Pfaffian Identities and Q-Functions

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Plan:

- Schur Q-functions
- Pfaffian identities and their applications to Schur Q-functions
- Generalized Q-functions
- Symplectic Q-functions and their factorial analogues

Schur functions	Schur Q-functions
partitions	strict partitions
linear representation of S_n	projective representation of S_n
representation of $\mathfrak{gl}(n)$	representation of $\mathfrak{q}(n)$
Grassmannian	Lagrangian Grassmannian
determinants	Pfaffians

Schur Q-Functions

Pfaffian

Let $A = (a_{ij})_{1 \le i,\, j \le 2m}$ be a $2m \times 2m$ skew-symmetric matrix. The Pfaffian of A is defined by

$$\mathrm{Pf}\, A = \sum_{\pi \in \mathfrak{F}_{2m}} \mathrm{sgn}(\pi) a_{\pi(1),\pi(2)} a_{\pi(3),\pi(4)} \cdots a_{\pi(2m-1),\pi(2m)},$$

where $\mathfrak{F}_{2m}$ is the subset of the symmetric group $\mathfrak{S}_{2m}$ given by

$$\mathfrak{F}_{2m} = \left\{ \pi \in \mathfrak{S}_{2m} : \begin{array}{ccccccc} \pi(1) & < & \pi(3) & < & \cdots & < & \pi(2m-1) \\ \wedge & & \wedge & & & & \wedge \\ \pi(2) & & \pi(4) & & & & \pi(2m) \end{array} \right\},$$

and $\mathrm{sgn}(\pi)$ denotes the signature of π.

Example If $2m = 4$, then

$$\mathrm{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Pfaffian and determinant

For an $r \times r$ skew-symmetric matrix C and an $r \times (2m-r)$ matrix B, we have

$$\mathrm{Pf}\begin{pmatrix} C & B \\ -{}^tB & O \end{pmatrix} = \begin{cases} 0 & \text{if } r > m, \\ (-1)^{m(m-1)/2} \det B & \text{if } r = m. \end{cases}$$

Schur Pfaffian

If n is even, then we have

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j + x_i}\right)_{1 \le i, j \le n} = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i}.$$

Schur P-function (Nimmo's formula)

Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a sequence of n indeterminates. For a strict partitio $\lambda = (\lambda_1, \cdots, \lambda_{l(\lambda)})$ ($\lambda_1 > \cdots > \lambda_{l(\lambda)} > 0$), the Schur P-function $P_\lambda(x_1, \cdots, x_n)$ corresponding to λ is defined by

$$P_\lambda(\boldsymbol{x}) = \frac{1}{D(\boldsymbol{x})} \operatorname{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \le i, j \le n} & \left(x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} \\ \hline -{}^t\left(x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} & O \end{array} \right),$$

where $r = l(\lambda)$ or $l(\lambda) + 1$ according to whether $n + l(\lambda)$ is even or odd, and

$$D(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i}.$$

Schur Q-function (Nimmo's formula)

The Schur Q-function $Q_\lambda(x_1, \cdots, x_n)$ corresponding to a strict partition λ is defined by

$$Q_\lambda(\boldsymbol{x}) = 2^{l(\lambda)} P_\lambda(\boldsymbol{x}).$$

Then we have

$$Q_\lambda(\boldsymbol{x})$$

$$= \frac{1}{D(\boldsymbol{x})} \mathrm{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \le i, j \le n} & \left(\chi(\lambda_j) x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} \\ \hline -{}^t\!\left(\chi(\lambda_j) x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} & O \end{array} \right),$$

where

$$\chi(k) = \begin{cases} 1 & \text{if } k = 0, \\ 2 & \text{if } k > 0. \end{cases}$$

