A new proof of a *q*-continued fraction of Ramanujan

Gaurav Bhatnagar (Wien)

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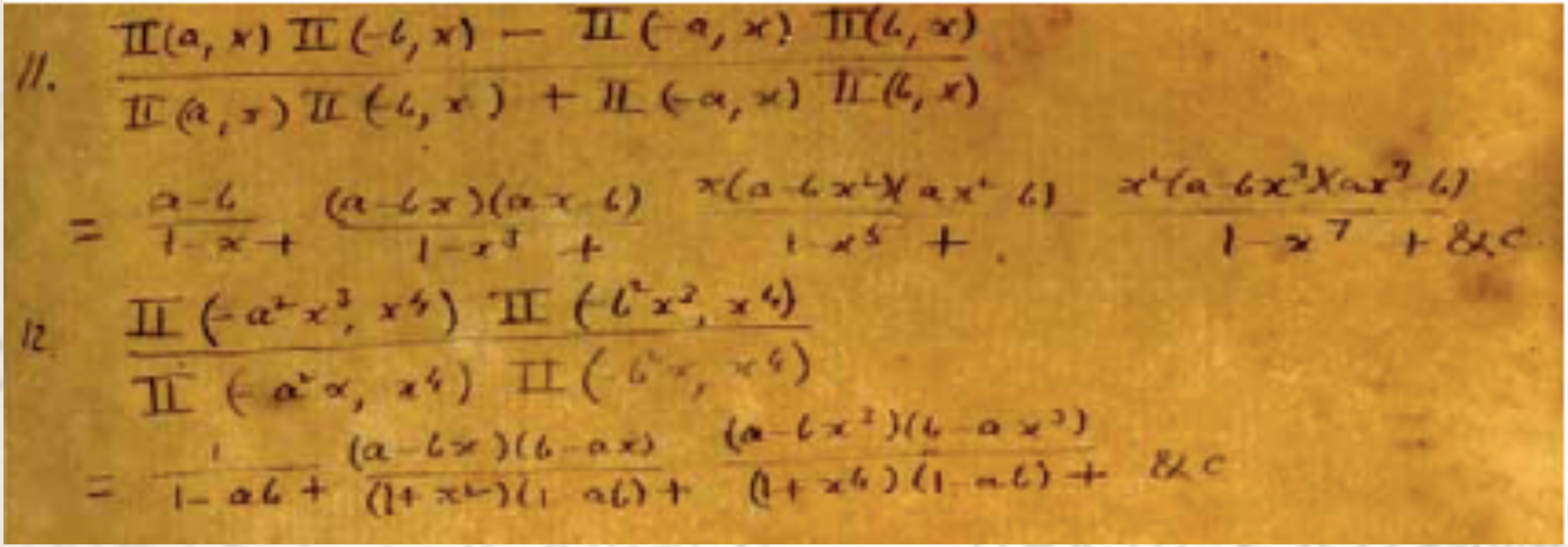
It is hoped that others will attempt to discover the pathways that Ramanujan took on his journey through his luxuriant labyrinthine forest of enchanting and alluring formulas.

Bruce Berndt (Page 1, Ramanujan's Notebooks, Part III)

Goal



11. $$\frac{\Pi(a,x)\,\Pi(-b,x)-\Pi(-a,x)\,\Pi(b,x)}{\Pi(a,x)\,\Pi(-b,x)+\Pi(-a,x)\,\Pi(b,x)} = \frac{a-b}{1-x}+\frac{(a-bx)(ax-b)}{1-x^3}+\frac{x(a-bx^2)(ax^2-b)}{1-x^5}+\frac{x^2(a-bx^3)(ax^3-b)}{1-x^7}+\&c$$

12. $$\frac{\Pi(-a^2x^3,x^4)\,\Pi(-b^2x^3,x^4)}{\Pi(-a^2x,x^4)\,\Pi(-b^2x,x^4)} = \frac{1}{1-ab}+\frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)}+\frac{(a-bx^2)(b-ax^2)}{(1+x^4)(1-ab)}+\&c$$

$$\prod(a,x) := \prod_{k=0}^{\infty}(1+ax^k)$$

Ch. 16, Entry 11, 12 Notebook 2 (Part III, Berndt)

