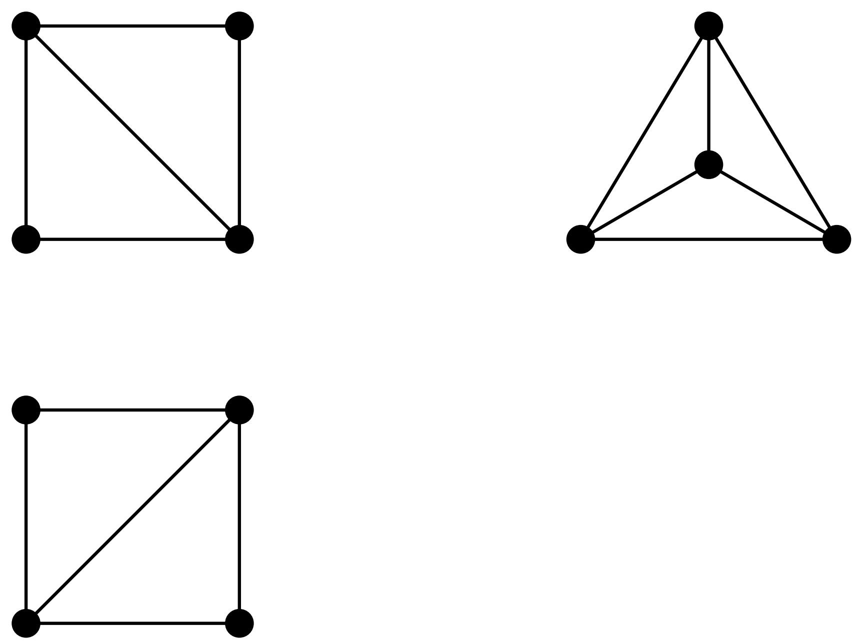
number of points

no three points lie on the same line



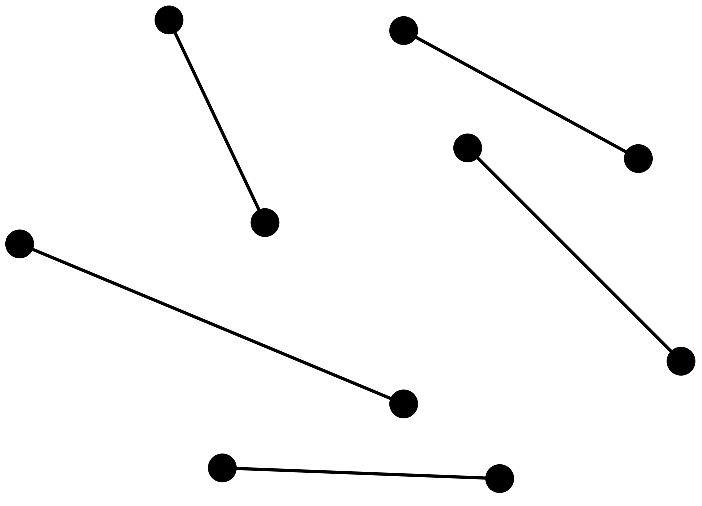
What is the minimum / maximum number of triangulations that a planar point set of size n in general position can have?

$n = 4$



non-crossing
perfect matchings

What is the minimum / maximum number of ~~triangulations~~ that a planar point set of size n in general position can have?



non-crossing spanning trees

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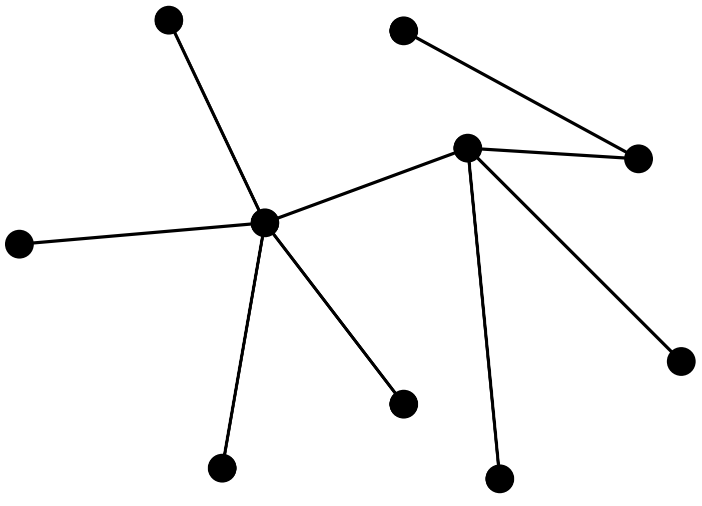

[some class of non-crossing graphs]

What is the minimum / maximum number of ~~triangulations~~ that a planar point set of size n in general position can have?

