Block decomposition of permutations and Schur Positivity

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Schur Positivity

A symmetric function is called "Schur positive" if its coordinates in the basis of Schur functions are non-negative.

Example

Given $\lambda \vdash k$ and $\mu \vdash \ell$, consider the product

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

The Littlewood-Richardson rule provides a combinatorial interpretation of the coefficients $c^\nu_{\lambda,\mu}$, proving that $s_\lambda s_\mu$ is Schur positive.

Statistics on $\mathcal{S}_n$

$\pi \in S_n$ is equipped with the statistics:

- The descent set:

$$Des(\pi) = \{i \mid \pi(i) > \pi(i+1)\}.$$

Example

$\pi = \bar{5}\bar{3}2\bar{6}14$, so $Des(\pi) = \{1, 2, 4\}$.

- The left to right maxima:

$$LtrMax(\pi) = \{i \mid \pi(i) > \pi(j) \text{ for all } j < i\}.$$

Example

$\pi = \bar{3}\bar{6}2415.$

Standard Young tableaux

λ is a partition of n, represented by a Young diagram.
A standard Young tableau of shape λ is a filling of the cells of λ such that:

- The entries in each row are strictly increasing.
- The entries in each column are strictly increasing.

Denote the set of all standard Young tableaux of shape λ by $SYT(\lambda)$

Example

$$SYT(3,2) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

The descent set of a SYT

The descent set of a standard Young tableaux T is

$$Des(T) = \{i \mid i+1 \text{ is in a lower row than } i\}.$$

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

$$Des(T) = \{2, 4\}.$$

Semi standard Tableaux

Definition

λ is a shape. A semistandard Young tableau of shape λ is a filling of the cells of λ such that

- The entries in each row are weakly increasing.
- The entries in each column are strictly increasing.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}.$$

The Schur function

To each semi standard Young tableau T, we associate the weight monomial:

$$\mathbf{x}^T = \prod_i x_i^{\text{number of i's in} T}.$$

Example

$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$ has $\mathbf{x}^T = x_1 x_2^2 x_3^2 x_4 x_5 x_6$.

For a partition λ, the Schur function s_λ is defined as:

$$s_\lambda = \sum_{T \in SSYT} \mathbf{x}^T$$

Proposition

$\{s_\lambda \mid \lambda \vdash n\}$ *is a basis for degree* n *homogenuous s.f.*

Example

For $\lambda = (2,1)$ the list of semistandard tableaux of shape λ with numbers $1,2,3$ is:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

The corresponding Schur polynomial is:

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Quasisymmetric functions

A formal power series $f(x_1, x_2, \cdots)$ is a quasisymmetric function if for every composition $(\alpha_1, \cdots, \alpha_k)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with indices $i_1 < i_2 < \cdots < i_k$ have the same coefficients.

Example

(In 3 variables)

$$f = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$$

is quasisymmetric but not symmetric.

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The fundamental basis

For each subset $D \subseteq [n-1]$ define the *quasi-symmetric function*

$$F_D(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

In the typical case, the sets are descents sets of permutations.

Example

$\pi = 132$, $Des(\pi) = \{2\}$.

$$\mathcal{F}_{Des\{132\}} = x_1x_1x_2 + x_1x_1x_3 + x_1x_2x_3 + x_2x_2x_3 + \cdots .$$

Proposition

The algebra of homogeneous quasisymetric functions n, Q_n,has $\{\mathcal{F}_D\}_{D\subseteq[n-1]}$ as a basis. This is Gessel's fundamental basis of Q_n.

Schur poitivity

For $A \subseteq \mathcal{S}_n$, let

$$\mathcal{Q}(A) = \sum_{\pi \in A} \mathcal{F}_{Des(\pi)}.$$

$\mathcal{Q}(A)$ is called **Schur positive** if it is symmetric and can be written as a linear combination of Schur functions with non-negative coefficients.

A is called Schur Positive if $\mathcal{Q}(A)$ is Schur positive.

Question

(Gessel, Reutenaur, '93) For which $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ symmetric?

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Some known Schur Positive sets

- Subsets closed under conjugation. (involutions, derangments,...).(Gessel, Reutenauer, '93).
- Permutations with prescribed number of inversions (Adin, Roichman, '15).
- Arc permutations. (each prefix of π forms an interval in $\mathbb{Z}_n$).