Ch. 16, Entry 11, Notebook 2 (Part III, Berndt)

Goal

$$\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}=$$

$$\frac{a-b}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3}\,+\,\frac{q(a-bq^2)(aq^2-b)}{1-q^5}\,+\,\frac{q^2(a-bq^3)(aq^3-b)}{1-q^7}\,+\cdots$$

Ch. 16, Entry 12

$$\frac{(a^2q^3,b^2q^3;q^4)_\infty}{(a^2q,b^2q;q^4)_\infty}=$$

$$\frac{1}{1-ab+}\,\frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+}\,\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)}\,+\,\frac{(a-bq^5)(b-aq^5)}{(1-ab)(1+q^6)}\,+\cdots.$$

$$(a;q)_\infty:=\prod_{j=0}^{\infty}(1-aq^j),\ \text{for } |q|<1$$

The approach comes from

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$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3$$
$$+ m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

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Euler's approach

$$\frac{N}{D} = 1 + \frac{N - D}{D}$$

$$\frac{1 + a_1x + a_2x^2 + a_3x^3 + \cdots}{1 + b_1x + b_2x^2 + b_3x^3 + \cdots} = 1 + \frac{(1 + a_1x + a_2x^2 + \cdots) - (1 + b_1x + b_2x^2 + \cdots)}{1 + b_1x + b_2x^2 + b_3x^3 + \cdots}$$

Example 1.
The Rogers-Ramanujan Continued fraction

Cor. Entry 15,
Chapter 16,
Notebook 2, Part 3

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} a^k}{\displaystyle\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} a^k} = \cfrac{1}{1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \cfrac{aq^3}{1 + \cdots}}}}$$

where $\quad (a;q)_n := \displaystyle\prod_{j=0}^{n-1} (1 - aq^j)$

The (formal) proof

$$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k}=\frac{1}{\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}}$$

Divide!

$$\frac{N}{D}=1+\frac{N-D}{D}$$

$$\frac{1}{1+\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k-\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}}$$

Consider the difference of sums

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} a^k - \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} a^k = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} a^k (1-q^k)$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q)_k} a^k (1-q^k) &= \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q)_{k-1}} a^k \\
&= \sum_{k=0}^{\infty} \frac{q^{(k+1)^2}}{(q;q)_k} a^{k+1} \\
&= aq \sum_{k=0}^{\infty} \frac{q^{k^2+2k}}{(q;q)_k} a^k
\end{aligned}$$

$$\frac{1}{1+\dfrac{aq\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}} = \frac{1}{1+}\;\frac{aq}{\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}}$$

$$\frac{1}{1+\cfrac{aq}{1+\cfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k(1-q^k)}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}}}$$

$$\frac{N}{D} = 1 + \frac{N-D}{D}$$

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k(1-q^k) &= \sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(q;q)_{k-1}}a^k \\
&= \sum_{k=0}^{\infty}\frac{q^{(k+1)^2+k+1}}{(q;q)_k}a^{k+1} \\
&= aq^2\sum_{k=0}^{\infty}\frac{q^{k^2+3k}}{(q;q)_k}a^k
\end{aligned}$$

$$\frac{1}{1}+\frac{aq}{1}+\frac{aq^2}{\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+3k}}{(q;q)_k}a^k}}$$

In general:

$$R(s) := \sum_{k=0}^{\infty} \frac{q^{k^2+sk}}{(q;q)_k} a^k$$

$$\frac{R(s)}{R(s+1)} = 1 + \frac{aq^{s+1}}{\frac{R(s+1)}{R(s+2)}}$$

$$\frac{R(1)}{R(0)} = \frac{1}{\frac{R(0)}{R(1)}} = \frac{1}{1+}\frac{aq}{1+}\frac{aq^2}{1+}\cdots+\frac{aq^{s+1}}{\frac{R(s+1)}{R(s+2)}}$$

Take limits to complete (formal) proof.