Typical situation for maximum:

No exact results but only upper and lower bounds are known:

Triangulations: $\Omega(8.65^n)$ and $O(30^n)$.

Non-crossing perfect matchings: $\Omega(3.09^n)$ and $O(10.05^n)$.

All non-crossing graphs: $\Omega(41.18^n)$ and $O(187.53^n)$.

Summary of such results: Adam Sheffer, Numbers of Plane Graphs:
`adamsheffer.wordpress.com/numbers-of-plane-graphs/`

[some class of non-crossing graphs]

What is the minimum / maximum number of ~~triangulations~~ that a planar point set of size n in general position can have?

Typical situation for minimum:

Attained by sets of points in convex position

for many classes of non-crossing graphs:

all non-crossing graphs; non-crossing connected graphs;

all the classes of cycle-free graphs.

(Aichholzer, Hackl, Huemer, Hurtado, Krasser, Vogtenhuber, 2007)

BUT NOT FOR TRIANGULATIONS

n points in convex position: $C_{n-2} = \Theta^*(4^n)$ triangulations

n points in convex position: $C_{n-2} = \Theta^*(4^n)$ triangulations

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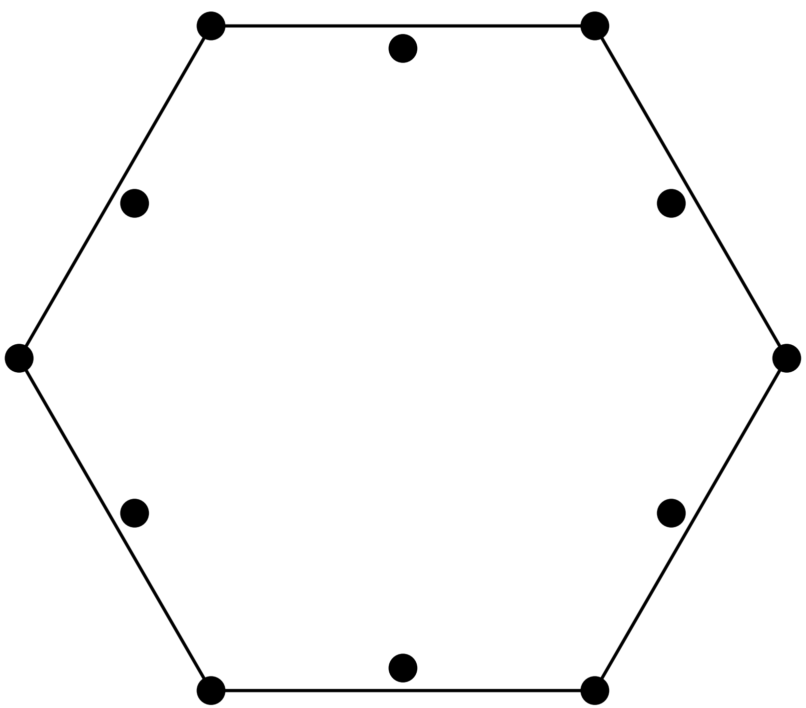
"Double circle":



