Snake graphs for generalised cluster algebras

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76th Séminaire Lotharingien de Combinatoire

Contents

1. Cluster algebras from surfaces and snake graphs
2. Generalised cluster algebras
3. Snake graphs for generalised cluster algebras

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$$\begin{array}{ccc} \{\text{almost positive roots}\} & \overset{\text{bijection}}{\leftrightarrow} & \{\text{cluster variables}\} \\ -\alpha_i & \mapsto & x_i \\ \sum n_i \alpha_i \in \Phi^+ & \mapsto & \frac{1}{\prod x_i^{n_i}}(\ldots) \end{array}$$

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Theorem (Keller, 2012)

The bijection also exists for cluster algebras of affine/twisted types.

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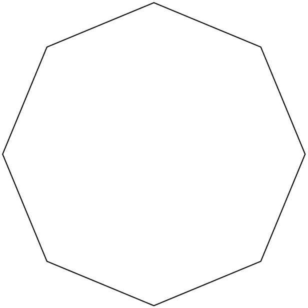

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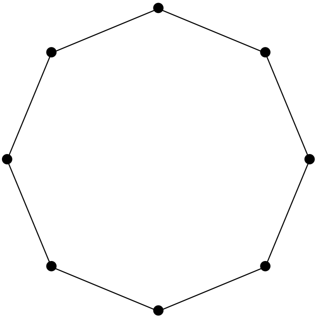

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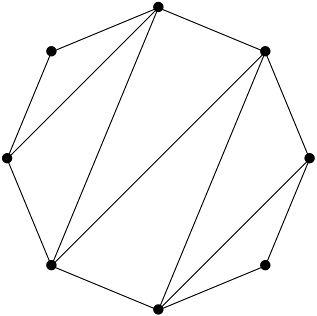

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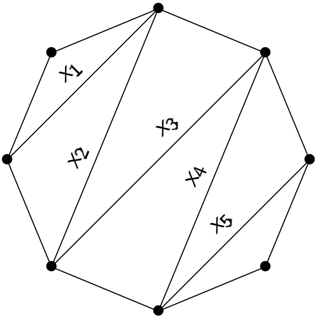

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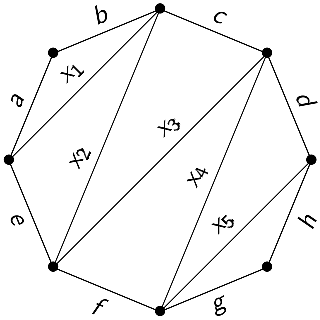

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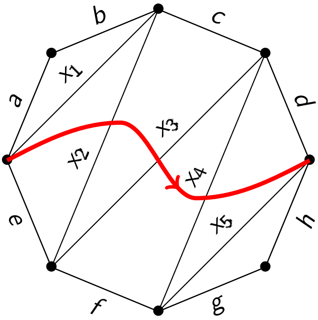

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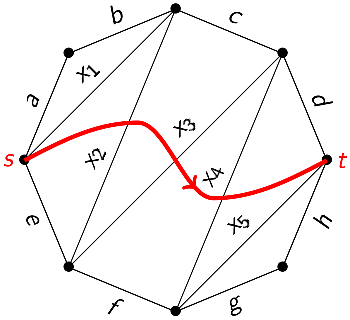

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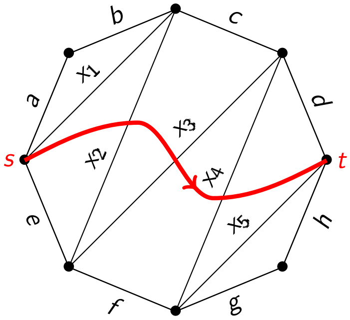