Notes

- Note this calculation:

$$\frac{1-q^k}{(q;q)_k} = \frac{1-q^k}{(1-q)(1-q^2)\cdots(1-q^k)} = \frac{1}{(q;q)_{k-1}}$$

- When we shift the index, a few terms come out of the sum, because we want the sum to have first term 1
- These sums can be written as infinite products. These are the famous Rogers-Ramanujan identities

Rogers-Ramanujan Identities

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})};$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+2})(1-q^{5m+3})}$$

Example 2

- Want a continued fraction where we know that the sums can be written in terms of infinite products.
- We can think of using the q-binomial theorem, the simplest example of a sum written as a ratio of products.
- Here's one approach.

Example 2. Entry 11

Entry 3. The q-binomial theorem

$$\frac{(b;q)_\infty}{(a;q)_\infty} = \sum_{k=0}^{\infty} \frac{(b/a;q)_k}{(q;q)_k} a^k$$

$$\frac{(-b;q)_\infty}{(-a;q)_\infty} = \sum_{k=0}^{\infty} \frac{(b/a;q)_k}{(q;q)_k} (-1)^k a^k$$

$$\frac{\dfrac{(b;q)_\infty}{(a;q)_\infty} - \dfrac{(-b;q)_\infty}{(-a;q)_\infty}}{\dfrac{(b;q)_\infty}{(a;q)_\infty} + \dfrac{(-b;q)_\infty}{(-a;q)_\infty}} = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k+1}}{(q;q)_{2k+1}} a^{2k+1}}{\displaystyle\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k}}$$

Ratio of odd part/even part of series

Entry 11- Product side

$$\frac{\dfrac{(b;q)_\infty}{(a;q)_\infty}-\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}{\dfrac{(b;q)_\infty}{(a;q)_\infty}+\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}=\frac{\sum_{k=0}^{\infty}\dfrac{(b/a;q)_{2k+1}}{(q;q)_{2k+1}}a^{2k+1}}{\sum_{k=0}^{\infty}\dfrac{(b/a;q)_{2k}}{(q;q)_{2k}}a^{2k}}$$

$$\frac{\dfrac{(b;q)_\infty}{(a;q)_\infty}-\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}{\dfrac{(b;q)_\infty}{(a;q)_\infty}+\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}=\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}$$

Rewrite the products to get one side of Entry 11

Ratio of sums

$$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{(b/a;q)_{2k+1}}{(q;q)_{2k+1}}a^{2k+1}}{\displaystyle\sum_{k=0}^{\infty}\frac{(b/a;q)_{2k}}{(q;q)_{2k}}a^{2k}}.$$

The ratio can be written as

$$\frac{a-b}{(1-q)\frac{\displaystyle\sum_{k=0}^{\infty}\frac{(b/a;q)_{2k}}{(q;q)_{2k}}a^{2k}}{\displaystyle\sum_{k=0}^{\infty}\frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}}a^{2k}}}$$

Apply Euler's approach

$$\frac{N_1}{D_1} \equiv \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k}}{\displaystyle\sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k}} = 1 + \frac{1}{D_1}\left(\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k} - \sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k}\right)$$

$$= 1 + \frac{1}{D_1}\left(\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k}\left[1 - \frac{1-bq^{2k}/a}{1-b/a} \cdot \frac{1-q}{1-q^{2k+1}}\right]\right)$$

$$= 1 + \frac{1}{D_1}\left(\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k}\left[\frac{(q-b/a)(1-q^{2k})}{(1-b/a)(1-q^{2k+1})}\right]\right).$$