Knuth classes

For every standard Young tableau T of size n, the set

$$\mathcal{C}_T := \{\pi \in \mathcal{S}_n : \; P_\pi = T\}$$

is a Knuth class corresponding to T, where P_π is given by the RSK correspondence: $\pi \mapsto (P_\pi, Q_\pi)$.

Example

$213 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\right)$ and $231 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\right)$, so that

$$\mathcal{C}_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}} = \{213, 231\}.$$

Proposition

For $\pi \in S_n$:

$$Des(\pi) = Des(Q_\pi), Des(\pi^{-1}) = Des(P_\pi).$$

Proposition

(Gessel, '84) Knuth classes are Schur-positive.

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Two questions

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(Sagan, Woo, '14) Is there a way to combine pattern avoidance and quasisymmetric functions?
In other words, which pattern avoiding sets of permutations $\mathcal{S}_n(\sigma_1, \sigma_2, \cdots)$ *are Schur positive?*

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Is there a way to combine pattern avoidance, quasisymmetric functions and permutation statistics?
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The blocks number

Definition

Let $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$. The direct sum *of π and σ is the permutation $\pi \oplus \sigma \in \mathcal{S}_{m+n}$ defined by*

$$(\pi \oplus \sigma)_i = \begin{cases} \pi(i), & \text{if } i \leq n; \\ \sigma(i-n)+n, & \text{otherwise.} \end{cases}$$

Example

If $\pi = 132$ and $\sigma = 4231$ then $\pi \oplus \sigma = 1327564$.

The direct sum is clearly associative.

A nonempty permutation which is not a direct sum of two nonempty permutations is called $\oplus$-*irreducible*.
Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the *blocks* of π.

$$bl(\pi) = \text{number of blocks.}$$

Example

$$\text{bl}(45321) = 1,$$

$$\text{bl}(312 \mid 54) = 2,$$

$$\text{bl}(1 \mid 2 \mid 3 \mid 4) = 4.$$

Another statistic: the last descent

Definition

For a permutation $\pi \in S_n$ let

$$\mathrm{ldes}(\pi) := \max\{i : \; i \in \mathrm{Des}(\pi)\},$$

with $\mathrm{ldes}(\pi) := 0$ *if* $\mathrm{Des}(\pi) = \emptyset$ *(i.e., if* π *is the identity permutation).*

The sets $Bl_{n,k}$ and $L_{n,k}$

Definition

Let

$$Bl_{n,k} := \{\pi \in \mathcal{S}_n(321) : \ \mathrm{bl}(\pi) = k\}.$$

Definition

Let

$$L_{n,k} = \{\pi \in \mathcal{S}_n(321) : \ \mathrm{ldes}(\pi^{-1}) = k\}.$$

Enumeration of $Bl_{n,k}$

Definition

Recall: The Catalan number is given by:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The corresponding generating function is

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Enumeratrion of $B_{n,k}$

Definition

For each $k \geq 0$, the n-th k-fold Catalan number is the coefficient of x^n in $(xc(x))^k$, given by:

$$C_{n,k} = \frac{k}{2n-k}\binom{2n-k}{n}.$$

Proposition

For positive integers $n \geq k \geq 1$:

$$C_{n,k} = |SYT(n-1, n-k)| = L_{n,n-k} = B_{n,k}$$

This result will be refined in a moment.

LtrMax determines Des in $\mathcal{S}_n(321)$

Definition

$\mathcal{S}_n(321)$ is the set of $321-$ avoiding permutations in $\mathcal{S}_n$.

Observation

For $\pi \in \mathcal{S}_n(321)$, the complement of LtrMax(π) is an increasing sequance.

Example

$$\pi = \bar{3}12\bar{5}4\bar{6}.$$

Observation

For each $\pi \in \mathcal{S}_n(321)$, the descents of π are placed exactly in the transitions from left to right maxima to non left to right maxima.

Example

$$\pi = \bar{3}12\bar{5}4\bar{6}$$

Proposition

For $\pi \in \mathcal{S}_n(321)$, height$(P_\pi) \leq 2$.

Equidistribution

We present a **left-to-right-maxima preserving** bijection from $Bl_{n,k}$ to $L_{n,n-k}$ which will give us:

Theorem (A.B.R. '16)

For every positive integer n:

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi)} t^{\pi^{-1}(n)} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi)} t^{\pi^{-1}(n)} q^{n-\mathrm{ldes}(\pi^{-1})}.$$

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The bijection

Definition

The map $f_n : \mathcal{S}_n(321) \mapsto \mathcal{S}_n(321)$ is defined recursively on n, as follows, distinguishing between 3 cases, according to the location of n in π or the relative order of $n-1$ and n.