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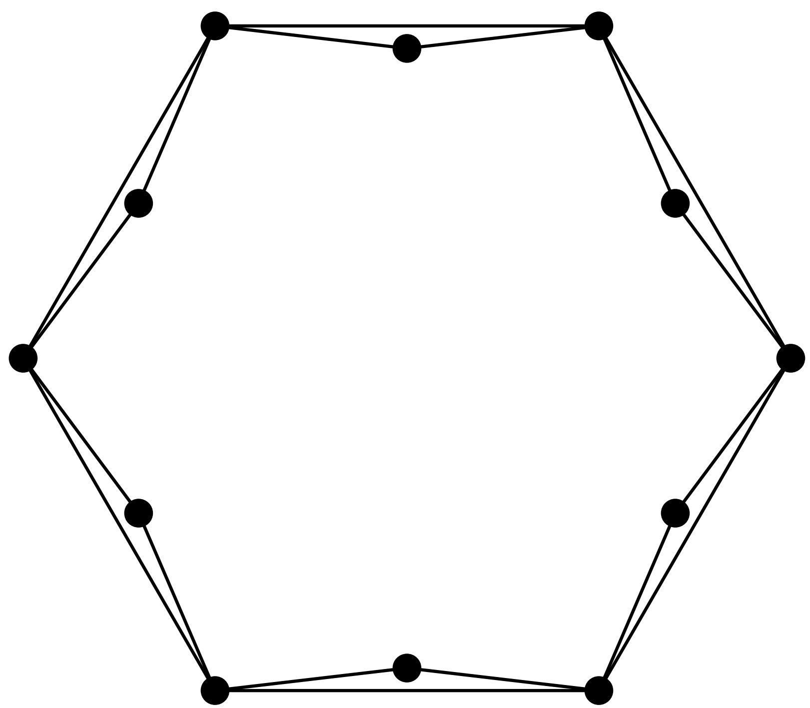

n points in convex position: $C_{n-2} = \Theta^*(4^n)$ triangulations

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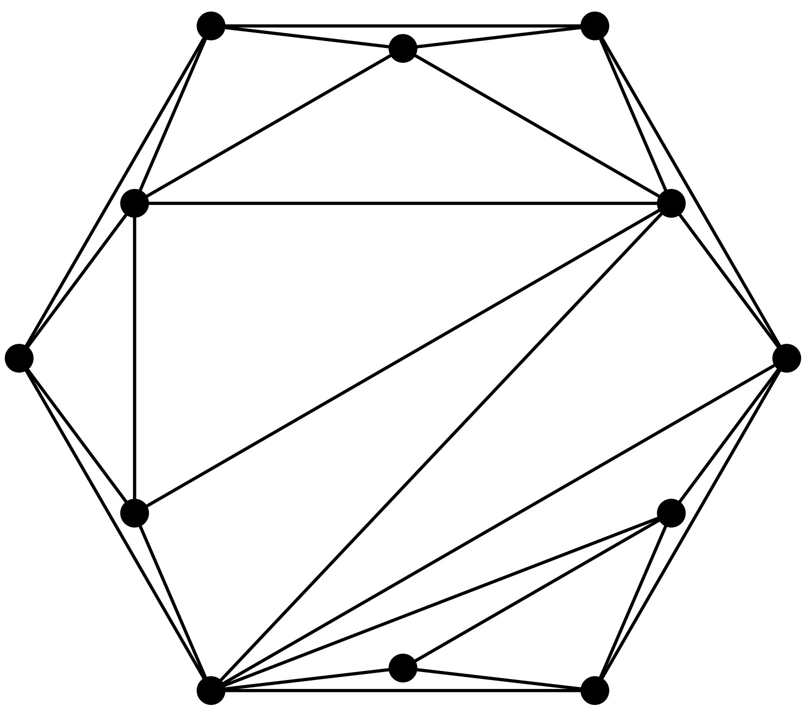
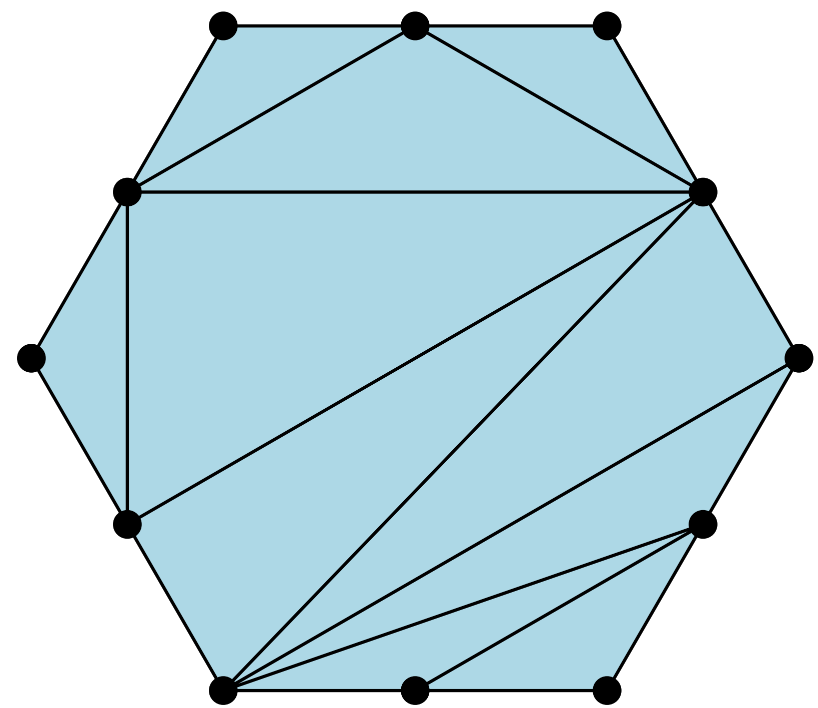
“Double circle”: $\Theta^*(\sqrt{12}^{\,n})$ triangulations