Recall

$$(q;q)_{2k} = (1-q)(1-q^2)\cdots(1-q^{2k})$$

Apply Euler's approach

$$\begin{aligned}\frac{N_1}{D_1} &= 1+\frac{1}{D_1}\left(\sum_{k=1}^{\infty}\frac{(bq/a;q)_{2k-1}}{(q;q)_{2k-1}}a^{2k}\left[\frac{q-b/a}{1-q^{2k+1}}\right]\right)\\
&= 1+\frac{1}{D_1}\left(\sum_{k=0}^{\infty}\frac{(bq/a;q)_{2k+1}}{(q;q)_{2k+1}}a^{2k+2}\left[\frac{q-b/a}{1-q^{2k+3}}\right]\right)\\
&= 1+\frac{1}{D_1}\left(\frac{a^2(1-bq/a)(q-b/a)}{(1-q)(1-q^3)}\sum_{k=0}^{\infty}\frac{(bq^2/a;q)_{2k}}{(q^2;q)_{2k}}a^{2k}\left[\frac{1-q^3}{1-q^{2k+3}}\right]\right)\end{aligned}$$

So we get

$$\cfrac{a-b}{1-q+\cfrac{(a-bq)(aq-b)}{(1-q^3)\cfrac{\sum_{k=0}^{\infty}\frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}}a^{2k}}{\sum_{k=0}^{\infty}\frac{(bq^2/a;q)_{2k}}{(q^2;q)_{2k}}a^{2k}\left[\frac{1-q^3}{1-q^{2k+3}}\right]}}}$$

Again, apply Euler's approach to the ratio of two series

$$
\begin{aligned}
\frac{N_2}{D_2} \equiv & \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k}}{\displaystyle\sum_{k=0}^{\infty} \frac{\left(bq^2/a;q\right)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[\frac{1-q^3}{1-q^{2k+3}}\right]} \\
&= 1 + \frac{1}{D_2}\left(\sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k} - \sum_{k=0}^{\infty} \frac{\left(bq^2/a;q\right)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[\frac{1-q^3}{1-q^{2k+3}}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[1 - \frac{1-bq^{2k+1}/a}{1-bq/a} \cdot \frac{1-q^3}{1-q^{2k+3}}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\sum_{k=0}^{\infty} \frac{(bq/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[\frac{(q^3-bq/a)(1-q^{2k})}{(1-bq/a)(1-q^{2k+3})}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\sum_{k=1}^{\infty} \frac{\left(bq^2/a;q\right)_{2k-1}}{(q^2;q)_{2k-2}} a^{2k} \left[\frac{q^3-bq/a}{(1-q^{2k+1})\,(1-q^{2k+3})}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\sum_{k=0}^{\infty} \frac{\left(bq^2/a;q\right)_{2k+1}}{(q^2;q)_{2k}} a^{2k+2} \left[\frac{q^3-bq/a}{(1-q^{2k+3})\,(1-q^{2k+5})}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\frac{a^2(1-bq^2/a)(q^3-bq/a)}{(1-q^3)(1-q^5)} \sum_{k=0}^{\infty} \frac{\left(bq^3/a;q\right)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[\frac{1-q^3}{1-q^{2k+3}}\,\frac{1-q^5}{1-q^{2k+5}}\right]\right) \\
&= 1 + \frac{1}{D_2}\left(\frac{q(a-bq^2)(aq^2-b)}{(1-q^3)(1-q^5)} \sum_{k=0}^{\infty} \frac{\left(bq^3/a;q\right)_{2k}}{(q^2;q)_{2k}} a^{2k} \left[\frac{1-q^3}{1-q^{2k+3}}\,\frac{1-q^5}{1-q^{2k+5}}\right]\right).
\end{aligned}
$$

We get

$$\cfrac{a-b}{1-q+\cfrac{(a-bq)(aq-b)}{(1-q^3)+\cfrac{q(a-bq^2)(aq^2-b)}{(1-q^5)\,\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq^2/a;q\right)_{2k}}{(q^2;q)_{2k}}a^{2k}\left[\frac{1-q^3}{1-q^{2k+3}}\right]}{\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq^3/a;q\right)_{2k}}{(q^2;q)_{2k}}a^{2k}\left[\frac{1-q^3}{1-q^{2k+3}}\cdot\frac{1-q^5}{1-q^{2k+5}}\right]}}}}$$

Pattern is clear now.