Pfaffian Identities
and
Applications to Schur's Q-Functions

Pfaffian analogue of Sylvester identity

Proposition (Knuth) Let n and r be even integers. For an $(n+r) \times (n+r)$ skew-symmetric matrix, we have

$$\operatorname{Pf}\left(\frac{\operatorname{Pf} X([n] \cup \{n+i, n+j\})}{\operatorname{Pf} X([n])}\right)_{1 \le i, j \le r} = \frac{\operatorname{Pf} X}{\operatorname{Pf} X([n])},$$

where $[n] = \{1, 2, \cdots, n\}$ and

$$X(I) = \left(x_{i,j}\right)_{i, j \in I}.$$

Remark (Sylevester identity) For an $(m+s) \times (m+s)$ matrix Y, we have

$$\det\left(\frac{\det Y([m] \cup \{i\}; [m] \cup \{j\})}{\det Y([m]; [m])}\right)_{1 \le i, j \le s} = \frac{\det Y}{\det Y([m]; [m])},$$

where

$$Y(I; J) = \left(y_{i,j}\right)_{i \in I, j \in J}.$$

Pfaffian analogue of Sylvester identity

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Applying this to

$$X = \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \le i, j \le n} & \left(x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} \\ \hline -{}^t\left(x_i^{\lambda_j} \right)_{1 \le i \le n, 1 \le j \le r} & O \end{array} \right),$$

we immediately recover Schur's original definition of Q-functions.

Schur's original definition of Q-functions

Proposition For a strict partition λ, we have

$$Q_\lambda(\boldsymbol{x}) = \mathrm{Pf}\left(Q_{\lambda_i,\lambda_j}(\boldsymbol{x}) \right)_{1 \le i, j \le r},$$

where $r = l(\lambda)$ or $l(\lambda) + 1$ according to whether $l(\lambda)$ is even or odd.

Remark We have

$$\sum_{r \ge 0} Q_r(\boldsymbol{x}) z^r = \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z},$$

$$\sum_{r,s \ge 0} Q_{r,s}(\boldsymbol{x}) z^r w^s = \frac{z - w}{z + w} \left(\prod_{i=1}^{n} \frac{(1 + x_i z)(1 + x_i w)}{(1 - x_i z)(1 - x_i w)} - 1 \right),$$

where $Q_{r,s} = -Q_{s,r}$ for $0 < r < s$, $Q_{r,0} = -Q_{0,r} = Q_r$ for $r > 0$ and $Q_{r,r} = 0$ for $r \ge 0$.

Pfaffian analogue of Cauchy–Binet identity

Theorem Let m and n be positive integers wirh $m \equiv n \bmod 2$. Let

A : an $m \times m$ skew symmetric matrix, S : an $m \times N$ matrix,
B : an $n \times n$ skew symmetric matrix, T : an $n \times N$ matrix.

Then we have

$$\sum_J \operatorname{Pf} \begin{pmatrix} A & S([m];J) \\ -{}^tS([m];J) & O \end{pmatrix} \operatorname{Pf} \begin{pmatrix} B & T([n],J) \\ -{}^tT([n],J) & O \end{pmatrix} = (-1)^{\binom{n}{2}} \operatorname{Pf} \begin{pmatrix} A & S{}^tT \\ -T{}^tS & -B \end{pmatrix},$$

where J runs over all subsets of $[N]$ with $\#J \equiv n \bmod 2$, and

$$S([m];J) = \left(s_{i,j}\right)_{1 \le i \le m,\, j \in J}, \quad T([n];J) = \left(t_{i,j}\right)_{1 \le i \le n,\, j \in J}.$$

Pfaffian analogue of Cauchy–Binet identity

Theorem Let m and n be positive integers wirh $m \equiv n \bmod 2$. Then we have

$$\sum_J \mathrm{Pf} \begin{pmatrix} A & S([m];J) \\ -{}^t S([m];J) & O \end{pmatrix} \mathrm{Pf} \begin{pmatrix} B & T([n],J) \\ -{}^t T([n],J) & O \end{pmatrix} = (-1)^{\binom{n}{2}} \mathrm{Pf} \begin{pmatrix} A & S{}^t T \\ -T{}^t S & -B \end{pmatrix},$$

where J runs over all subsets of $[N]$ with $\# J \equiv n \bmod 2$.