The double circle with n points has $\Theta^*(\sqrt{12}^n)$ triangulations.

(Santos and Seidel, 2003; the case $r = 2$ of our result)

For $n \leq 15$, the double circle has indeed the minimum number of triangulations over all sets of n points in general position.

(Aichholzer et al., 2001–2016)

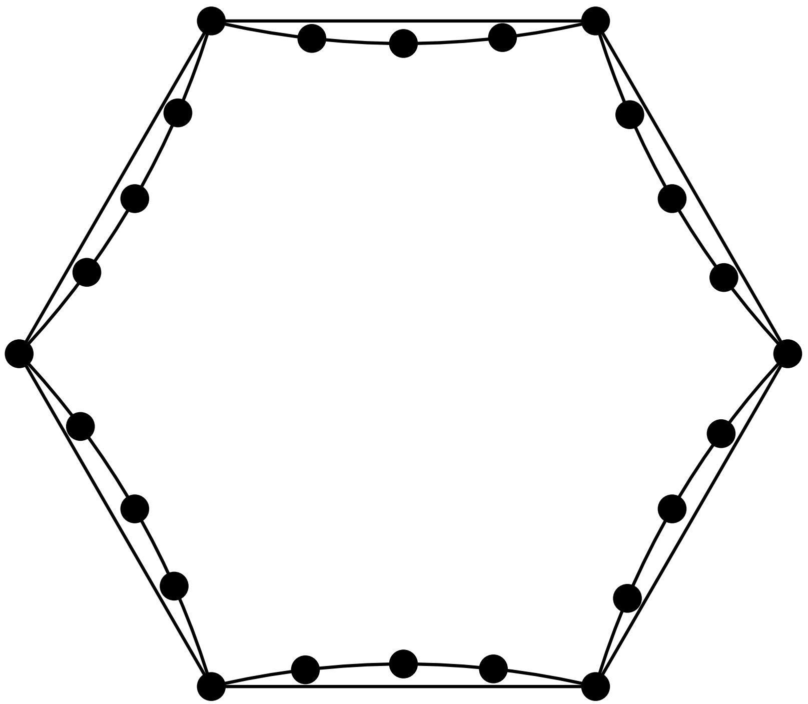
Conjecture: This is true for any n.

(Aichholzer, Hurtado, Noy, 2004)

Any set of n points in general position has $\Omega(2.63^n)$ triangulations.