Define, for $s = 1, 2, 3, \ldots$

$$C(s) := \sum_{k=0}^{\infty} \frac{(bq^s/a;q)_{2k}}{(q^2;q)_{2k}} a^{2k} \prod_{i=1}^{s-1} \frac{1-q^{2i+1}}{1-q^{2k+2i+1}}.$$

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k+1}}{(q;q)_{2k+1}} a^{2k+1}}{\displaystyle\sum_{k=0}^{\infty} \frac{(b/a;q)_{2k}}{(q;q)_{2k}} a^{2k}} = \frac{a-b}{1-q+\cfrac{(a-bq)(aq-b)}{(1-q^3)\cfrac{C(1)}{C(2)}}}$$

$$(1-q^{2s+1})\frac{C(s)}{C(s+1)} = 1-q^{2s+1} + q^s \frac{(a-bq^{s+1})(aq^{s+1}-b)}{(1-q^{2s+3})\cfrac{C(s+1)}{C(s+2)}}$$

Iterate to obtain

Proposition:
A "finite form" of Entry 11

For: $|q| < 1 \; and \; |a| < 1$

$$\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}=$$

$$\cfrac{a-b}{1-q+\cfrac{(a-bq)(aq-b)}{1-q^3+\cfrac{q(a-bq^2)(aq^2-b)}{1-q^5}+\cdots}}$$

$$+\cfrac{q^{s-1}(a-bq^s)(aq^s-b)}{1-q^{2s+1}+\cfrac{q^s(a-bq^{s+1})(aq^{s+1}-b)}{(1-q^{2s+3})\frac{C(s+1)}{C(s+2)}}}$$

As s goes to infinity, we get "Modified Covnergence" of the infinite continued fraction of Entry 11

Next

- Entry 11 involves $1-q, 1-q^3, 1-q^5$
- One can ask: Is there a similar continued fraction with even powers?
- We had $(q^2;q)_{2k}$ in the denominator.
- One can try with $(q^2;q^2)_k$
- In view of the q-binomial theorem, one can try with sums of the type: $$\sum_{k=0}^{\infty}\frac{(b/a;q^2)_k}{(q^2;q^2)_k}a^k$$

- After some messing around, we end up with $\dfrac{\sum_{k=0}^{\infty} \frac{(b/aq;q^2)_k}{(q^2;q^2)_k}(aq)^k}{\sum_{k=0}^{\infty} \frac{(bq/a;q^2)_k}{(q^2;q^2)_k} a^k}$

- The calculations involved become

$$\frac{(1-q^k)}{(q^2;q^2)_k} = \frac{(1-q^k)}{(q,-q;q)_k} = \frac{1}{(q;q)_{k-1}(-q;q)_k}$$

We use

$$(a;q^2)_k = (\sqrt{a},-\sqrt{a};q^2)_k$$

$$(a,b;q)_k = (a;q)_k(b;q)_k$$

- On shifting index, we will get

$$\frac{1}{(q;q)_k(1+q)(-q^2;q)_k}$$

- We get $(1+q)$ not $1-q^2$

- So we take $q \mapsto q^2$ and squares of other parameters too, and make some more minor adjustments.

Example 3: Entry 12

We begin with

$$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{((bq/a)^2;q^4)_k}{(q^4;q^4)_k}(a^2q)^k}{\displaystyle\sum_{k=0}^{\infty}\frac{((b/aq)^2;q^4)_k}{(q^4;q^4)_k}(a^2q^3)^k}=\frac{(b^2q^3;q^4)_\infty/(a^2q;q^4)_\infty}{(b^2q;q^4)_\infty/(a^2q^3;q^4)_\infty}$$

$$=\frac{(a^2q^3,b^2q^3;q^4)_\infty}{(a^2q,b^2q;q^4)_\infty}$$

Product side is OK. One can now try what happens in Euler's approach.