Remark (Cauchy–Binet formula) For two $n \times N$ matrices S and T, we have

$$\sum_J \det S([n];J) \det T([n];J) = \det \left(S{}^t T \right).$$

Cauchy-type formula for Q-functions

Theorem (Schur) For $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, we have

$$\sum_{\lambda} \frac{1}{2^{l(\lambda)}} Q_\lambda(\boldsymbol{x}) Q_\lambda(\boldsymbol{y}) = \prod_{i,j=1}^{n} \frac{1 + x_i y_j}{1 - x_i y_j},$$

where λ runs over all strict partitions of length $\leq n$.

Proof Apply the Pfaffian version of Cauchy–Binet identity to

$$A = \left(\frac{x_j - x_i}{x_j + x_i} \right)_{1 \leq i, j \leq n}, \quad S = \begin{pmatrix} 1 & x_1 & x_1^2 & x_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots \end{pmatrix},$$

$$B = \left(\frac{y_j - y_i}{y_j + y_i} \right)_{1 \leq i, j \leq n}, \quad T = \begin{pmatrix} 1 & 2y_1 & 2y_1^2 & 2y_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 2y_n & 2y_n^2 & 2y_n^3 & \cdots \end{pmatrix}.$$

Jozefiak–Pragacz formula for skew Q-functions

Theorem (Jozefiak–Pragacz) For two strict partitions λ and μ, we define the skew Q-function $Q_{\lambda/\mu}(\boldsymbol{x})$ by

$$Q_{\lambda/\mu}(\boldsymbol{x}) = \operatorname{Pf}\left(\begin{array}{c|c} \left(Q_{\lambda_i,\lambda_j}(\boldsymbol{x})\right)_{1\le i,j\le l} & \left(Q_{\lambda_i-\mu_{r+1-j}}(\boldsymbol{x})\right)_{1\le i\le l,1\le j\le r} \\ \hline -{}^t\left(Q_{\lambda_i-\mu_{r+1-j}}(\boldsymbol{x})\right)_{1\le i\le l,1\le j\le r} & O \end{array}\right),$$

where $r = l(\mu)$ or $l(\mu)+1$ according to whether $l(\lambda)+l(\mu)$ is even or odd. Then we have

$$Q_\lambda(\boldsymbol{x},\boldsymbol{y}) = \sum_\mu Q_{\lambda/\mu}(\boldsymbol{x})Q_\mu(\boldsymbol{y}).$$

Proof Apply (a variant of) the Pfaffian analogue of Cauchy–Binet identity to

$$A = \Big(Q_{\lambda_i,\lambda_j}(\boldsymbol{x}) \Big)_{1 \le i,\, j \le l}, \quad S = \Big(Q_{\lambda_i - k}(\boldsymbol{x}) \Big)_{1 \le i \le l,\, k \ge 0},$$

$$B = \left(\frac{y_j - y_i}{y_j + y_i} \right)_{1 \le i,\, j \le n}, \quad T = \begin{pmatrix} 1 & 2y_1 & 2y_1^2 & 2y_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 2y_n & 2y_n^2 & 2y_n^3 & \cdots \end{pmatrix}.$$

Then we have

$$\sum_{\mu} Q_{\lambda/\mu}(\boldsymbol{x}) Q_{\mu}(\boldsymbol{y})$$

$$= \frac{1}{D(\boldsymbol{y})} \operatorname{Pf} \left(\begin{array}{c|c} \Big(Q_{\lambda_i,\lambda_j}(\boldsymbol{x}) \Big)_{1 \le i,\, j \le l} & \Big(Q_{\lambda_i}(\boldsymbol{x}, y_j) \Big)_{1 \le i \le l,\, 1 \le j \le n} \\ \hline -{}^t\Big(Q_{\lambda_i}(\boldsymbol{x}, y_j) \Big)_{1 \le i \le l,\, 1 \le j \le n} & \left(\frac{y_j - y_i}{y_j + y_i} \right)_{1 \le i,\, j \le n} \end{array} \right).$$