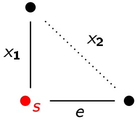

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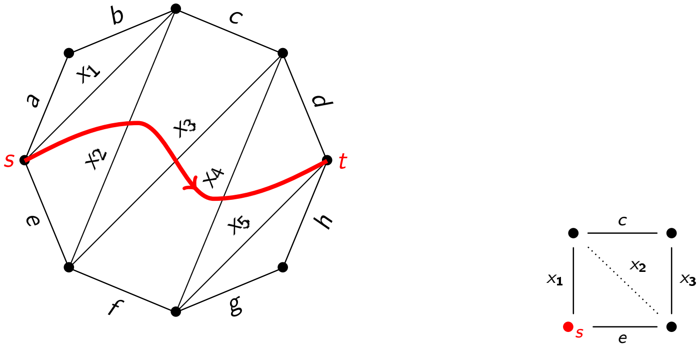

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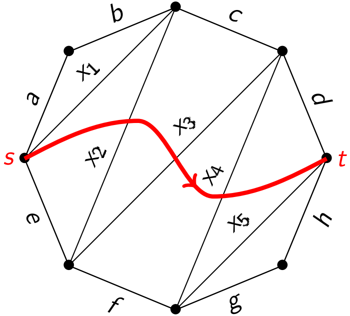

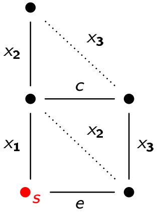

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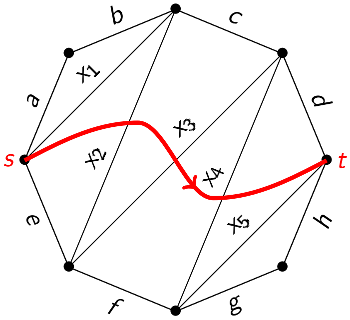

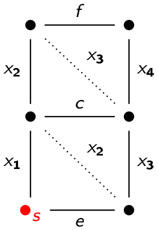

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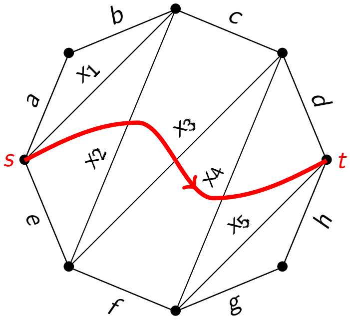

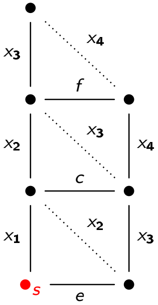

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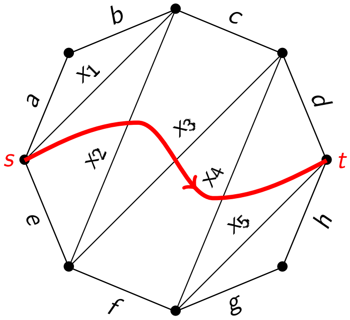
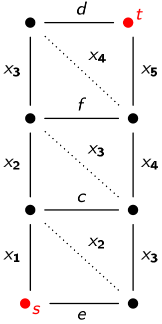
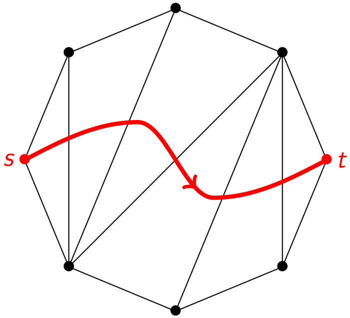




Type A_5 : another triangulation



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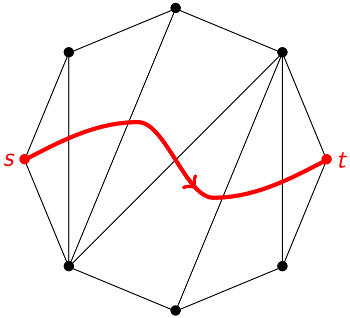

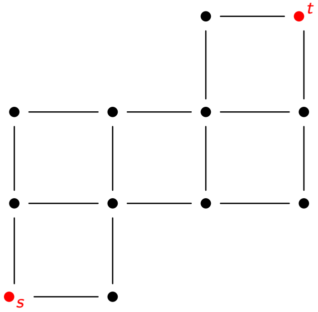

Cluster algebras, surfaces, and snake graphs

(S,M) surface with boundary and marked points

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$\downarrow$

triangulation

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triangulation

internal arcs $\tau_i \quad \leftrightarrow \quad$ initial cluster variables x_i

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internal arcs τ_i	$\leftrightarrow$	initial cluster variables x_i
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triangles crossed by γ