Entry 12: Euler's approach

We begin with

$$\frac{\sum_{k=0}^{\infty}\frac{((bq/a)^2;q^4)_k}{(q^4;q^4)_k}(a^2q)^k}{\sum_{k=0}^{\infty}\frac{((b/aq)^2;q^4)_k}{(q^4;q^4)_k}(a^2q^3)^k}=\frac{\sum_{k=0}^{\infty}\frac{(bq/a,-bq/a;q^2)_k}{(q^2,-q^2;q^2)_k}(a^2q)^k}{\sum_{k=0}^{\infty}\frac{(b/aq,-b/aq;q^2)_k}{(q^2,-q^2;q^2)_k}(a^2q^3)^k}$$

$$=\frac{1}{\dfrac{\sum_{k=0}^{\infty}\frac{(b/aq,-b/aq;q^2)_k}{(q^2,-q^2;q^2)_k}(a^2q^3)^k}{\sum_{k=0}^{\infty}\frac{(bq/a,-bq/a;q^2)_k}{(q^2,-q^2;q^2)_k}(a^2q)^k}}$$

We use

$$(a;q^2)_k=(\sqrt{a},-\sqrt{a};q^2)_k$$

$$(a,b;q)_k=(a;q)_k(b;q)_k$$

$$
\begin{aligned}
\frac{N}{D} &= \frac{\displaystyle\sum_{k=0}^{\infty} \frac{\left(b/aq, -b/aq; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q^3)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q)^k} \\
&= 1 + \frac{1}{D} \left(\sum_{k=0}^{\infty} \frac{\left(b/aq, -b/aq; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q^3)^k - \sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q)^k \right) \\
&= 1 + \frac{1}{D} \left(\sum_{k=0}^{\infty} \frac{\left(b/aq, -b/aq; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q)^k \left(q^{2k} - \frac{\left(1 - bq^{2k}/aq\right)\left(1 + bq^{2k}/aq\right)}{(1 - b/aq)(1 + b/aq)} \right) \right) \\
&= 1 + \frac{1}{D} \left(\sum_{k=0}^{\infty} \frac{\left(b/aq, -b/aq; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q)^k \left((-1) \frac{\left(1 - q^{2k}\right)\left(1 + b^2 q^{2k}/a^2 q^2\right)}{(1 - b/aq)(1 + b/aq)} \right) \right) \\
&= 1 + \frac{(-1)}{D} \left(\sum_{k=1}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_{k-1}}{(q^2; q^2)_{k-1} (-q^2; q^2)_k} (a^2 q)^k \left(1 + b^2 q^{2k}/a^2 q^2\right) \right) \\
&= 1 + \frac{(-1)}{D} \left(\sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2 q)^k \left[\frac{a^2 q \left(1 + b^2 q^{2k}/a^2\right)}{1 + q^{2k+2}} \right] \right).
\end{aligned}
$$

Some issues

The factor $(1+q^{2k+2})$ is not a problem, we can use

$$\left(-q^2;q^2\right)_k\left(1+q^{2k+2}\right)=\left(1+q^2\right)\left(-q^4;q^2\right)_k$$

But we cannot absorb the factor $\left(1+b^2q^{2k}/a^2\right)$ in the sum.

However, if we just had $1-bq^{2k+1}/a$, we could proceed by using

$$\left(bq/a;q^2\right)_k\left(1-bq^{2k+1}/a\right)=\left(1-bq/a\right)\left(bq^3/a;q^2\right)_k.$$

So we add and subtract enough terms to get this factor and see what happens

$$
\begin{aligned}
\frac{a^2q\left(1+b^2q^{2k}/a^2\right)}{1+q^{2k+2}} &= \frac{\left(a^2q+b^2q^{2k+1}\right)}{1+q^{2k+2}} \\
&= \frac{\left(a^2q-abq^{2k+2}+b^2q^{2k+1}-ab+abq^{2k+2}+ab\right)}{1+q^{2k+2}} \\
&= \frac{\left(a^2q\left(1-bq^{2k+1}/a\right)-ab\left(1-bq^{2k+1}/a\right)+ab\left(1+q^{2k+2}\right)\right)}{1+q^{2k+2}} \\
&= \frac{\left(a^2q-ab\right)\left(1-bq^{2k+1}/a\right)+ab\left(1+q^{2k+2}\right)}{1+q^{2k+2}} \\
&= ab+\frac{a\left(aq-b\right)\left(1-bq^{2k+1}/a\right)}{1+q^{2k+2}}.
\end{aligned}
$$