Generalized Q-Functions

Generalized Q functions

Let $\mathcal{G} = \{g_d(t)\}_{d \geq 0}$ be a sequence of univariate polynomials satisfying

$$g_0(t) = 1, \quad \deg g_d(t) = d$$

Then we define generalized Q functions $Q^{\mathcal{G}}_{\lambda}(\boldsymbol{x})$ associated to $\mathcal{G}$ by

$$Q^{\mathcal{G}}_{\lambda}(\boldsymbol{x}) = \frac{1}{D(\boldsymbol{x})} \operatorname{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \leq i, j \leq n} & \left(g_{\lambda_j}(x_i) \right)_{1 \leq i \leq n, 1 \leq j \leq r} \\ \hline -^t\left(g_{\lambda_j}(x_i) \right)_{1 \leq i \leq n, 1 \leq j \leq r} & O \end{array} \right),$$

where $r = l(\lambda)$ or $l(\lambda) + 1$ according to whether $n + l(\lambda)$ is even or odd, and

$$D(\boldsymbol{x}) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}.$$

Generalized Q functions

$$Q_\lambda^{\mathcal{G}}(\boldsymbol{x}) = \frac{1}{D(\boldsymbol{x})} \mathrm{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \le i, j \le n} & \left(g_{\lambda_j}(x_i) \right)_{1 \le i \le n, 1 \le j \le r} \\ \hline -{}^t\left(g_{\lambda_j}(x_i) \right)_{1 \le i \le n, 1 \le j \le r} & O \end{array} \right).$$

Example

(1) If $g_d(t) = t^d$ (resp. $2t^d$) for $d \geq 1$, then $Q_\lambda^{\mathcal{G}}$ is the Schur P-function (resp. Q-function).

(2) If $g_d(t) = \prod_{i=1}^d (t + a_i)$ (resp. $2 \prod_{i=1}^d (t + a_i)$) for $d \geq 1$, then $Q_\lambda^{\mathcal{G}}$ is Ivanov's factorial P-function (resp. Q-function).

Generalized Q-functions

$$Q^{\mathcal{G}}_{\lambda}(\boldsymbol{x})$$

$$= \frac{1}{D(\boldsymbol{x})} \mathrm{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i} \right)_{1 \le i, j \le n} & \left(g_{\lambda_j}(x_i) \right)_{1 \le i \le n, 1 \le j \le r} \\ \hline -{}^t\left(g_{\lambda_j}(x_i) \right)_{1 \le i \le n, 1 \le j \le r} & O \end{array} \right).$$

By applying the Pfaffian analogue of Sylvester identity, we have

Proposition For a strict partition λ, we have

$$Q^{\mathcal{G}}_{\lambda}(\boldsymbol{x}) = \mathrm{Pf} \left(Q^{\mathcal{G}}_{\lambda_i, \lambda_j}(\boldsymbol{x}) \right)_{1 \le i, j \le r}.$$

Jozefiak–Pragacz formula for generalized Q-functions

Theorem For simplicity, we assume that $g_d(0) = 0$ for $d \geq 1$. For two strict partitions λ and μ, we put

$$Q^{\mathcal{G}}_{\lambda/\mu}(\boldsymbol{x}) = \mathrm{Pf}\left(\begin{array}{c|c} \left(Q^{\mathcal{G}}_{\lambda_i,\lambda_j}(\boldsymbol{x})\right)_{1\leq i,j\leq l} & \left(Q^{\mathcal{G}}_{\lambda_i/\mu_{r+1-j}}(\boldsymbol{x})\right)_{1\leq i\leq l,1\leq j\leq r} \\ \hline -{}^t\left(Q^{\mathcal{G}}_{\lambda_i/\mu_{r+1-j}}(\boldsymbol{x})\right)_{1\leq i\leq l,1\leq j\leq r} & O \end{array}\right),$$

where $r = l(\mu)$ or $l(\mu) + 1$ according to whether $l(\lambda) + l(\mu)$ is even or odd, and

$$Q^{\mathcal{G}}_{r}(\boldsymbol{x}, t) = \sum_{k=0}^{r} Q^{\mathcal{G}}_{r/k}(\boldsymbol{x}) g_k(t).$$