- L : n is positioned in the **L**ast location.
- D: n is not positioned in the last slot and $n-1$ preceds n.
- R: $n-1$ is to the right of n.

Case L: n is the last letter.

- Omit n
- Apply f_{n-1};
- Insert n at the last position.

Case D: $n-1$ is left of n, but n is not the last letter.

- Omit n.
- Apply f_{n-1}.
- Multiply from left by the transposition $(n-k-1, n-k)$.
- Insert n at the same position as in π.

Case R: $n-1$ is right of n.
In this case $n-1$ must be the last letter.

- Exchange $n-1$ and n in π, then omit n.
- Apply f_{n-1}
- Multiply (from the left) the resulting permutation by the cycle $(n-k, n-k+1, ..., n-1, n)$.

Example

Let $\pi_8 = \pi = 31254786$

$$\pi_8 = 31254786 \xrightarrow[(45)]{D} \pi_7 = 3125476 \xrightarrow[(4567)]{R} \pi_6 = 312546 \xrightarrow{L}$$

$$\pi_5 = 31254 \xrightarrow[(345)]{R} \pi_4 = 3124$$

$$\xrightarrow{L} \pi_3 = 312 \xrightarrow[(23)]{R} \pi_2 = 21$$

Now we make the other way around.

$$f(\pi_2) = 21 \xrightarrow{(23)} f(\pi_3) = 312 \rightarrow f(\pi_4) = 3124 \xrightarrow{(345)}$$

$$f(\pi_5) = 41253 \xrightarrow{(45)} f(\pi_6) = 412536$$

$$\xrightarrow{(4567)} f(\pi_7) = 5126374 \xrightarrow{(45)} f(\pi_8) = 41263785$$

Schur positivity of $Bl_{n,k}$

Theorem: (A.B.R) $\mathcal{Q}(Bl_{n,k})$ is Schur positive.

Proof:

Recall that in $\mathcal{S}_n(321)$ The *LtrMax* determines the *Des* and let $t = 1$ in

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi)} t^{\pi^{-1}(n)} q^{\mathrm{bl}(\pi)} = \mathbf{x}^{\mathrm{ltrMax}(\pi)} t^{\pi^{-1}(n)} q^{n-\mathrm{Ides}(\pi^{-1})},$$

to get:

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} q^{n-\mathrm{Ides}(\pi^{-1})}.$$

Hence

$$\mathcal{Q}(Bl_{n,k}) = \sum_{\pi \in Bl_{n,k}} \mathcal{F}_{Des(\pi)} = \sum_{\pi \in L_{n,n-k}} \mathcal{F}_{Des(\pi)} = \mathcal{Q}(L_{n,n-k}).$$

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On the other hand,

$$L_{n,n-k} = \{\pi \in \mathcal{S}_n(321) \mid Ides(\pi^{-1}) = n-k\} =$$

$$\{\pi \in \mathcal{S}_n \mid \text{height}(P_\pi) < 3 \text{ and } \text{Ides}(P_\pi) = n-k\}$$

is a disjoint union of Knuth classes, thus is Schur-positive.

Characters

Recall that the *Frobenius image* of an $\mathcal{S}_n$-character $\chi = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda$ is the symmetric function $f = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, denoted by $ch(\chi)$.

Theorem

For every positive integer $1 \le k \le n$

$$\mathcal{Q}(Bl_{n,k}) = ch(\chi^{n-1,n-k} \downarrow_{S_n}),$$

where ch is the Frobenius characteristic map from class functions on $\mathcal{S}_n$ *to symmetric functions.*

Open questions

1. Find a non-recursive definition for the bijection.
2. A patterns-statistics pair $(\Pi, stat)$ consisting of $\Pi \subseteq \mathcal{S}_m$ and a permutations statistic $stat : \mathcal{S}_n \longrightarrow \mathbb{N}$ is Schur-positive if

$$\mathcal{Q}(\{\pi \in \mathcal{S}_n(\Pi) \mid stat(\pi) = k\})$$

is Schur-positive for all positive integers n and k.
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Thank you

Corollary

Thank you for your attention!