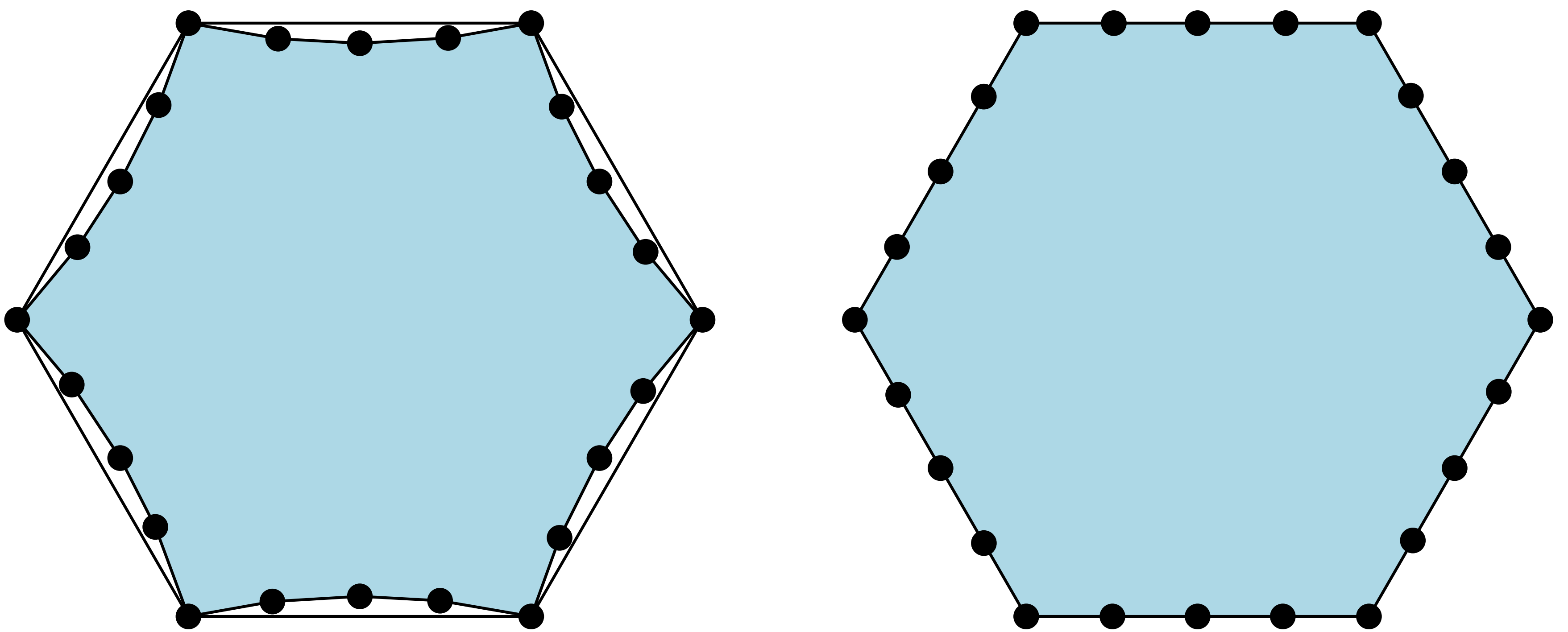
(Aichholzer et al., 2016)

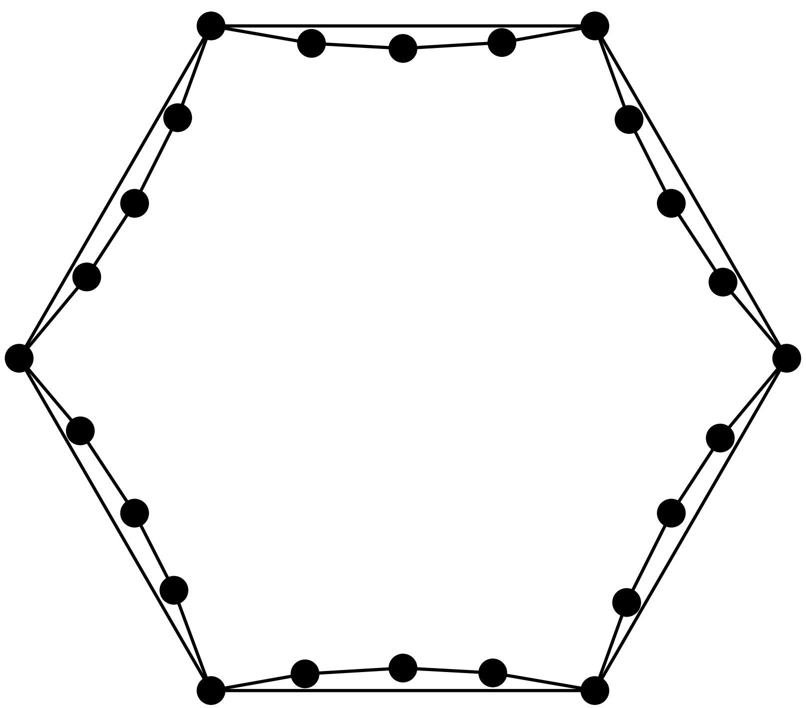
Generalized double circle:



Generalized double circle:

We have k regions that consist of $r + 1$ points in convex position, and one central region equivalent to a subdivided balanced convex polygon. (The shown edges are unavoidable in any triangulation.)