Then we have

$$Q^{\mathcal{G}}_{\lambda}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mu} Q^{\mathcal{G}}_{\lambda/\mu}(\boldsymbol{x}) Q^{\mathcal{G}}_{\mu}(\boldsymbol{y}).$$

Remark Note that $Q^{\mathcal{G}}_{r/k} \neq Q^{\mathcal{G}}_{r-k}$ in general.

Symplectic Q-Functions

Symplectic Hall–Littlewood functions

The Hall–Littlewood functions associated to the root system of type C_n are defined by

$$P_\lambda(\boldsymbol{x};t) = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(\boldsymbol{x}^\lambda \prod_{\alpha \in R^+} \frac{1 - t\boldsymbol{x}^{-\alpha}}{1 - \boldsymbol{x}^{-\alpha}} \right)$$

where $\lambda = \sum_{i=1}^n \lambda_i e_i$ is a dominant weight, W is the Weyl group of type C_n and

$$W_\lambda = \{w \in W : w\lambda = \lambda\}, \quad W_\lambda(t) = \sum_{w \in W_\lambda} t^{l(w)},$$

$$R^+ = \left\{e_i \pm e_j : 1 \le i < j \le n\right\} \cup \{2e_i : 1 \le i \le n\}.$$

It is known that

$$P_\lambda(\boldsymbol{x};t) \in \mathbb{Z}[t][x_1^{\pm 1}, \cdots, x_n^{\pm 1}]^W.$$

Symplectic Q-functions

For a strict partition, we define

$$P_{\langle\lambda\rangle}(\boldsymbol{x}) = P_\lambda(\boldsymbol{x}; -1), \quad Q_{\langle\lambda\rangle}(\boldsymbol{x}) = 2^{l(\lambda)} P_{\langle\lambda\rangle}(\boldsymbol{x}).$$

and call them symplectic P-functions and symplectic Q-functions respectively.

Theorem Let $\mathcal{G} = \{g_d(t)\}_{d\geq 0}$ be a polynomial sequence given by

$$g_0(t) = 1, \quad g_d(x + x^{-1}) = 2\left(x^d - x^{-d}\right)\frac{x + x^{-1}}{x - x^{-1}} \quad (d \geq 1).$$

Then we have

$$Q_{\langle\lambda\rangle}(\boldsymbol{x}) = Q_\lambda^{\mathcal{G}}(\boldsymbol{x} + \boldsymbol{x}^{-1}),$$

where $\boldsymbol{x} + \boldsymbol{x}^{-1} = (x_1 + x_1^{-1}, \cdots, x_n + x_n^{-1})$.

Tableaux description of symplectic Q-functions

Definition (Hamel–King) A symplectic primed shifted tableau of shape λ is a filling of the boxes in the shifted diagram $S(\lambda)$ with entries from

$$1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{1} < \cdots < n' < n < \overline{n}' < \overline{n}$$

satisfying the following conditions:

- the entries in each row and in each column are weakly increasing;
- each unprimed entry appears at most once in every column;
- each primed entry appears at most once in every row;
- for each k, at most one of $\{k', k, \overline{k'}, \overline{k}\}$ appears on the main diagonal.

Example

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \overline{2}' & 3' \\ \hline \multicolumn{1}{c|}{} & 2' & \overline{2}' & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array} .$$

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- each unprimed entry appears at most once in every column;
- each primed entry appears at most once in every row;
- for each k, at most one of $\{k', k, \overline{k'}, \overline{k}\}$ appears on the main diagonal.