So we add and subtract enough terms to get this factor and see what happens

$$
\begin{aligned}
\frac{N}{D} &= 1 - ab - \frac{a(aq-b)(1-bq/a)}{(1+q^2)\,D}\left(\sum_{k=0}^{\infty}\frac{\left(bq^3/a,-bq/a;q^2\right)_k}{(q^2,-q^4;q^2)_k}(a^2q)^k\right)\\
&= 1 - ab + \frac{(b-aq)(a-bq)}{\dfrac{\left(1+q^2\right)\left(\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq/a,-bq/a;q^2\right)_k}{(q^2,-q^4;q^2)_k}(a^2q)^k\right)}{\left(\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq^3/a,-bq/a;q^2\right)_k}{(q^2,-q^2;q^2)_k}(a^2q)^k\right)}}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{N_1}{D_1} &= \frac{\displaystyle\sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{\left(q^2, -q^2; q^2\right)_k} (a^2 q)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{\left(bq^3/a, -bq/a; q^2\right)_k}{\left(q^2, -q^4; q^2\right)_k} (a^2 q)^k} \\
&= 1 + \frac{1}{D_1}\left(\sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{\left(q^2, -q^2; q^2\right)_k} (a^2 q)^k \left(1 - \frac{\left(1 - bq^{2k+1}/a\right)\left(1+q^2\right)}{\left(1-bq/a\right)\left(1+q^{2k+2}\right)} \right) \right) \\
&= 1 + \frac{1}{D_1}\left(\sum_{k=0}^{\infty} \frac{\left(bq/a, -bq/a; q^2\right)_k}{\left(q^2, -q^2; q^2\right)_k} (a^2 q)^k \left((-1) \frac{\left(1 - q^{2k}\right)\left(q^2 + bq/a\right)}{\left(1-bq/a\right)\left(1+q^{2k+2}\right)} \right) \right) \\
&= 1 + \frac{(-1)}{D_1}\left(\sum_{k=1}^{\infty} \frac{\left(bq^3/a; q^2\right)_{k-1}\left(-bq/a; q^2\right)_k}{\left(q^2; q^2\right)_{k-1}\left(-q^2; q^2\right)_k} (a^2 q)^k \left(\frac{q^2 + bq/a}{1+q^{2k+2}} \right) \right) \\
&= 1 + \frac{(-1)}{D_1}\left(\sum_{k=0}^{\infty} \frac{\left(bq^3/a; q^2\right)_{k}\left(-bq/a; q^2\right)_{k+1}}{\left(q^2; q^2\right)_{k}\left(-q^2; q^2\right)_{k+1}} (a^2 q)^k \left[\frac{a^2 q\left(q^2 + bq/a\right)}{1+q^{2k+4}} \right] \right) \\
&= 1 + \frac{(-1)}{D_1}\left(\sum_{k=0}^{\infty} \frac{\left(bq^3/a, -bq/a; q^2\right)_k}{\left(q^2, -q^4; q^2\right)_k} (a^2 q)^k \left[\frac{\left(1 + bq^{2k+1}/a\right)\left(a^2 q^3 + abq^2\right)}{\left(1+q^2\right)\left(1+q^{2k+4}\right)} \right] \right).
\end{aligned}
$$

We now have an extra factor

$$\left[\frac{\left(1+bq^{2k+1}/a\right)\left(a^2q^3+abq^2\right)}{(1+q^2)\,(1+q^{2k+4})}\right].$$

But given our experience, let us try to obtain the factor

$$\left(1-bq^{2k+3}/a\right)$$

$$
\begin{aligned}
&\frac{\left(1+bq^{2k+1}/a\right)\left(a^2q^3+abq^2\right)}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}\\
&=\frac{a^2q^3+abq^{2k+4}+abq^2+b^2q^{2k+3}}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}\\
&=\frac{\left(a^2q^3-abq^{2k+6}\right)+abq^{2k+4}+abq^2+\left(b^2q^{2k+3}-ab\right)+ab+abq^{2k+6}}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}\\
&=\frac{\left(a^2q^3-ab\right)\left(1-bq^{2k+3}/a\right)+ab\left(1+q^2+q^{2k+4}+q^{2k+6}\right)}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}\\
&=\frac{a\left(aq^3-b\right)\left(1-bq^{2k+3}/a\right)+ab\left(1+q^2\right)\left(1+q^{2k+4}\right)}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}\\
&=ab+\frac{a\left(aq^3-b\right)\left(1-bq^{2k+3}/a\right)}{\left(1+q^2\right)\left(1+q^{2k+4}\right)}
\end{aligned}
$$

