On the Intersection of G-Set Colourings

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1. Introduction - A problem for starters

Problem

If $A \subseteq \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, what is

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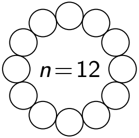

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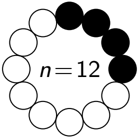

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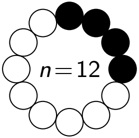

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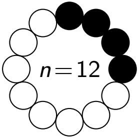

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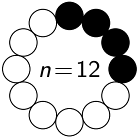

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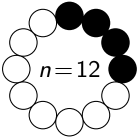

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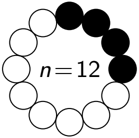

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But what if A is not an interval?

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- There is another induced action on the power set $\mathcal{P}(X)$ of X:

$$G \times \mathcal{P}(X) \longrightarrow \mathcal{P}(X) \ , \ (g, A) \longmapsto g.A := \{g.a | a \in A\}.$$

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Orbit-stabilizer relation

$$|G.x| \cdot |G_x| = |G| \text{ for all } x \in X.$$

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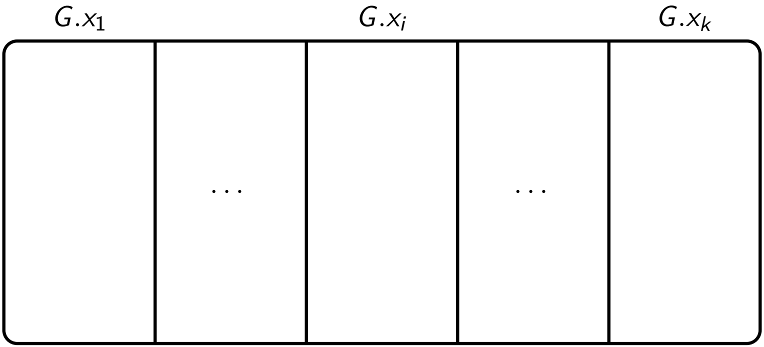
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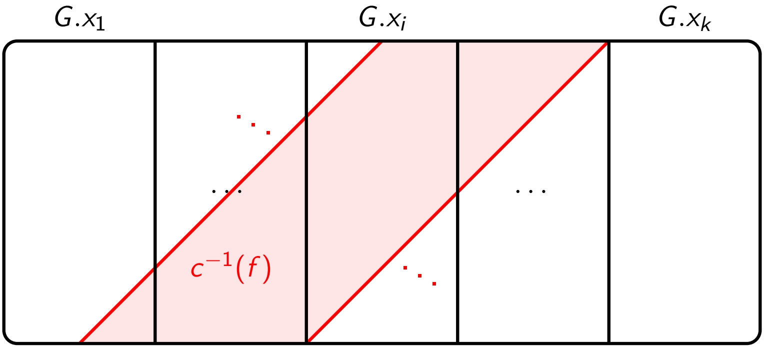
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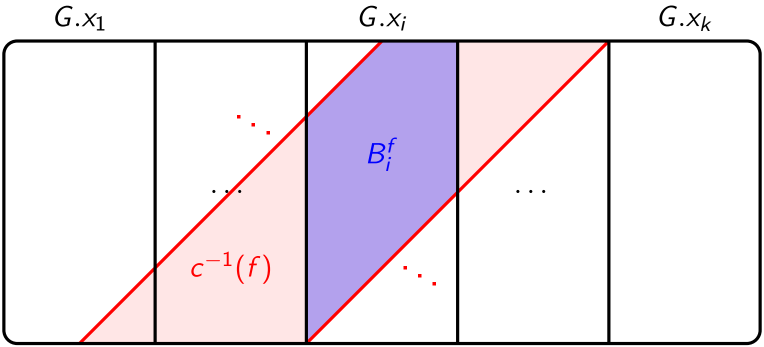

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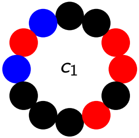

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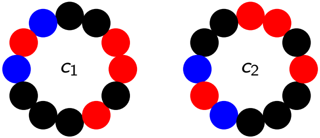

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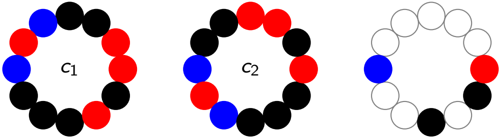

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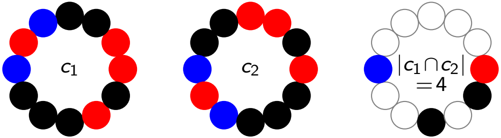

2. Main result

Lemma

For all $(g_1, ..., g_m) \in G^m$ we have

$$\left|\bigcap_{i=1}^{m} g_i.c\right| = \sum_{f \in F}\left|\bigcap_{i=1}^{m} g_i.(c^{-1}(f))\right|.$$

2. Main result

Theorem (subset version)

Let G be a finite group, X a finite G-set and $A \subseteq X$. Then

$$\frac{1}{|G|^m} \sum_{g \in G^m} \left| \bigcap_{i=1}^{m} g_i.A \right| = \sum_{A_i \neq \emptyset} \frac{|A_i|^m}{|G.x_{A_i}|^{m-1}}.$$

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Note that the right-most sum is always taken over $i \in \{1, ..., k\}$.