This configuration has

tr$(k, r) \cdot C_{r-1}^{k}$ triangulations

r fixed, $k \to \infty$:

$\Theta^* \left((2(r+1)^{1/r} C_{r-1}^{1/r})^n \right)$

minimum for $r = 2$:

$\Theta^*(\sqrt{12}^{\,n})$

k fixed, $r \to \infty$:

$\Theta^*(8^n)$

$$2(r+1)^{1/r}C_{r-1}^{1/r}$$

For **integer** r, the minimum is at $r = 2$.

However, for **real** r ($C_n = \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}$),

the minimum is at $r \approx 1.4957$.

This leads to the idea to "mix" non-subdivided sides with sides subdivided by one point.

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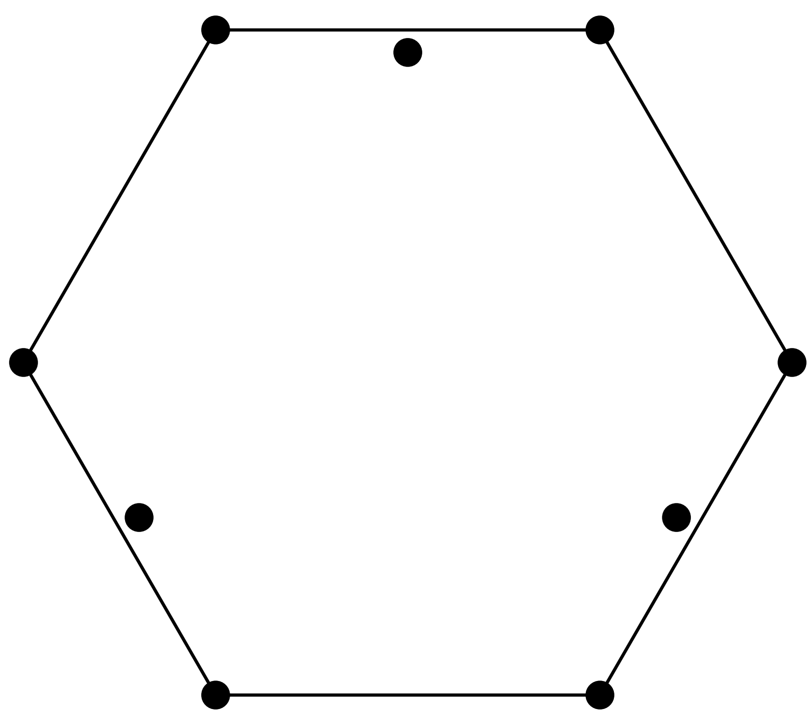

n the total number of points

s the number of subdivided sides

$n, s \to \infty, \quad s/n \to \alpha$:

$\text{tr} = \Theta^*((4^{1-\alpha}3^{\alpha})^n)$

minimum for $\alpha = 1/2$ –

again the double circle with $\sqrt{12}$.

SUMMARY

For the number of triangulations of the convex k-gon with sides subdivided by $r-1$ points, we found:

An inclusion-exclusion formula, a double sum formula,
the asymptotic behaviour for $k \to \infty$ or/and $r \to \infty$.

We proved that "vertical" (r is fixed) and "'horizontal" (k is fixed) generating functions are algebraic.

For $k = 3$, we also found formulas for the non-balanced case.

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For the number of triangulations of the convex k-gon with sides subdivided by $r-1$ points, we found:

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For $k = 3$, we also found formulas for the non-balanced case.

Our results imply that for the problem of characterizing a planar point set in general position of size n with the minimal number of triangulations, it is impossile to "beat" the bound of $\Theta((\sqrt{12})^n)$ attained by double circle, using balanced subdivided polygons, in whatever way $n \to \infty$; or using the "mixed " construction.

END