For such a tableau T, we define

$$\boldsymbol{x}^T = \prod_{k=1}^{n} x_k^{\#\{k', k \text{ in } T\} - \#\{\overline{k}', \overline{k} \text{ in } T\}}.$$

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Example

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \overline{2}' & 3' \\ \hline \multicolumn{1}{c|}{} & 2' & \overline{2}' & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}, \quad \boldsymbol{x}^T = x_1^2 x_2^{-1} x_3^2 x_4.$$

Theorem (Conjectured by Hamel–King) For a strict partition λ, we have

$$Q_{\langle\lambda\rangle}(\boldsymbol{x}) = \sum_{T} \boldsymbol{x}^T$$

where T runs over all symplectic primed shifted tableaux of shape λ.

Idea of Proof Both sides satisfy

- $Q_{\langle\lambda\rangle}(x_1, \cdots, x_{n-1}, x_n) = \sum_{\mu} Q_{\langle\mu\rangle}(x_1, \cdots, x_{n-1}) Q_{\langle\lambda/\mu\rangle}(x_n)$,
- $Q_{\langle\lambda/\mu\rangle}(x_n) = 0$ unless $\lambda \supset \mu$ and $l(\lambda) - l(\mu) \leq 1$,
- $Q_{\langle\lambda/\mu\rangle}(x_n) = \det \left(Q_{\langle\lambda_i - \mu_j\rangle}(x_n) \right)_{1 \leq i, j \leq l(\lambda)}$ if $l(\lambda) - l(\mu) \leq 1$.

Hence the proof is reduced to the case where $\lambda = (r)$ and $\boldsymbol{x} = (x_n)$.

Factorial symplectic Q-functions

Let $\mathcal{G} = \{g_d(t)\}_{d\geq 0}$ be a polynomial sequence given by $g_0(t) = 1$ and

$$g_d(x + x^{-1}) = \left(\prod_{i=0}^{d-1}(x + a_i) - \prod_{i=0}^{d-1}(x^{-1} + a_i)\right)\frac{x + x^{-1}}{x - x^{-1}},$$

where $\boldsymbol{a} = (a_0, a_1, a_2, \cdots)$ is a factorial parameter. Then we define factorial symplectic Q-functions $Q_{\langle\lambda\rangle}(\boldsymbol{x}|\boldsymbol{a})$ to be the generalized Q-functions associated to $\mathcal{G}$ given above:

$$Q_{\langle\lambda\rangle}(\boldsymbol{x}|\boldsymbol{a}) = Q_\lambda^{\mathcal{G}}(\boldsymbol{x} + \boldsymbol{x}^{-1}).$$

If $a_0 = a_1 = a_2 = \cdots = 0$, then we have

$$Q_{\langle\lambda\rangle}(\boldsymbol{x}|0) = Q_{\langle\lambda\rangle}(\boldsymbol{x}) = 2^{l(\lambda)}P_\lambda(\boldsymbol{x}; -1).$$

Factorial symplectic Q-functions

Theorem Assume that $a_0 = 0$. Then we have

$$Q_{\langle\lambda\rangle}(\boldsymbol{x}|\boldsymbol{a}) = \sum_{T} (\boldsymbol{x}|\boldsymbol{a})^T$$

where T runs over all symplectic primed shifted tableaux of shape λ, and

$$(\boldsymbol{x}|\boldsymbol{a})^T = \prod_{(i,j)\in S(\lambda)} \mathrm{wt}(T_{i,j}; a_{j-i})$$

with

$$\mathrm{wt}(\gamma; a) = \begin{cases} x_k - a & \text{if } \gamma = k', \\ x_k + a & \text{if } \gamma = k, \\ x_k^{-1} - a & \text{if } \gamma = \overline{k'}, \\ x_k^{-1} + a & \text{if } \gamma = \overline{k}. \end{cases}$$

Remark The right hand (combinatorics) side was introduced by King–Hamel.