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snake graph

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triangles crossed by γ $\mapsto$ snake graph

Laurent expansion of x_γ

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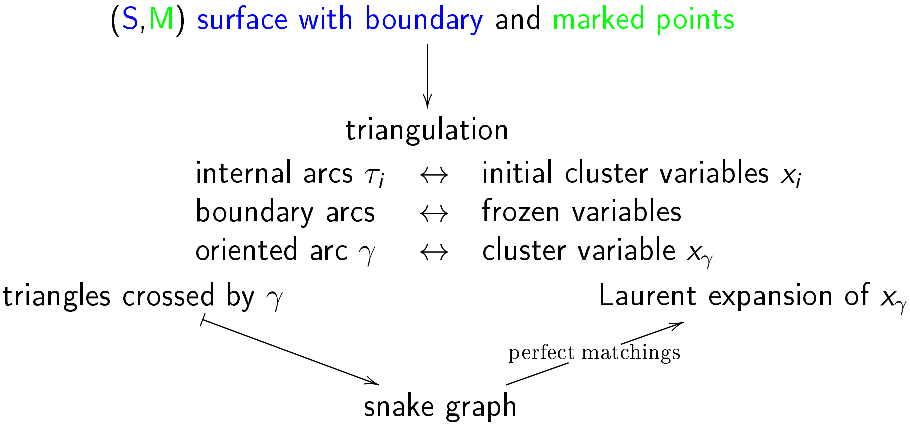

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For a graph $G = (G_0, G_1)$, a *perfect matching* of G is a subgraph $\Gamma = (G_0, \Gamma_1)$ such that each vertex of Γ is the endpoint of exactly **one** edge.

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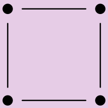 :

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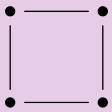
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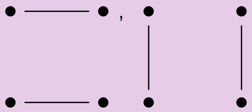

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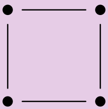

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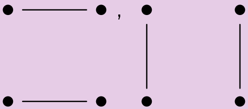

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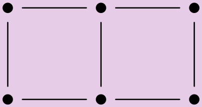 :

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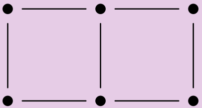

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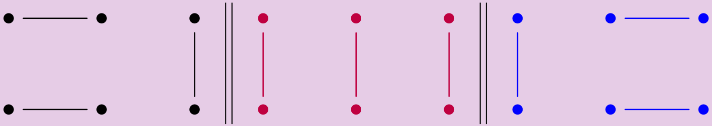

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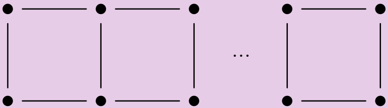

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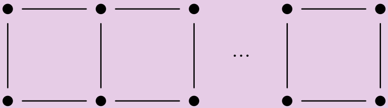

(n aligned squares) : Fibonacci$(n+1)$.

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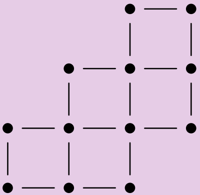

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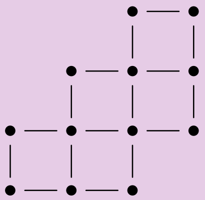

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Suppose that the arc γ crosses a_i times each internal arc τ_i, and yields the snake graph G.

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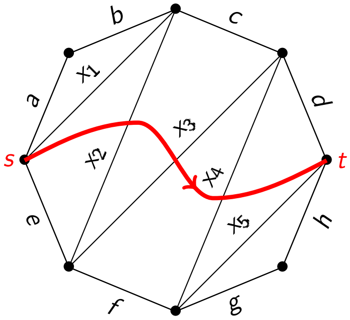
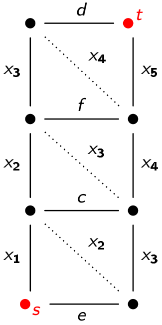
The cluster variable x_γ can then be written :

$$x_\gamma = \frac{1}{x_1^{a_1} \dots x_n^{a_n}} \sum_{\Gamma \text{ perf. mat. of } G} \left(\prod_{w \in \Gamma_1} \mathrm{label}(w) \right).$$