$$\left(1+q^2\right)\frac{N_1}{D_1}=(1-ab)\left(1+q^2\right)-\frac{a(aq^3-b)(1-bq^3/a)}{(1+q^4)\,D_1}\left(\sum_{k=0}^{\infty}\frac{\left(bq^5/a,-bq/a;q^2\right)_k}{(q^2,-q^6;q^2)_k}(a^2q)^k\right)$$

$$=(1-ab)\left(1+q^2\right)+\frac{(b-aq^3)(a-bq^3)}{\dfrac{(1+q^4)\left(\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq^3/a,-bq/a;q^2\right)_k}{(q^2,-q^4;q^2)_k}(a^2q)^k\right)}{\left(\displaystyle\sum_{k=0}^{\infty}\frac{\left(bq^5/a,-bq/a;q^2\right)_k}{(q^2,-q^6;q^2)_k}(a^2q)^k\right)}}.$$

The pattern is clear now.

$$D(s) := \sum_{k=0}^{\infty} \frac{\left(aq^{2s-1}/b, -aq/b; q^2\right)_k}{(q^2, -q^{2s}; q^2)_k} (a^2q)^k$$

$$\frac{1}{N_1/D_1} = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{\left(aq/b, -aq/b; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2q)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{\left(a/bq, -a/bq; q^2\right)_k}{(q^2, -q^2; q^2)_k} (a^2q^3)^k} = \frac{1}{1 - ab + \cfrac{(a-bq)(b-aq)}{(1+q^2)\cfrac{D(1)}{D(2)}}}$$

Further, for $s = 1, 2, \ldots,$ we have

$$(1+q^{2s})\frac{D(s)}{D(s+1)} = (1-ab)(1+q^{2s}) + \frac{(a-bq^{2s+1})(b-aq^{2s+1})}{(1+q^{2s+2})\cfrac{D(s+1)}{D(s+2)}}$$

Entry 12: Iterate to obtain

Proposition:
A “finite form” of Entry 12

$$\frac{(a^2q^3, b^2q^3; q^4)_\infty}{(a^2q, b^2q; q^4)_\infty} =$$

$$\frac{1}{1-ab+}\frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+}\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)}+\cdots$$

$$+\frac{(a-bq^{2s-1})(b-aq^{2s-1})}{(1-ab)(1+q^{2s})}+\frac{(a-bq^{2s+1})(b-aq^{2s+1})}{(1+q^{2s+2})\frac{D(s+1)}{D(s+2)}}$$

As s goes to infinity, we get “Modified Covnergence” of the infinite continued fraction of Entry 12

For: $|q| < 1 \ and \ |a| < 1$

Overview

- Entry 11: Was first proved by Adiga, Berndt, Bhargava and Watson (1985)
  - Used Heine's continued fraction, q-binomial theorem
- Entry 12: Adiga, Berndt, Bhargava and Watson (1985), Jacobsen (1989), Ramanathan (1987)
  - Adiga, Berndt et.al. thank Askey and Bressoud for ideas on how to prove Entry 12.
  - Used Heine's continued fraction, Heine's transformation, Bailey-Daum summation
- Our proof uses only the q-binomial theorem, which is Entry 2 in chapter 16 of Ramanujan's second notebook (Berndt, Part III)
- It's a "discovery" proof.
- Given Ramanujan's talents, not too farfetched to think he may have thought like this.

I have often been asked whether Ramanujan had any special secret; whether his methods differed in kind from those of other mathematicians; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception.

G. H. Hardy

Thank you

Gaurav Bhatnagar