2. Main result - Proof of subset version

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\end{aligned}
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$$= \sum_{x \in X} \left(\frac{|G_x|}{|G|} |G.x \cap A| \right)^m$$

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&= \sum_{x\in X} \left(\frac{|G_x|}{|G|} |G.x \cap A| \right)^m \quad = \sum_{i=1}^{k} \sum_{x\in G.x_i} \left(\frac{|G.x \cap A|}{|G.x|} \right)^m \\
&= \sum_{\substack{i=1 \\ G.x_i\cap A\neq\emptyset}}^{k} |G.x_i| \left(\frac{|G.x_i \cap A|}{|G.x_i|} \right)^m
\end{aligned}
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$$= \frac{1}{|G|^m}\sum_{x\in X}\left(\sum_{h\in G}[h.x\in A]\right)^m \quad = \frac{1}{|G|^m}\sum_{x\in X}\left(|G_x|\sum_{y\in G.x}[y\in A]\right)^m$$

$$= \sum_{x\in X}\left(\frac{|G_x|}{|G|}|G.x\cap A|\right)^m \quad = \sum_{i=1}^{k}\sum_{x\in G.x_i}\left(\frac{|G.x\cap A|}{|G.x|}\right)^m$$

$$= \sum_{\substack{i=1\\ G.x_i\cap A\neq\emptyset}}^{k}|G.x_i|\left(\frac{|G.x_i\cap A|}{|G.x_i|}\right)^m \quad = \sum_{A_i\neq\emptyset}\frac{|A_i|^m}{|G.x_{A_i}|^{m-1}}$$

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Theorem (subset version)

Let G be a finite group, X a finite G-set and $A \subseteq X$. Then

$$\frac{1}{|G|^m} \sum_{g \in G^m} \left| \bigcap_{i=1}^{m} g_i.A \right| = \sum_{A_i \neq \emptyset} \frac{|A_i|^m}{|G.x_{A_i}|^{m-1}}.$$

Theorem (colouring version)

Let G be a finite group, X a finite G-set and $c \in F^X$. Then

$$\frac{1}{|G|^m} \sum_{g \in G^m} \left| \bigcap_{i=1}^{m} g_i.c \right| = \sum_{f \in F} \sum_{B_i^f \neq \emptyset} \frac{|B_i^f|^m}{|G.x_{B_i^f}|^{m-1}}.$$

Note that the right-most sum is always taken over $i \in \{1, ..., k\}$.

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Proposition

Let G be a finite group, $A \subsetneq G$ a proper subset and $l \in \mathbb{N}$ with $l \leq |A|$ such that

$$\frac{(|A| \quad - 1)^{\underline{l-1}}}{(|A|_G - 1)^{\underline{l-1}}} \geq \frac{|A|}{|G_A|}.$$

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For example, if $|A| = 100, |A|_G = 85$ and $|G_A| = 2$ then the actual movement of A by means of $G \setminus G_A$ is already reflected in a subset $B \subseteq A$ with at most $l = 23$ elements.

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2. Main result - Injective colourings

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A generalization of Burnside's lemma (alternative formulation)

Let G be a finite group and X a finite G-set. Then

$$\sum_{d \mid |G|} n_d \, d^{2-m} = \underbrace{\frac{1}{|G|^{m-1}} \sum_{g \in G^{m-1}} |\operatorname{fix}(g_1) \cap \ldots \cap \operatorname{fix}(g_{m-1})|}_{=:\operatorname{Fix}(G,m)}.$$

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Let G be a finite group and X a finite transitive G-set with $|X|>1$. Then there is some $g \in G$ with $\text{fix}(g) = \emptyset$.

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Let G be a finite group and X a finite G-set with $|X| > 1$. If

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This is precisely Jordan's theorem if we choose $m = 2$.

Let us assume that

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Task: Generalize the results by Cameron, Cohen and Serre to the non-transitive case!

3. Applications - Refined enumeration of necklaces

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Let $n, a \in \mathbb{N}$. The *necklaces* with n beads and a colours are defined as the orbits of $G = \mathbb{Z}/n\mathbb{Z}$ acting on $X = \{1, \ldots, a\}^{\{0,\ldots,n-1\}}$ via cyclic permutations. Their number equals

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Let us see if it is possible to refine necklace enumeration using our generalization of Burnside's lemma.

Generalization of Burnside's lemma

Let G be a finite group and X a finite G-set. Then

$$\sum_{d \mid |G|} n_d \, d^{2-m} = \underbrace{\frac{1}{|G|^{m-1}} \sum_{g \in G^{m-1}} |\operatorname{fix}(g_1) \cap \ldots \cap \operatorname{fix}(g_{m-1})|}_{=\operatorname{Fix}(G,m)} .$$

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Strategy: Evaluate $\operatorname{Fix}(G, m)$ for suitable choices of m to obtain a system of linear equations for the variables n_d.

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$$J_k(x) := |\{r \in \{1,\dots,x\}^k \,|\, \gcd(r_1,\dots,r_k,x) = 1\}| = x^k \prod_{\substack{p|x \\ p \text{ prime}}} \left(1 - \frac{1}{p^k}\right)$$

3. Applications - Refined enumeration of necklaces

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The unique solution is

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An orbit of length d means a symmetry group of order n/d by the orbit-stabilizer relation.
For example, we read off that among the $3 + 3 + 48 + 5880 = 5934$ necklaces on $n = 10$ points with up to $a = 3$ colours there are precisely 48 which possess a two-element symmetry group.

4. Conclusion remarks

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- This in turn enabled us to generalize an old theorem by Jordan.
- Further generalizations of the results by Cameron, Cohen and Serre - all based on Jordan's theorem - might be possible.
- As we have demonstrated, applications of our results are varied. It would be interesting to find even more of them.

References

- P. J. Cameron, A. M. Cohen, *On the number of fixed point free elements in a permutation group*. Discrete Mathematics **106/107** (1992), 135-138.
- J-P. Serre. *On a theorem of Jordan*, Bulletin of the American Mathematical Society **40** (2003), 429-440.

Thank you very much!