Example : Type A_5





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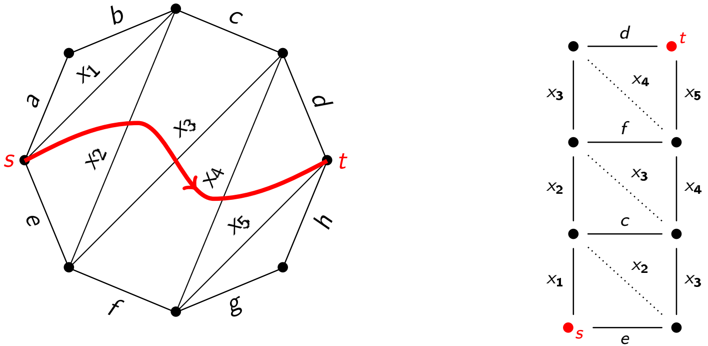

$$x_\gamma = \frac{1}{x_2x_3x_4}(ecfd + ecx_3x_5 + edx_2x_4 + x_1x_3^2x_5 + fdx_1x_3).$$

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$$x_k x_k' = p_k^+ (m_k^+)^n + \lambda_1^{(k)} (m_k^+)^{n-1} m_k^- + \cdots + + \lambda_{n-1}^{(k)} m_k^+ (m_k^-)^{n-1} + p_k^- (m_k^-)^n$$

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The *generalised cluster algebra* of *initial seed* $(\mathbf{x}, B)$ is the subring of $\mathbb{Q}(\mathbf{x}, (p_k^{\pm}), (\lambda_i^{(k)}))$ generated by all the cluster variables obtained through every possible sequence of mutations.

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Generalised cluster algebras were initially used by Chekhov and Shapiro to study triangulations of Riemann surfaces with orbifold points, giving us a motivation to find generalised snake graphs !

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Example : type C_3

Initial seed : $\Pi_0 := (\mathbf{x}^0, B)$, with

$$\mathbf{x}^0 = (x_1, x_2, x_3), \quad B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}.$$

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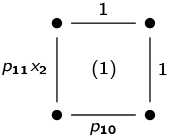

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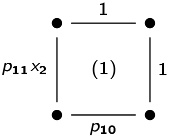

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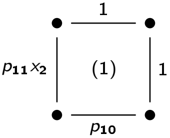

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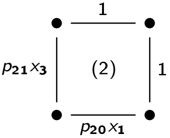

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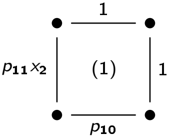

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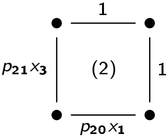

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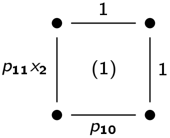

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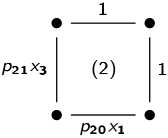

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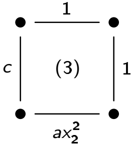

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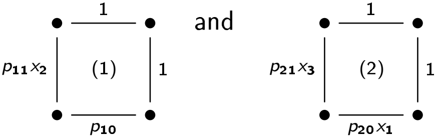

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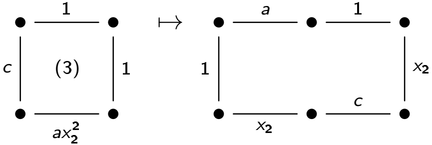

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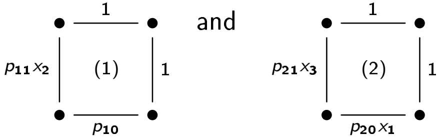

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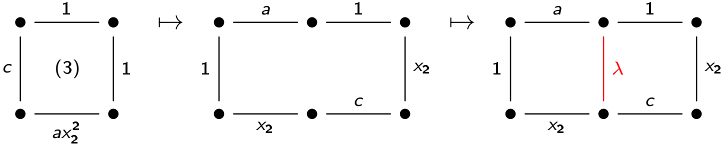

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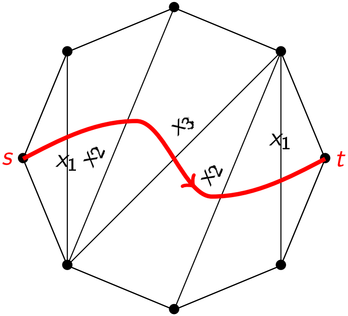

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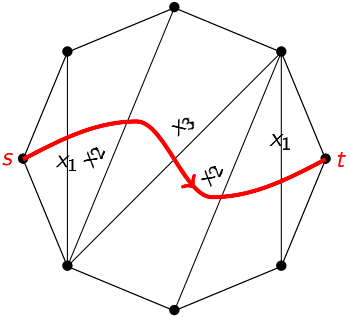
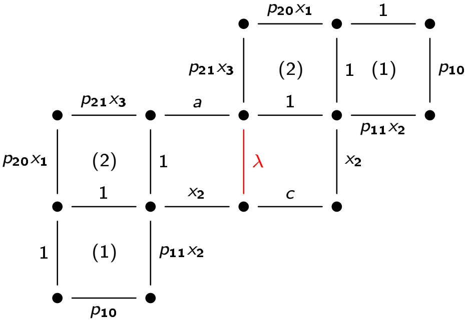


Corresponding snake graph :



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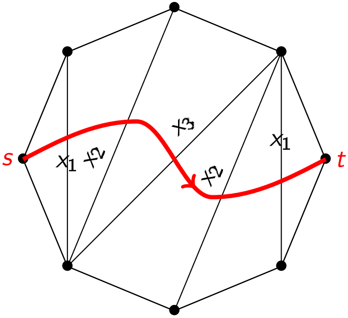

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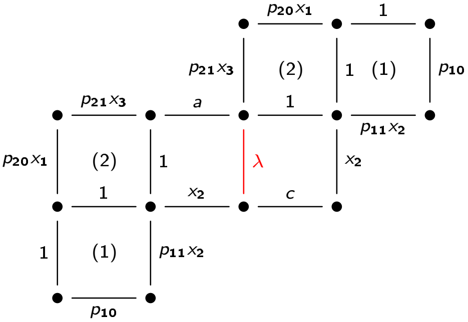

$$x_\gamma = \frac{1}{x_1^2x_2^2x_3}\left(\begin{array}{l} ap_{10}^2p_{20}^2x_1^2x_2^2 + cp_{11}^2p_{21}^2x_2^2x_3^2 + 2cp_{10}p_{11}p_{21}^2x_2x_3^2 \\ \quad +2cp_{10}p_{11}p_{20}p_{21}x_1x_2x_3 + cp_{10}^2p_{21}^2x_3^2 + 2cp_{10}^2p_{20}p_{21}x_1x_3 \\ \quad +cp_{10}^2p_{20}^2x_1^2 + \lambda p_{10}^2p_{20}^2x_1^2x_2 + \lambda p_{10}^2p_{20}p_{21}x_1x_2x_3 \\ \quad +\lambda p_{10}p_{11}p_{20}p_{21}x_1x_2^2x_3 \end{array}\right).$$

Type C_n

Theorem (G., Musiker)

Let $\mathcal{A}_n$ be the generalised cluster algebra of type C_n, with initial cluster $(x_1, \ldots, x_n)$ and initial exchange polynomials
$\theta_i^0(u,v) = p_{i,0}u + p_{i,1}v \, (i \in [\![1, n-1]\!])$ and $\theta_n^0(u,v) = au^2 + \lambda uv + cv^2$,

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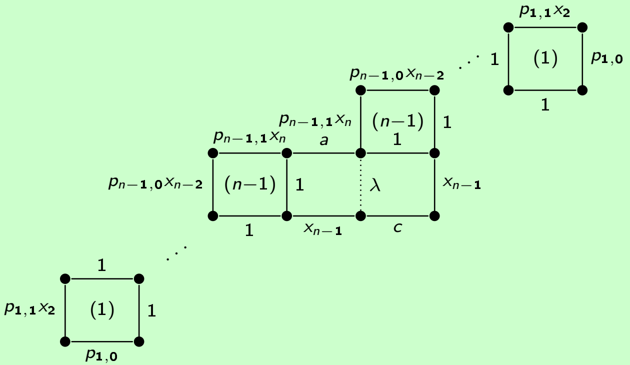

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By recursion : generalised exchange relation after mutating k times in direction $n, \ldots, 1$:

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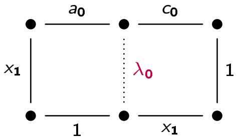
$$\chi_\varepsilon\left(W_\varepsilon(n,1)\right)\chi_\varepsilon\left(W_\varepsilon(n,\varepsilon^2)\right) = \begin{aligned} &\chi_\varepsilon\left(W_\varepsilon(n-1,\varepsilon^2)\right)^2 \\ &+ \mathrm{Fr}^*(V(\varpi))\chi_\varepsilon\left(W_\varepsilon(n-1,\varepsilon^2)\right) + 1. \end{aligned}$$

Type $C_n^{(1)}$

Snake graph pattern : periodically alternating the tile

$T_0 =$ 

and the longest snake graph from type C_n.

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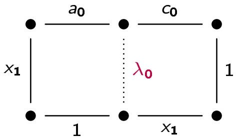
$T_0 =$ 

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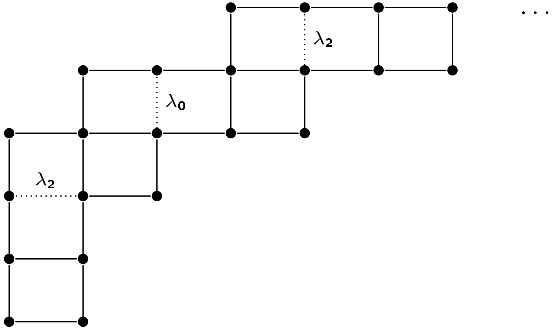
For $C_2^{(1)}$:



Type CD_n

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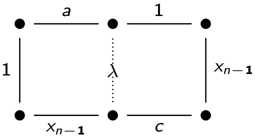
$T_n =$



and the longest type D_n snake graph.

Type CD_n

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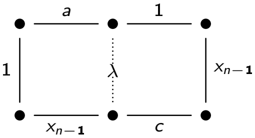

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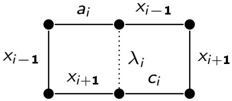
For CD_4 :

1 2 . . .
l
l r
2 1
(λ) 3
2 3
l 1
1 l r
3 2
(λ)

Markov quiver : $0 \rightleftarrows 1 \Rightarrow 2$

Tiles : for $i \in \{0, 1, 2\} \bmod 3$, $T_i =$ 

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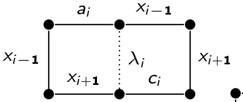
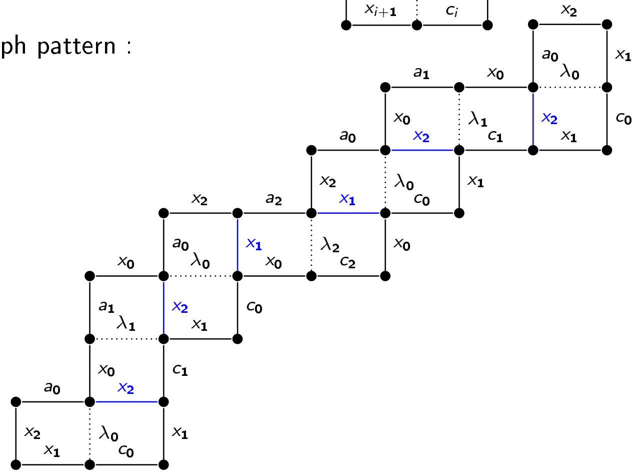


Snake graph pattern :



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- What about degree 3 ? degree 4 ? It gets a lot harder, because the number of terms in the Laurent expansion formulas grow exponentially !

: −)

Thank you for your attention !